MULTIVARIÁTNA ANALÝZA 2

1. Kvadratické formy

Definí cia 1.1. *Nech $X_1, X_2, ..., X_n$ sú nezávislé, $N(0,1)$ rozdelené náhodné veličiny. Potom*

$$Y = X_1^2 + X_2^2 + ... + X_n^2$$

má rozdelenie χ_n^2 (centrálne chí kvadrát rozdelenie s n stupňami voľnosti).

Veta 1.2. *Nech $Y \sim \chi_n^2$. Y má hustotu*

$$f_n(y) = \begin{cases} \dfrac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} e^{-\frac{y}{2}} y^{\frac{n}{2}-1} & \textit{pre } y > 0, \\ 0 & \textit{inde.} \end{cases}$$

Dôkaz. Pozri [Anděl, str. 79].

Poznámka. χ_n^2 rozdelenie je špeciálny prípad gama rozdelenia s parametrami a, p $(a > 0, p > 0)$, ktoré má hustotu

$$f(x) = \begin{cases} \dfrac{a^p}{\Gamma(p)} e^{-ax} x^{p-1} & \text{pre } x > 0, \\ 0 & \text{inde.} \end{cases}$$

Označujeme ho $\Gamma(a,p)$. Platí, že χ_n^2 je rozdelenie $\Gamma\left(\frac{1}{2}, \frac{n}{2}\right)$.
($\Gamma(p) = \int_0^\infty e^{-t} t^{p-1} dt,\ p > 0.$)

Definícia 1.3. *Nech $X_1, X_2, ..., X_n$ sú nezávislé, $X_i \sim N(\mu_i, 1)$, $i = 1, 2, ..., n$. Nech $\lambda = \sum_{i=1}^n \mu_i^2 \neq 0$. Náhodná veličina*

$$Y = X_1^2 + X_2^2 + ... + X_n^2$$

má necentrálne χ^2 rozdelenie s n stupňami voľnosti a koeficientom necentrality λ. Označujeme ho $\chi_{n,\lambda}^2$.

Veta 1.4. *Nech $X_1, X_2, ..., X_n$ sú nezávislé, $X_i \sim N(\mu_i, 1)$, $i = 1, 2, ..., n$. Rozdelenie pravdepodobnosti náhodnej veličiny $Y = \sum_{i=1}^n X_i^2$, (teda $\chi_{n,\lambda}^2$, kde $\lambda = \sum_{i=1}^n \mu_i^2$) závisí len od n a λ (nezávisí od jednotlivých $\mu_1, ..., \mu_n$).*

Dôkaz. Pozri [Anděl, str. 80].

Lema 1.5. *Nech $X \sim N(\mu, 1)$. Náhodná veličina X^2 má hustotu*

$$f_{X^2}(t) = \begin{cases} \dfrac{e^{-\frac{t+\mu^2}{2}}}{\sqrt{2\pi}\sqrt{t}} \left(1 + \dfrac{\mu^2 t}{2!} + \dfrac{(\mu^2 t)^2}{4!} + ...\right) & t > 0, \\ 0 & t \leqq 0. \end{cases}$$

Dôkaz. X^2 má distribučnú funkciu pre $t > 0$

$$F_{X^2}(t) = P\{X^2 < t\} = P\{-\sqrt{t} < X < \sqrt{t}\} = \int_{-\sqrt{t}}^{\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}} dx.$$

Preto je hľadaná hustota pre $t > 0$

$$f_{X^2}(t) = \frac{dF_{X^2}(t)}{dt} = \frac{1}{2\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{t}-\mu)^2}{2}} + \frac{1}{2\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(-\sqrt{t}-\mu)^2}{2}} =$$

$$= \frac{e^{-\frac{t+\mu^2}{2}}}{2\sqrt{t}\sqrt{2\pi}} \left(e^{\mu\sqrt{t}} + e^{-\mu\sqrt{t}} \right) =$$

$$= \frac{e^{-\frac{t+\mu^2}{2}}}{\sqrt{2\pi}\sqrt{t}} \left(1 + \frac{\mu^2 t}{2!} + \frac{(\mu^2 t)^2}{4!} + ... \right).$$

Samozrejme pre $t \leqq 0$ je $f_{X^2}(t) = 0$. $\square$

Poznámka. Použili sme vzorec

$$\frac{d}{dy} \int_{\alpha(y)}^{\beta(y)} f(x, y) dx = \int_{\alpha(y)}^{\beta(y)} \frac{\partial f(x, y)}{\partial y} dx + \beta'(y) f[\beta(y), y] - \alpha'(y) f[\alpha(y), y].$$

Počítajme teraz charakteristickú funkciu náhodnej veličiny $\xi = X^2$.

$$\psi_\xi(t) = \mathcal{E}(e^{it\xi}) = \int_0^\infty e^{itx} f_\xi(x) dx =$$

$$= \int_0^\infty e^{itx} \frac{e^{-\frac{x+\mu^2}{2}}}{\sqrt{2\pi}\sqrt{x}} \left(1 + \frac{\mu^2 x}{2!} + \frac{(\mu^2 x)^2}{4!} + ... \right) dx.$$

Postupne pre prvý člen

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{itx} e^{-\frac{x+\mu^2}{2}} x^{-\frac{1}{2}} dx = \frac{e^{-\frac{\mu^2}{2}}}{\sqrt{2\pi}} \int_0^\infty e^{-x(\frac{1}{2}-it)} x^{-\frac{1}{2}} dx =$$

(substitúcia $x(\frac{1}{2} - it) = w$)

$$= \frac{e^{-\frac{\mu^2}{2}}}{\sqrt{2\pi}} \int_0^\infty e^{-w} \frac{w^{\frac{1}{2}-1}}{\sqrt{\frac{1}{2} - it}} dw = \frac{1}{\sqrt{2\pi}\sqrt{\frac{1}{2} - it}} e^{-\frac{\mu^2}{2}} \Gamma(\tfrac{1}{2}).$$

