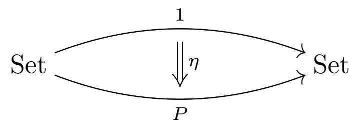
EXERCISES IN CATEGORY THEORY 3

Try either (2) or (2*) depending upon whether you prefer monoids or vector spaces.

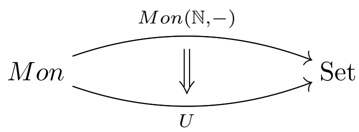
(1) Let $PX = \{U : U \subseteq X\}$ be the power set. Extend this construction to a functor $P : \mathrm{Set} \to \mathrm{Set}$. Show that the functions $\eta_X : X \to PX : x \mapsto \{x\}$ give a natural transformation



(2) The *free monoid* FX on a set X has elements given by words (lists) $[x_1 x_2 \dots x_n]$. Multiplication in FX is given by joining words together (eg. $[xy] \circ [z] = [xyz]$), and the identity element is the *empty word* $[-]$. Show that F is part of a functor $F : \mathrm{Set} \to Mon$.
Let $U : Mon \to \mathrm{Set}$ be the forgetful functor and consider the composite $W = UF : \mathrm{Set} \to \mathrm{Set}$. Describe a natural transformation $1 \Rightarrow W$.

(2*) The *free vector space* FX on a set X is the vector space with basis $\{e_i : i \in X\}$: thus elements of FX are finite sum $\Sigma_i r_i e_i$ where each $r_i \in \mathbb{R}$. Show that F extends to a functor $F : \mathrm{Set} \to Vect$. Letting $U : Vect \to \mathrm{Set}$ denote the forgetful functor, describe a natural transformation $1 \to UF$.

(3) Let $\mathbb{N} = (\mathbb{N}, +, 0)$ be the monoid of natural numbers with addition. Describe a natural transformation

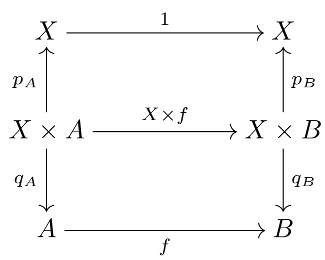


and show that it is a natural isomorphism.

(4) Let $\mathcal{K}$ be a category with products and $X \in \mathcal{K}$. Given $f : B \to C$ there exists a unique map

$$X \times f : X \times A \to X \times B$$

making the diagram



commute. Use the universal property of products to show that this gives a functor $X \times - : \mathcal{K} \to \mathcal{K}$. (*More generally product gives a functor* $- \times - : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$.)

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(5) Show that the product projections $q_A : X \times A \to A$ form the components of a natural transformation