EXERCISES IN CATEGORY THEORY 9

1. Adjoint functors

1.1. **Algebra of adjoint functors.**

(1) Consider $U_1 : \mathcal{A} \to \mathcal{B}$ and $U_2 : \mathcal{B} \to \mathcal{C}$ and adjunctions $F_1 \dashv U_1$ and $F_2 \dashv U_2$. Show that we have a composite adjunction $F_1F_2 \dashv U_2U_1$.
(2) Show that the left adjoint of a functor $U : \mathcal{A} \to \mathcal{B}$ is unique up to natural isomorphism.

1.2. **Adjoints and representables.** Let $\mathcal{C}$ be a locally small category. Recall that a functor $U : \mathcal{C} \to \mathrm{Set}$ is said to be *representable* if there exists a natural isomorphism $U \cong \mathcal{C}(X, -)$ for some $X \in \mathcal{C}$.

(1) Let $\mathcal{C}$ and U be as above. Prove that if U has a left adjoint then U is representable.
(2) Assuming that $\mathcal{C}$ has all (infinite) coproducts show that the converse holds: U is representable $\implies$ U has a left adjoint.

1.3. **Examples.**

(1) We have seen that $U : Mon \to \mathrm{Set}$ has a left adjoint (constructed using word monoids). Prove that U does *not* have a right adjoint.
(2) Show that the forgetful functor $U : Grp \to Mon$ from groups to monoids *does* have a right adjoint – construct it!

1.4. **Further questions.**

(1) Consider the forgetful functor $O : \mathrm{Cat} \to \mathrm{Set}$ which sends a small category to its set of objects. Show that there is a string of four adjoints $C \dashv D \dashv O \dashv I$.
(2) Show that $U : \mathcal{A} \to \mathcal{B}$ has a left adjoint $\iff$ each category X/U has an initial object $\iff$ each functor $\mathcal{B}(X, U-) : \mathcal{A} \to \mathrm{Set}$ is representable.[1] [2]
(3) Given a category $\mathcal{C}$ a *weakly initial set* consists of a *set* of objects $\{X_i \in \mathcal{C}, i \in I\}$ such that for each $A \in \mathcal{C}$ there exists some $i \in I$ and a morphism $X_i \to A$. Now given a functor $U : \mathcal{A} \to \mathcal{B}$ we say that U satisfies the *solution set condition* if for each $X \in \mathcal{B}$ the category X/U has a weakly initial set of objects. The general adjoint functor theorem asserts that if $\mathcal{C}$ is a locally small category with limits then $U : \mathcal{C} \to \mathcal{D}$ has a left adjoint if and only if U preserves limits and satisfies the solution set condition. Use the adjoint functor theorem to verify that $U : Grp \to Mon$ (or your favourite forgetful functor between algebraic categories) has a left adjoint.
(*The key point here is to think about how to construct solution sets in algebraic categories.*)

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[1] Objects of the category X/U are pairs $(f : X \to UA, A)$. A morphism $h : (f : X \to UA, A) \to (g : X \to UB, B)$ in X/U is given by a morphism $h : A \to B \in \mathcal{A}$ such that $Uh \circ f = g$.

[2] The functor $\mathcal{B}(X, U-) : \mathcal{A} \to \mathrm{Set}$ is the composite of $U : \mathcal{A} \to \mathcal{B}$ and $\mathcal{B}(X, -) : \mathcal{B} \to \mathrm{Set}$.