Pre druhý člen

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{itx} e^{-\frac{x+\mu^2}{2}} x^{-\frac{1}{2}} \frac{\mu^2 x}{2!} dx = \frac{\mu^2 e^{-\frac{\mu^2}{2}}}{2!\sqrt{2\pi}} \int_0^\infty e^{-x(\frac{1}{2}-it)} x^{\frac{3}{2}-1} dx =$$

(substitúcia $x(\frac{1}{2} - it) = w$)

$$= \frac{\mu^2 e^{-\frac{\mu^2}{2}}}{2!\sqrt{2\pi}} \int_0^\infty e^{-w} \frac{w^{\frac{3}{2}-1}}{\sqrt{(\frac{1}{2} - it)^3}} dw = \frac{\mu^2}{2!\sqrt{2\pi}\sqrt{(\frac{1}{2} - it)^3}} e^{-\frac{\mu^2}{2}} \Gamma(\tfrac{3}{2}).$$

Pre tretí člen

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{itx}e^{-\frac{x+\mu^2}{2}}x^{-\frac{1}{2}}\frac{(\mu^2)^2x^2}{4!}dx = \frac{(\mu^2)^2e^{-\frac{\mu^2}{2}}}{4!\sqrt{2\pi}}\int_0^\infty e^{-x(\frac{1}{2}-it)}x^{\frac{5}{2}-1}dx =$$

(substitúcia $x(\frac{1}{2}-it)=w$)

$$= \frac{(\mu^2)^2e^{-\frac{\mu^2}{2}}}{4!\sqrt{2\pi}}\int_0^\infty e^{-w}\frac{w^{\frac{5}{2}-1}}{\sqrt{(\frac{1}{2}-it)^5}}dw = \frac{(\mu^2)^2}{4!\sqrt{2\pi}\sqrt{(\frac{1}{2}-it)^5}}e^{-\frac{\mu^2}{2}}\Gamma(\tfrac{5}{2}),$$

atď.

Dostávame

$$\psi_\xi(t) = \frac{e^{-\frac{\mu^2}{2}}}{\sqrt{2\pi}}\left[\frac{\Gamma\left(\frac{1}{2}\right)(\mu^2)^0}{0!\left(\frac{1}{2}-it\right)^{\frac{1}{2}}} + \frac{\Gamma\left(\frac{3}{2}\right)(\mu^2)^1}{2!\left(\frac{1}{2}-it\right)^{\frac{3}{2}}} + \frac{\Gamma\left(\frac{5}{2}\right)(\mu^2)^2}{4!\left(\frac{1}{2}-it\right)^{\frac{5}{2}}} + ...\right] =$$

$$= \frac{e^{-\frac{\mu^2}{2}}}{\sqrt{\pi}\sqrt{1-2it}}\left[\frac{\sqrt{\pi}(\mu^2)^0}{0!\left(\frac{1}{2}-it\right)^0} + \frac{\sqrt{\pi}\frac{1}{2}(\mu^2)^1}{2.1!\left(\frac{1}{2}-it\right)^1} + \frac{\sqrt{\pi}\frac{3}{2}\frac{1}{2}(\mu^2)^2}{4.3.2!\left(\frac{1}{2}-it\right)^2} + \right.$$
$$\left. + \frac{\sqrt{\pi}\frac{5}{2}\frac{3}{2}\frac{1}{2}(\mu^2)^3}{6.5.4.3!\left(\frac{1}{2}-it\right)^3} + \frac{\sqrt{\pi}\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}(\mu^2)^4}{8.7.6.5.4!\left(\frac{1}{2}-it\right)^4} + ...\right] =$$

$$= \frac{e^{-\frac{\mu^2}{2}}}{\sqrt{1-2it}}\left[\frac{(\mu^2)^0}{0!4^0\left(\frac{1}{2}-it\right)^0} + \frac{(\mu^2)^1}{1!4^1\left(\frac{1}{2}-it\right)^1} + \frac{(\mu^2)^2}{2!4^2\left(\frac{1}{2}-it\right)^2} + ...\right] =$$

$$= \frac{e^{-\frac{\mu^2}{2}}e^{\frac{\mu^2}{2(1-2it)}}}{\sqrt{1-2it}} = \frac{e^{\frac{it\mu^2}{1-2it}}}{\sqrt{1-2it}}. \tag{1.1}$$

Ak máme $X_1, X_2, ..., X_k$ nezávislé, $X_i \sim N(\mu_i, 1)$, tak charakteristická funkcia

$$\psi_{X_i^2}(t) = \frac{e^{\frac{it\mu_i^2}{1-2it}}}{\sqrt{1-2it}} \tag{1.2}$$

a charakteristická funkcia náhodnej veličiny $Y = X_1^2 + X_2^2 + ... + X_k^2$ je

$$\psi_Y(t) = \psi_{X_1^2}(t)\psi_{X_2^2}(t)...\psi_{X_k^2}(t) = \frac{e^{\frac{it}{1-2it}\sum_{j=1}^k \mu_j^2}}{(1-2it)^{\frac{k}{2}}} = \frac{e^{\frac{it\lambda}{1-2it}}}{(1-2it)^{\frac{k}{2}}}, \tag{1.3}$$

kde $\lambda = \sum_{j=1}^k \mu_j^2$.

Veta 1.6. *Nech náhodné premenné* $X_1, X_2, ..., X_n$ *sú nezávislé,* $X_i \sim N(\mu_i, 1),\ i = 1, 2, ..., n$. *Potom*

$$T = \sum_{i=1}^{n} \gamma_i X_i^2 + 2\sum_{i=1}^{n} b_i X_i + c$$

má $\chi^2_{k,\delta}$ *rozdelenie práve vtedy ak*
 (i) $\gamma_i = 0$ *alebo 1 pre* $i = 1, 2, ..., n,$
 (ii) *ak* $\gamma_i = 0 \ \Rightarrow\ b_i = 0$ *pre* $i = 1, 2, ..., n,$
 (iii) $c = \sum_{i=1}^{n} b_i^2.$
Ak sú podmienky (i),(ii) a (iii) splnené, tak $k = \sum_{i=1}^{n} \gamma_i$ *a* $\delta = \sum_{i=1}^{n} \gamma_i (b_i + \mu_i)^2.$

