EXERCISES IN CATEGORY THEORY 5

1. Equalisers and coequalisers, pullbacks and pushouts

1.1. Some questions about Set**.**

(1) Let $f : X \to X$ be a function. Can you describe $Fix(f) = \{x \in X : f(x) = x\}$ as the equaliser in Set of f and another function?
(2) Let $U, V \subseteq X$ be subsets of X and consider the inclusions $i : U \to X$ and $j : V \to X$. Show that the intersection of U and V is the pullback of i and j:

$$\begin{array}{ccc} U \cap V & \xrightarrow{p} & U \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle i} \\ V & \xrightarrow{j} & X \end{array}$$

Can you find a nice description of the *pushout* of p and q as a subset of X too?
(3) Let $U \subset Y$ and $f : X \to Y$. Describe the preimage $f^{-1}(U) = \{x \in X : f(x) \in U\}$ as a pullback.

1.2. General categorical questions.

(1) Show that a category with a terminal object and pullbacks also has products.
(2) In a general category, prove that an arrow $f : A \to B$ is mono if and only if the square

$$\begin{array}{ccc} A & \xrightarrow{1} & A \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{f} & B \end{array}$$

(3) Let $f, g : A \rightrightarrows B$ have coequaliser $h : B \to C$. Show that h is an epi.

1.3. Quotients and coequalisers.

(1) Consider an equivalence relation E on X, denoted $E = \{(x, y) \in X^2 : xEy\}$. Then we have projections $p, q : E \rightrightarrows X$ where $p(x, y) = x$ and $q(x, y) = y$. Show that the coequaliser of p and q is $X \to X/E$ where X/E is the set of equivalence classes of E.
(2) Let $\mathcal{C}$ be a category with pullbacks and coequalisers. Given $f : A \to B$ consider the pullback of f with itself:

$$\begin{array}{ccc} K(f) & \xrightarrow{p} & A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{f} & V \end{array} \qquad\qquad \begin{array}{ccccc} K(f) & \overset{p}{\underset{q}{\rightrightarrows}} & A & \xrightarrow{f} & B \\ & & {\scriptstyle g}\searrow & & \nearrow{\scriptstyle h} \\ & & & C & \end{array}$$

and then the coequaliser $g : A \to C$ of p and q. Show that there exist a unique arrow $h : C \to B$ such that $hg = f$ as drawn above.
Show that in Set the above factorisation agrees with the factorisation of a function through its image:

$$X \xrightarrow{g} im(f) \xrightarrow{h} Y$$

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where $im(f) = \{y \in Y : \exists x \text{ such that } fx = y\}$.
(You may find it helpful to use that $K(f) = \{(x,y) : fx = fy\}$ is an equivalence relation and use the previous question).

(3) If you are enthusiastic, show that in the category of monoids/groups/vector spaces, the same factorisation coincides with the factorisation of a homomorphism through its image.

(4) Let $f : G \to H$ be a group homomorphism. Show that the kernel of f is the equaliser

$$ker(f) \longrightarrow A \underset{0}{\overset{f}{\rightrightarrows}} B$$

where $0 : G \to H$ is the homomorphism sending every element to the unit.
Given a group G and normal subgroup N describe the quotient group G/N as a coequaliser in a similar way.

1.4. **Coequalisers and gluing topological spaces.** These exercises concern the relationship between coequalisers, pushouts and gluing in topology.

(1) Let $1 = \{*\}$ be the 1-point space and consider the real interval $[0,1]$. There are two continuous maps $f, g : 1 \rightrightarrows [0,1]$ given by $f(*) = 0$ and $g(*) = 1$ respectively. Show that the coequaliser in the category of topological spaces of f and g is the circle S^1.

(2) Can you describe the cylinder $S^1 \times [0,1]$ as a coequaliser? How about the Mobius strip or more complex spaces?
Can you describe the 2-dimensional sphere $S^2 = \{(x,y,z) : x^2 + y^2 + z^2 = 1\}$ as a pushout by gluing two disks along their boundary?