EXERCISES IN CATEGORY THEORY 5

1. Equalisers and coequalisers, pullbacks and pushouts

1.1. Some questions about Set.

- (1) Let $f: X \to X$ be a function. Can you describe $Fix(f) = \{x \in X : f(x) = x\}$ as the equaliser in Set of f and another function?
- (2) Let $U, V \subseteq X$ be subsets of X and consider the inclusions $i: U \to X$ and $j: V \to X$. Show that the intersection of U and V is the pullback of i and j:

$$\begin{array}{ccc} U \cap V \xrightarrow{p} & U \\ q & & \downarrow i \\ V \xrightarrow{j} & X \end{array}$$

Can you find a nice description of the *pushout* of p and q as a subset of X too?

(3) Let $U \subset Y$ and $f: X \to Y$. Describe the preimage $f^{-1}(U) = \{x \in X : f(x) \in U\}$ as a pullback.

1.2. General categorical questions.

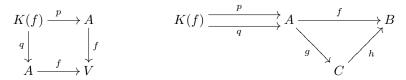
- (1) Show that a category with a terminal object and pullbacks also has products.
- (2) In a general category, prove that an arrow $f: A \to B$ is mono if and only if the square



(3) Let $f, g: A \rightrightarrows B$ have coequaliser $h: B \to C$. Show that h is an epi.

1.3. Quotients and coequalisers.

- (1) Consider an equivalence relation E on X, denoted $E = \{(x, y) \in X^2 : xEy\}$. Then we have projections $p, q : E \rightrightarrows X$ where p(x, y) = x and q(x, y) = y. Show that the coequaliser of p and q is $X \rightarrow X/E$ where X/E is the set of equivalence classes of E.
- (2) Let \mathcal{C} be a category with pullbacks and coequalisers. Given $f: A \to B$ consider the pullback of f with itself:



and then the coequaliser $g: A \to C$ of p and q. Show that there exist a unique arrow $h: C \to B$ such that hg = f as drawn above.

Show that in Set the above factorisation agrees with the factorisation of a function through its image:

$$X \xrightarrow{g} im(f) \xrightarrow{h} Y$$

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where $im(f) = \{y \in Y : \exists x \text{ such that } fx = y\}.$

(You may find it helpful to use that $K(f) = \{(x, y) : fx = fy\}$ is an equivalence relation and use the previous question).

- (3) If you are enthusiastic, show that in the category of monoids/groups/vector spaces, the same factorisation coincides with the factorisation of a homomorphism through its image.
- (4) Let $f: G \to H$ be a group homomorphism. Show that the kernel of f is the equaliser

$$ker(f) \longrightarrow A \xrightarrow[]{f} B$$

where $0: G \to H$ is the homomorphism sending every element to the unit. Given a group G and normal subgroup N describe the quotient group G/N as a coequaliser in a similar way.

1.4. Coequalisers and gluing topological spaces. These exercises concern the relationship between coequalisers, pushouts and gluing in topology.

- (1) Let $1 = \{*\}$ be the 1-point space and consider the real interval [0, 1]. There are two continuous maps $f, g: 1 \Rightarrow [0, 1]$ given by f(*) = 0 and g(*) = 1 respectively. Show that the coequaliser in the category of topological spaces of f and g is the circle S^1 .
- (2) Can you describe the cylinder S¹ × [0, 1] as a coequaliser? How about the Mobius strip or more complex spaces?
 Can you describe the 2-dimensional sphere S² = {(x, y, z) : x² + y² + z² = 1} as a pushout by gluing two disks along their boundary?

 $\mathbf{2}$