Dôkaz. Porovnáme charakteristické funkcie $\psi_T(.)$ a $\psi_Y(.)$, kde $Y \sim \chi^2_{k,\delta}$. Platí

$$\psi_T(t) = \mathcal{E}\left(e^{itT}\right) = \mathcal{E}\left(e^{it\left[\sum_{\substack{j=1\\ \gamma_j\neq 0}}^{n} \gamma_j X_j^2 + 2\sum_{\substack{j=1\\ \gamma_j\neq 0}}^{n} b_j X_j + c + 2\sum_{\substack{j=1\\ \gamma_j = 0}}^{n} b_j X_j\right]}\right) =$$

$$= \mathcal{E}\left(e^{it\left[\sum_{\substack{j=1\\ \gamma_j\neq 0}}^{n} \gamma_j (X_j + \frac{b_j}{\gamma_j})^2 + c - \sum_{\substack{j=1\\ \gamma_j\neq 0}}^{n} \frac{b_j^2}{\gamma_j} + 2\sum_{\substack{j=1\\ \gamma_j = 0}}^{n} b_j X_j\right]}\right) =$$

$$= e^{it\left[c - \sum_{\substack{j=1\\ \gamma_j\neq 0}}^{n} \frac{b_j^2}{\gamma_j}\right]} \mathcal{E}\left(e^{it2\sum_{\substack{j=1\\ \gamma_j = 0}}^{n} b_j X_j}\right) \prod_{\substack{j=1\\ \gamma_j\neq 0}}^{n} \mathcal{E}\left(e^{it\gamma_j\left(X_j + \frac{b_j}{\gamma_j}\right)^2}\right) =$$

$$= e^{it\left[c - \sum_{\substack{j=1\\ \gamma_j\neq 0}}^{n} \frac{b_j^2}{\gamma_j}\right]} \prod_{\substack{j=1\\ \gamma_j = 0}}^{n} \psi_{\xi_j}(2b_j t) \prod_{\substack{j=1\\ \gamma_j\neq 0}}^{n} \psi_{\xi_j^2}(\gamma_j t),$$

kde $\xi_i \sim N\left(\mu_i + \frac{b_i}{\gamma_i}, 1\right)$ ak $\gamma_i \neq 0$ a $\xi_i \sim N(\mu_i, 1)$ ak $\gamma_i = 0$. Podľa (1.2) je

$$\psi_T(t) = e^{it\left(c - \sum_{\substack{j=1\\ \gamma_j\neq 0}}^{n} \frac{b_j^2}{\gamma_j}\right)} e^{i2t\sum_{\substack{j=1\\ \gamma_j = 0}}^{n} \mu_j b_j - 2t^2 \sum_{\substack{j=1\\ \gamma_j = 0}}^{n} b_j^2} \times \frac{1}{\prod_{\substack{j=1\\ \gamma_j\neq 0}}^{n} \sqrt{1 - 2it\gamma_j}} e^{it\sum_{\substack{j=1\\ \gamma_j\neq 0}}^{n} \frac{\gamma_j\left(\mu_j + \frac{b_j}{\gamma_j}\right)^2}{1 - 2it\gamma_j}}. \tag{1.4}$$

Podľa (1.3) pre charakteristickú funkciu $Y \sim \chi^2_{k,\delta}$ platí

$$\psi_Y(t) = \frac{1}{\prod_{j=1}^{k} \sqrt{1 - 2it}} e^{\frac{it\delta}{1 - 2it}}. \tag{1.5}$$

Porovnaním (1.4) a (1.5) musí platiť pre každé $t \in \mathcal{R}$

$$\prod_{\substack{j=1\\ \gamma_j \neq 0}}^{n} \sqrt{1-2it\gamma_j} = \prod_{l=1}^{k} \sqrt{1-2it}$$

a súčasne

$$e^{it\left(c-\sum_{\substack{j=1\\ \gamma_j\neq 0}}^{n} \frac{b_j^2}{\gamma_j}\right)} e^{i2t\sum_{\substack{j=1\\ \gamma_j=0}}^{n} \mu_j b_j - 2t^2\sum_{\substack{j=1\\ \gamma_j=0}}^{n} b_j^2} e^{it\sum_{\substack{j=1\\ \gamma_j\neq 0}}^{n} \frac{\gamma_j\left(\mu_j+\frac{b_j}{\gamma_j}\right)^2}{1-2it\gamma_j}} = e^{\frac{it\delta}{1-2it}},$$

z čoho je jasne vidieť, ako dokončíme dôkaz. $\square$

Veta 1.7. *Nech* $\boldsymbol{\xi} \sim N_n(\boldsymbol{\mu}, \mathbf{I})$, $\mathbf{A}_{n,n}$ *je symetrická,* $\mathbf{b} \in \mathcal{R}^n$ *a* $c \in \mathcal{R}$. *Náhodná premenná* $T = \boldsymbol{\xi}'\mathbf{A}\boldsymbol{\xi} + 2\mathbf{b}'\boldsymbol{\xi} + c$ *má* $\chi^2_{k,\delta}$ *rozdelenie práve vtedy ak*

(i) $\mathbf{A}^2 = \mathbf{A}$,

(ii) $\mathbf{b} \in \mu(\mathbf{A})$,

(iii) $c = \mathbf{b}'\mathbf{b}$.

Ak sú podmienky (i),(ii) a (iii) splnené, tak $k = h(\mathbf{A})$, $\delta = (\mathbf{b}+\boldsymbol{\mu})'\mathbf{A}(\mathbf{b}+\boldsymbol{\mu})$.

