EXERCISES IN CATEGORY THEORY 7

1. Equivalence of categories

An equivalence of categories consists of categories $\mathcal{C}$ and $\mathcal{D}$ together with functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $1_{\mathcal{C}} \to GF$ and $1_{\mathcal{D}} \to FG$.[1] We also call the functors involved $F : \mathcal{C} \to \mathcal{D}$ and $\mathcal{D} \to \mathcal{C}$ equivalences of categories.

(1) Let Cat_1 denote the category of categories with one object and functors between them. There is a functor $U : \mathrm{Cat}_1 \to Mon$ which sends a category C with one object x to the monoid $C(x, x)$. Show that U is part of an equivalence of categories.
(2) Show that equivalence of categories is an equivalence relation on categories.

A functor $U : \mathcal{C} \to \mathcal{D}$ is said to be *essentially surjective* if given $a \in \mathcal{D}$ there exists $b \in \mathcal{C}$ such that Fb is isomorphic to a in $\mathcal{D}$. A functor $U : \mathcal{C} \to \mathcal{D}$ is said to be *fully faithful* if given $f : Ua \to Ub$ there exists a unique arrow $g : a \to b$ such that $Ug = f$. (In other words, the function $F_{a,b} : \mathcal{C}(a, b) \to \mathcal{D}(Fa, Fb)$ is a bijection.)
In fact a functor $U : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if it is essentially surjective and fully faithful.
Use this result where useful in the following exercises.

(3) Set_{par} is the category of sets and partial functions. A *partial function* $(U, f) : X \to Y$ consists of a subset $U \subseteq X$ and function $f : U \to Y$. Thus a function from X to Y which is only defined on U. The composite of $(U, f) : X \to Y$ and $(V, g) : Y \to Z$ is the partial function $(W, gf) : X \to Z$ where $W = \{x \in U : fx \in V\}$. (Draw some diagrams to see this).
The category of pointed sets Set_* has objects (X, x) where $x \in X$ and morphisms $f : (X, x) \to (Y, y)$ are functions such that $fx = y$.
There is a functor $F : \mathrm{Set}_{par} \to \mathrm{Set}_*$ defined by $FX = (X + 1, * \in 1)$ and sends $(U, f) : X \to Y$ to $k : (X + 1, *) \to (Y + 1, *)$ defined by $kx = fx$ if $x \in U$ and $k(y) = *$ otherwise. Show that F is a functor and an equivalence of categories.
(4) Show that if $F : \mathcal{C} \to \mathcal{D}$ is an equivalence then it preserves terminal and initial objects. (In fact, an equivalence preserves any limits or colimits that exist.)
(5) (*Harder*) Prove that a functor is an equivalence of categories if and only if it is essentially surjective on objects and fully faithful.

Date: October 29, 2014.

[1]Note that the direction of the natural isomorphisms does not matter since if $1_{\mathcal{C}} \to GF$ is a natural isomorphism then we get a natural isomorphism $GF \to 1_{\mathcal{C}}$ by taking inverses.