Dôkaz. Pre $\mathbf{A}$ existuje ortogonálna matica $\mathbf{P}$, že platí $\mathbf{P}'\mathbf{A}\mathbf{P} = \boldsymbol{\Lambda}$ (diagonálna matica), $\mathbf{P}'\mathbf{P} = \mathbf{P}\mathbf{P}' = \mathbf{I}$ (pozri napr. Rao, str. 62). Potom $\boldsymbol{\eta} = \mathbf{P}'\boldsymbol{\xi} \sim N(\mathbf{P}'\boldsymbol{\mu}, \mathbf{I})$ a $\boldsymbol{\xi} = \mathbf{P}\boldsymbol{\eta}$. Preto

$$T = \boldsymbol{\xi}'\mathbf{A}\boldsymbol{\xi} + 2\mathbf{b}'\boldsymbol{\xi} + c = \boldsymbol{\eta}'\mathbf{P}'\mathbf{A}\mathbf{P}\boldsymbol{\eta} + 2\mathbf{b}'\mathbf{P}\boldsymbol{\eta} + c = \boldsymbol{\eta}'\boldsymbol{\Lambda}\boldsymbol{\eta} + 2\mathbf{b}'\mathbf{P}\boldsymbol{\eta} + c.$$

Podľa vety 1.6 má T rozdelenie $\chi^2_{k,\delta}$ práve vtedy ak

(i) $\{\boldsymbol{\Lambda}\}_{ii} = 0$ alebo 1 pre $i = 1, 2, ..., n$ $\Leftrightarrow \boldsymbol{\Lambda}^2 = \boldsymbol{\Lambda} \Leftrightarrow \mathbf{P}'\mathbf{A}\mathbf{P}\mathbf{P}'\mathbf{A}\mathbf{P} = \mathbf{P}'\mathbf{A}^2\mathbf{P} = \mathbf{P}'\mathbf{A}\mathbf{P} \Leftrightarrow \mathbf{A}^2 = \mathbf{A}$,

(ii) $\{\boldsymbol{\Lambda}\}_{ii} = 0 \Rightarrow \{\mathbf{P}'\mathbf{b}\}_i = 0$, čo je ekvivalentné s tým, že $\mathbf{P}'\mathbf{b} \in \mu(\boldsymbol{\Lambda}) \Leftrightarrow \mathbf{P}\mathbf{P}'\mathbf{b} = \mathbf{b} \in \mu(\mathbf{P}\mathbf{P}'\mathbf{A}\mathbf{P}) = \mu(\mathbf{A}\mathbf{P}) = \mu(\mathbf{A})$,

(iii) $c = (\mathbf{b}'\mathbf{P})\mathbf{P}'\mathbf{b} = \mathbf{b}'\mathbf{b}$.

Ak sú podmienky (i),(ii) a (iii) splnené, potom podľa vety 1.6 $k = \sum_{i=1}^{n}\{\boldsymbol{\Lambda}\}_{ii} = tr\boldsymbol{\Lambda} = h(\boldsymbol{\Lambda}) = h(\mathbf{P}\boldsymbol{\Lambda}\mathbf{P}') = h(\mathbf{A})$ a $\delta = \sum_{i=1}^{n}\{\boldsymbol{\Lambda}\}_{ii}(\{\mathbf{P}'\mathbf{b}\}_i + \{\mathbf{P}'\boldsymbol{\mu}\}_i)^2 = (\mathbf{b}'\mathbf{P} + \boldsymbol{\mu}'\mathbf{P})\boldsymbol{\Lambda}(\mathbf{P}'\mathbf{b} + \mathbf{P}'\boldsymbol{\mu}) = (\mathbf{b}+\boldsymbol{\mu})'\mathbf{P}\boldsymbol{\Lambda}\mathbf{P}'(\mathbf{b}+\boldsymbol{\mu}) = (\mathbf{b}+\boldsymbol{\mu})'\mathbf{A}(\mathbf{b}+\boldsymbol{\mu})$. $\square$

Veta 1.8. *Nech* $\boldsymbol{\xi} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $\mathbf{A}_{n,n}$ *je symetrická,* $\mathbf{b} \in \mathcal{R}^n$ *a* $c \in \mathcal{R}$. *Náhodná premenná* $T = \boldsymbol{\xi}'\mathbf{A}\boldsymbol{\xi} + 2\mathbf{b}'\boldsymbol{\xi} + c$ *má* $\chi^2_{k,\delta}$ *rozdelenie práve vtedy ak*

(i) $\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma} = \boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma} \Leftrightarrow (\boldsymbol{\Sigma}\mathbf{A})^3 = (\boldsymbol{\Sigma}\mathbf{A})^2$,

(ii) $\boldsymbol{\Sigma}(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}) \in \mu(\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma})$,

(iii) $(\mathbf{A}\boldsymbol{\mu} + \mathbf{b})'\boldsymbol{\Sigma}(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}) = \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu} + 2\mathbf{b}'\boldsymbol{\mu} + c$.

Ak sú podmienky (i),(ii) a (iii) splnené, tak $k = tr(\mathbf{A}\boldsymbol{\Sigma})$ *a* $\delta = (\mathbf{b}+\mathbf{A}\boldsymbol{\mu})'\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma}(\mathbf{b}+\mathbf{A}\boldsymbol{\mu})$.

Dôkaz. Faktorizujeme maticu $\boldsymbol{\Sigma} = \mathbf{J}\mathbf{J}'$, kde $\mathbf{J}$ je typu $n \times h(\boldsymbol{\Sigma})$ (pozri Anděl, str. 64). Vieme, že $P\{\boldsymbol{\xi} = \boldsymbol{\mu} + \mathbf{J}\boldsymbol{\eta}\} = 1$, kde $\boldsymbol{\eta} \sim N_{h(\boldsymbol{\Sigma})}(\mathbf{0}, \mathbf{I})$ (Anděl, str. 76). Teda

$$T = \boldsymbol{\xi}'\mathbf{A}\boldsymbol{\xi} + 2\mathbf{b}'\boldsymbol{\xi} + c = (\boldsymbol{\mu} + \mathbf{J}\boldsymbol{\eta})'\mathbf{A}(\boldsymbol{\mu} + \mathbf{J}\boldsymbol{\eta}) + 2\mathbf{b}'(\boldsymbol{\mu} + \mathbf{J}\boldsymbol{\eta}) + c =$$

$$= \boldsymbol{\eta}'\mathbf{J}'\mathbf{A}\mathbf{J}\boldsymbol{\eta} + 2(\mathbf{A}\boldsymbol{\mu} + \mathbf{b})'\mathbf{J}\boldsymbol{\eta} + \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu} + 2\mathbf{b}'\boldsymbol{\mu} + c.$$

Podľa vety 1.7 má T rozdelenie $\chi^2_{k,\delta}$ práve vtedy ak
(1) $\mathbf{J}'\mathbf{A}\mathbf{J}\mathbf{J}'\mathbf{A}\mathbf{J} = \mathbf{J}'\mathbf{A}\mathbf{J}$,
(2) $\mathbf{J}'(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}) \in \mu(\mathbf{J}'\mathbf{A}\mathbf{J})$,
(3) $(\mathbf{A}\boldsymbol{\mu} + \mathbf{b})'\mathbf{J}\mathbf{J}'(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}) = \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu} + 2\mathbf{b}'\boldsymbol{\mu} + c$.
Ďalej platí

$$\mathbf{J}'\mathbf{A}\mathbf{J}\mathbf{J}'\mathbf{A}\mathbf{J} = \mathbf{J}'\mathbf{A}\mathbf{J} \Rightarrow \mathbf{J}\mathbf{J}'\mathbf{A}\mathbf{J}\mathbf{J}'\mathbf{A}\mathbf{J}\mathbf{J}' = \mathbf{J}\mathbf{J}'\mathbf{A}\mathbf{J}\mathbf{J}',$$

čiže

$$\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma} = \boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma},$$

a tiež naopak

$$\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma} = \boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma} \;\Rightarrow\; \mathbf{J}\mathbf{J}'\mathbf{A}\mathbf{J}\mathbf{J}'\mathbf{A}\mathbf{J}\mathbf{J}' = \mathbf{J}\mathbf{J}'\mathbf{A}\mathbf{J}\mathbf{J}' \;\Rightarrow$$

$$(\mathbf{J}'\mathbf{J})^{-1}\mathbf{J}'.\mathbf{J}\mathbf{J}'\mathbf{A}\mathbf{J}\mathbf{J}'\mathbf{A}\mathbf{J}\mathbf{J}'.\mathbf{J}(\mathbf{J}'\mathbf{J})^{-1} = (\mathbf{J}'\mathbf{J})^{-1}\mathbf{J}'.\mathbf{J}\mathbf{J}'\mathbf{A}\mathbf{J}\mathbf{J}'.\mathbf{J}(\mathbf{J}'\mathbf{J})^{-1},$$

čiže

$$\mathbf{J}'\mathbf{A}\mathbf{J}\mathbf{J}'\mathbf{A}\mathbf{J} = \mathbf{J}'\mathbf{A}\mathbf{J},$$

čo dokazuje prvú časť (i).

Ekvivalencia

$$\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma} = \boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma} \;\Leftrightarrow\; (\boldsymbol{\Sigma}\mathbf{A})^3 = (\boldsymbol{\Sigma}\mathbf{A})^2$$

je jedným smerom ($\Rightarrow$) zrejmá. Ku opaku potrebujeme nasledovné tvrdenie

$$\exists\ \mathbf{D}_{n,n} :\ \boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma} = \boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}\mathbf{D}. \tag{1.6}$$

Tvrdenie (1.6) dokážeme takto:

$$h(\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma}) = h(\mathbf{J}\mathbf{J}'\mathbf{A}\mathbf{J}\mathbf{J}') \geqq h((\mathbf{J}'\mathbf{J})^{-1}\mathbf{J}'.\mathbf{J}\mathbf{J}'\mathbf{A}\mathbf{J}\mathbf{J}'.\mathbf{J}(\mathbf{J}'\mathbf{J})^{-1}) = h(\mathbf{J}'\mathbf{A}\mathbf{J}).$$

Podľa Anděl, str. 62 je

$$h(\mathbf{J}'\mathbf{A}\mathbf{J}) = h(\mathbf{J}'\mathbf{A}\mathbf{J}\mathbf{J}'\mathbf{A}\mathbf{J}) \geqq h(\mathbf{J}\mathbf{J}'\mathbf{A}\mathbf{J}\mathbf{J}'\mathbf{A}\mathbf{J}) = h(\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}\mathbf{J}), \tag{1.7}$$

ale

$$h(\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}\mathbf{J}) = h(\mathbf{J}\mathbf{J}'\mathbf{A}\mathbf{J}\mathbf{J}'\mathbf{A}\mathbf{J}) \geqq h((\mathbf{J}'\mathbf{J})^{-1}\mathbf{J}'.\mathbf{J}\mathbf{J}'\mathbf{A}\mathbf{J}\mathbf{J}'\mathbf{A}\mathbf{J}) =$$

$$= h(\mathbf{J}'\mathbf{A}\mathbf{J}\mathbf{J}'\mathbf{A}\mathbf{J}) = h(\mathbf{J}'\mathbf{A}\mathbf{J}), \tag{1.8}$$

a preto z (1.7) a (1.8)

$$h(\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma}) = h(\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}\mathbf{J}) \leqq h(\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}) \leqq h(\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma}),$$

teda

$$h(\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}) = h(\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma}).$$

Pretože zrejme $\mu(\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}) \subset \mu(\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma})$ a hodnosti matíc vytvárajúcich tieto podpriestory sa rovnajú, platí

$$\mu(\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}) = \mu(\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma})$$

a dostávame vzťah (1.6).

Z predpokladu $(\mathbf{\Sigma A})^3 = (\mathbf{\Sigma A})^2$ pomocou (1.6) dostávame

$$\mathbf{\Sigma A \Sigma A \Sigma A D} = \mathbf{\Sigma A \Sigma A D} \Rightarrow \mathbf{\Sigma A \Sigma A \Sigma} = \mathbf{\Sigma A \Sigma},$$

čím sme (i) úplne dokázali.

Poďme teraz dokázať (ii), čiže dokázať, že

$$\mathbf{J}'(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}) \in \mu(\mathbf{J}'\mathbf{A}\mathbf{J}) \Leftrightarrow \mathbf{\Sigma}(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}) \in \mu(\mathbf{\Sigma A \Sigma}).$$

Ak $\mathbf{J}'(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}) \in \mu(\mathbf{J}'\mathbf{A}\mathbf{J})$, tak $\mathbf{J}\mathbf{J}'(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}) \in \mu(\mathbf{J}\mathbf{J}'\mathbf{A}\mathbf{J}) = \mu(\mathbf{\Sigma A J}) =$
$= \mu(\mathbf{\Sigma A J J' A \Sigma}) = \mu(\mathbf{\Sigma A \Sigma A \Sigma}) = \mu(\mathbf{\Sigma A \Sigma})$ (podľa (i)).

Naopak ak $\mathbf{\Sigma}(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}) \in \mu(\mathbf{\Sigma A \Sigma})$, tak $(\mathbf{J}'\mathbf{J})^{-1}\mathbf{J}'.\mathbf{J}\mathbf{J}'(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}) = \mathbf{J}'(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}) \in \mu((\mathbf{J}'\mathbf{J})^{-1}\mathbf{J}'.\mathbf{J}\mathbf{J}'\mathbf{A}\mathbf{J}\mathbf{J}') = \mu(\mathbf{J}'\mathbf{A}\mathbf{J}\mathbf{J}') \subset \mu(\mathbf{J}'\mathbf{A}\mathbf{J})$, čím sme dokázali (ii). Samozrejme (iii) už máme dokázané (je ekvivalentné (1)). Dôkaz vety už dokončíme jednoducho. Podľa vety 1.7 je totiž $k = h(\mathbf{J}\mathbf{J}'\mathbf{A}) = tr(\mathbf{\Sigma A}) = tr(\mathbf{A\Sigma})$ a $\delta = ([\mathbf{J}'(\mathbf{A}\boldsymbol{\mu} + \mathbf{b})]'\mathbf{J}'\mathbf{A}\mathbf{J}[\mathbf{J}'(\mathbf{A}\boldsymbol{\mu} + \mathbf{b})]) = (\mathbf{A}\boldsymbol{\mu} + \mathbf{b})'\mathbf{\Sigma A \Sigma}(\mathbf{A}\boldsymbol{\mu} + \mathbf{b})$. □

Uvedieme bez dôkazu vety o nezávislosti kvadratických foriem. Podrobnejšie pozri [Rao, Mitra, kapitola 9].

Veta 1.9. *Nech* $\mathbf{Y} \sim N_p(\boldsymbol{\mu}, \mathbf{\Sigma})$ *a* $Q_1 = \mathbf{Y}'\mathbf{A}\mathbf{Y}, Q_2 = \mathbf{Y}'\mathbf{B}\mathbf{Y}$ *dve kvadratické formy. Nutné a postačujúce podmienky nezávislosti* Q_1 *a* Q_2 *sú*

(a) $\mathbf{\Sigma A \Sigma B \Sigma} = \mathbf{0}, \mathbf{\Sigma A \Sigma B}\boldsymbol{\mu} = \mathbf{0}, \mathbf{\Sigma B \Sigma A}\boldsymbol{\mu} = \mathbf{0}$ *a* $\boldsymbol{\mu}'\mathbf{A \Sigma B}\boldsymbol{\mu} = \mathbf{0}$, *ak* $\mathbf{A}$ *a* $\mathbf{B}$ *sú symetrické, nemusia byť pozitívne semidefinitné, pričom* $\mathbf{\Sigma}$ *nemusí byť regulárna.*

(b) $\mathbf{A \Sigma B \Sigma} = \mathbf{0}, \mathbf{A \Sigma B}\boldsymbol{\mu} = \mathbf{0}$, *ak* $\mathbf{A}$ *je pozitívne semidefinitná.*

(c) $\mathbf{A \Sigma B} = \mathbf{0}$, *ak* $\mathbf{A}$ *aj* $\mathbf{B}$ *sú pozitívne semidefinitné.*

(d) $\mathbf{A \Sigma B} = \mathbf{0}$, *ak* $\mathbf{\Sigma}$ *je regulárna,* $\mathbf{A}$ *a* $\mathbf{B}$ *sú symetrické, nemusia byť pozitívne semidefinitné.*

Veta 1.10. *Nech* $\mathbf{Y} \sim N_p(\boldsymbol{\mu}, \mathbf{\Sigma})$ *a* $Q_1 = \mathbf{Y}'\mathbf{A}\mathbf{Y} + 2\mathbf{a}'\mathbf{Y} + \alpha$, $Q_2 = \mathbf{Y}'\mathbf{B}\mathbf{Y} + 2\mathbf{b}'\mathbf{Y} + \beta$ *dve lineárne-kvadratické formy. Nutné a postačujúce podmienky nezávislosti* Q_1 *a* Q_2 *sú*

(a) $\mathbf{\Sigma A \Sigma B \Sigma} = \mathbf{0}, \mathbf{\Sigma A \Sigma b} = \mathbf{0}, \mathbf{\Sigma B \Sigma a} = \mathbf{0}$ *a* $\mathbf{a}'\mathbf{\Sigma b} = 0$, *ak* $\boldsymbol{\mu} = \mathbf{0}$, *pričom* $\mathbf{\Sigma}$ *nemusí byť regulárna.*

(b) $\mathbf{A \Sigma B} = \mathbf{0}, \mathbf{B \Sigma a} = \mathbf{0}, \mathbf{A \Sigma b} = \mathbf{0}$ *a* $\mathbf{a}'\mathbf{\Sigma b} = 0$, *ak* $\mathbf{\Sigma}$ *je regulárna, pričom* $\boldsymbol{\mu}$ *môže byť aj nenulový vektor.*

2. Wishartovo rozdelenie

2.1. Úvodné poznámky a definícia

Majme $\mathbf{U}_i \sim N_p(\boldsymbol{\mu}_i, \mathbf{\Sigma})$, $i = 1, 2, ..., k$, ktoré sú nezávislé, $\mathbf{\Sigma}$ je pozitívne definitná matica. Označme $\mathbf{U}_i = (U_{1i}, U_{2i}, ..., U_{pi})'$, $\mathbf{Y}_j = (U_{j1}, U_{j2}, ..., U_{jk})'$, $j = 1, 2, ..., p$ a

$$\begin{pmatrix} U_{11} & U_{12} & U_{13} & \dots & U_{1k} \\ U_{21} & U_{22} & U_{23} & \dots & U_{2k} \\ \vdots & & & & \\ U_{p1} & U_{p2} & U_{p3} & \dots & U_{pk} \end{pmatrix} = \mathcal{U}'_{p,k}.$$

Teda

$$\mathcal{U}' = (\mathbf{U}_1 \vdots \mathbf{U}_1 \vdots \ldots \vdots \mathbf{U}_k) = \begin{pmatrix} \mathbf{Y}_1' \\ \mathbf{Y}_2' \\ \vdots \\ \mathbf{Y}_p' \end{pmatrix}.$$

ďalej označme

$$\mathbf{M}'_{p,k} = \begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} & \ldots & \mu_{1k} \\ \mu_{21} & \mu_{22} & \mu_{23} & \ldots & \mu_{2k} \\ \vdots & & & & \\ \mu_{p1} & \mu_{p2} & \mu_{p3} & \ldots & \mu_{pk} \end{pmatrix} = (\boldsymbol{\mu}_1 \vdots \boldsymbol{\mu}_2 \vdots \ldots \vdots \boldsymbol{\mu}_k).$$

Pre pevný vektor $\mathbf{l} \in \mathcal{R}^p$ sú náhodné veličiny

$$\mathbf{l}'\mathbf{U}_i \sim N(\mathbf{l}'\boldsymbol{\mu}_i, \mathbf{l}'\boldsymbol{\Sigma}\mathbf{l} = \sigma_l^2), \quad i = 1, 2, ..., k$$

nezávislé (lebo $\mathbf{U}_i$ sú nezávislé). Náhodný vektor $\mathcal{U}\mathbf{l} = {}_{\mathbf{l}}\mathbf{Y}_{k,1}$ je lineárna kombinácia normálne rozdelených nezávislých náhodných vektorov, pričom

(2.1) $${}_{\mathbf{l}}\mathbf{Y} \sim N_k(\mathbf{M}\mathbf{l}, \sigma_l^2 \mathbf{I}_{k,k}).$$

ak $\mathbf{b} = (b_1, b_2, ..., b_k)'$ je vektor konštánt, tak

(2.2) $$\mathcal{U}'\mathbf{b} = b_1\mathbf{U}_1 + ... + b_k\mathbf{U}_k \sim N_p(\mathbf{M}'\mathbf{b}, \mathbf{b}'\mathbf{b}\boldsymbol{\Sigma}).$$

Poznámka. Nech

$$\mathbf{A}_{m,n} = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2n} \\ \vdots & & \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}, \mathbf{B}_{r,s} = \begin{pmatrix} b_{11} & \ldots & b_{1s} \\ b_{21} & \ldots & b_{2s} \\ \vdots & & \\ b_{r1} & \ldots & b_{rs} \end{pmatrix}.$$

Kroneckerov súčin matíc $\mathbf{A}$ a $\mathbf{B}$ je

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \ldots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \ldots & a_{2n}\mathbf{B} \\ \vdots & & & \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \ldots & a_{mn}\mathbf{B} \end{pmatrix}_{mr,ns}.$$

Vlastnosti Kroneckerovho súčinu matíc pozri napr. v [Rao].

ak napíšeme "pod seba" stĺpce matice $\mathbf{K}$, povieme, že sme vykonali na matici operáciu *vec.* Teda

$$vec\mathcal{U}' = \mathbf{U}_{kp,1} = \begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \vdots \\ \mathbf{U}_k \end{pmatrix}.$$

Ukážte, že

(2.3) $$vec\mathcal{U}' = \mathbf{U} \sim N_{kp}(vec\mathbf{M}', \mathbf{I}_{k,k} \otimes \mathbf{\Sigma}_{p,p})$$

a (2.2) sa dá zapísať ako

(2.4)

$$\mathcal{U}'\mathbf{b} = (\mathbf{b}' \otimes \mathbf{I}_{p,p})vec\mathcal{U}' \sim N_p((\mathbf{b}' \otimes \mathbf{I}_{p,p})vec\mathbf{M}', (\mathbf{b}' \otimes \mathbf{I}_{p,p})(\mathbf{I}_{p,p} \otimes \mathbf{\Sigma}_{p,p})(\mathbf{b} \otimes \mathbf{I}_{p,p})).$$

Poznámka. Nech $\mathbf{b}_1 \neq \mathbf{b}_2,\ \mathbf{b}_1, \mathbf{b}_2 \in \mathcal{R}^k$. Platí

$$cov(\mathcal{U}'\mathbf{b}_1, \mathcal{U}'\mathbf{b}_2) = (\mathbf{b}_1' \otimes \mathbf{I}_{p,p})(\mathbf{I} \otimes \mathbf{\Sigma})(\mathbf{b}_2 \otimes \mathbf{I}_{p,p}) = \mathbf{b}_1'\mathbf{b}_2 \otimes \mathbf{\Sigma} = \mathbf{b}_1'\mathbf{b}_2\mathbf{\Sigma}.$$

ak $\mathbf{b}_1'\mathbf{b}_2 = 0$, t.j. ak $\mathbf{b}_1$ a $\mathbf{b}_2$ sú ortogonálne, tak $\mathcal{U}'\mathbf{b}_1$ a $\mathcal{U}'\mathbf{b}_2$ sú neskorelované, t.j. v tomto prípade nezávislé.

Podľa predchádzajúcej poznámky ľahko dokážeme nasledujúcu lemu

Lema 2.1. *Ak $\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_r,\ r \leqq k$ tvorí ortonormálny systém v $\mathcal{R}^k$, tak*

$$\mathbf{V}_1 = \mathcal{U}'\mathbf{b}_1, ..., \mathbf{V}_r = \mathcal{U}'\mathbf{b}_r$$

sú navzájom nezávislé a majú normálne rozdelenie, pričom $\mathbf{V}_i \sim N_p(\mathbf{M}'\mathbf{b}_i, \mathbf{\Sigma})$.

ľahko dostaneme aj nasledujúci dôsledok

Dôsledok 2.2. *Ak $\mathbf{B}_{k,k}$ je ortogonálna matica ($\mathbf{B}\mathbf{B}' = \mathbf{B}'\mathbf{B} = \mathbf{I}$), tak $\mathbf{V}_i = (\mathbf{U}_1 \vdots ... \vdots \mathbf{U}_k)\{\mathbf{B}\}_{.i} = \mathcal{U}'\{\mathbf{B}\}_{.i} \sim N_p(\mathbf{M}'\{\mathbf{B}\}_{.i}, \mathbf{\Sigma}),\quad i = 1, 2, ..., k$ a $cov(\mathbf{V}_i, \mathbf{V}_j) = (\{\mathbf{B}\}_{.i}' \otimes \mathbf{I}_{p,p})(\mathbf{I} \otimes \mathbf{\Sigma})(\{\mathbf{B}\}_{.j} \otimes \mathbf{I}) = \{\mathbf{B}\}_{.i}'\{\mathbf{B}\}_{.j} \otimes \mathbf{\Sigma} = \mathbf{0}$ pre $i \neq j$, teda $\mathbf{V}_1, ..., \mathbf{V}_k$ sú nezávislé.*

Definícia 2.3. *Združené rozdelenie prvkov matice $\mathbf{S}_{p,p} = \sum_{i=1}^k \mathbf{U}_i\mathbf{U}_i' = \mathcal{U}'\mathcal{U}$ sa nazýva Wishartovo rozdelenie s k stupňami voľnosti a značí $W_p(k, \mathbf{\Sigma}, \mathbf{M})$. Ak $\mathbf{M} = \mathbf{0}$, jedná sa o centrálne rozdelenie, označujeme ho $W_p(k, \mathbf{\Sigma})$.*

Poznámka.

(i) $\{\mathbf{S}\}_{ij} = \left\{\sum_{l=1}^k \mathbf{U}_l\mathbf{U}_l'\right\}_{ij} = \sum_{l=1}^k U_{il}U_{jl} = \mathbf{Y}_i'\mathbf{Y}_j = \{\mathcal{U}'\mathcal{U}\}_{ij}$, lebo

$$\mathbf{S}_{p,p} = \begin{pmatrix} \sum_{l=1}^k U_{1l}^2 & \sum_{l=1}^k U_{1l}U_{2l} & \dots & \sum_{l=1}^k U_{1l}U_{pl} \\ \sum_{l=1}^k U_{2l}U_{1l} & \sum_{l=1}^k U_{2l}^2 & \dots & \sum_{l=1}^k U_{2l}U_{pl} \\ \vdots & & & \\ \sum_{l=1}^k U_{pl}U_{1l} & \sum_{l=1}^k U_{pl}U_{2l} & \dots & \sum_{l=1}^k U_{pl}^2 \end{pmatrix}.$$

(ii) Pre $p = 1$ a $\mu_{11} = \mu_{12} = ... = \mu_{1k} = 0$ sú $\mathbf{U}_i = U_{1i} \sim N(0, \sigma^2),\ i = 1, 2, ..., k$ nezávislé, $\mathcal{U}'\mathcal{U} = \sum_{i=1}^k U_{1i}^2 \sim W_1(k, \sigma^2)$. Pretože $\frac{\mathbf{U}_{1i}}{\sigma} \sim N(0, 1)$, má $\sum_{i=1}^k \frac{U_{1i}^2}{\sigma^2} \sim \chi_k^2$ rozdelenie a $\mathcal{U}'\mathcal{U} \sim \sigma^2\chi_k^2$ rozdelenie.

(iii) Pre $k \geqq p$ existuje hustota $W_p(k, \mathbf{\Sigma}, \mathbf{M})$ rozdelenia, ináč nie. Dôkaz je naznačený v [Rao, str. 641].

2.2. Niektoré vlastnosti Wishartovho rozdelenia

Lema 2.4. *Nech* $\mathbf{S} \sim W_p(k, \mathbf{\Sigma}, \mathbf{M})$ *a* $\mathbf{l} \in \mathcal{R}^p$ *je vektor konštánt. Potom* $\mathbf{l}'\mathbf{S}\mathbf{l} \sim \sigma_{\mathbf{l}}^2 \chi^2_{k,\delta}$ $(\sigma_{\mathbf{l}}^2 = \mathbf{l}'\mathbf{\Sigma}\mathbf{l},\ \delta = \frac{\mathbf{l}'\mathbf{M}'\mathbf{M}\mathbf{l}}{\sigma_l^2})$.

Dôkaz. $\mathbf{S} = \sum_{i=1}^k \mathbf{U}_i\mathbf{U}_i'$, preto $\mathbf{l}'\mathbf{S}\mathbf{l} = \sum_{i=1}^k \mathbf{l}'\mathbf{U}_i\mathbf{U}_i'\mathbf{l} = \sum_{i=1}^k (\mathbf{l}'\mathbf{U}_i)^2 = {}_{\mathbf{l}}\mathbf{Y}'\ {}_{\mathbf{l}}\mathbf{Y} \sim \sigma_{\mathbf{l}}^2 \chi^2_{k,\delta}$, kde $\delta = \frac{\mathbf{l}'\mathbf{M}'\mathbf{M}\mathbf{l}}{\sigma_l^2}$, lebo ${}_{\mathbf{l}}\mathbf{Y} \sim N_k(\mathbf{M}\mathbf{l}, \sigma_{\mathbf{l}}^2 \mathbf{I}_{k,k})$. ak $\mathbf{M} = \mathbf{0}$, tak $\delta = 0$. □

Lema 2.5. *Nech* $\mathbf{U}_i \sim N_p(\mathbf{0}, \mathbf{\Sigma})$, $\quad i = 1, 2, ..., k$ *sú nezávislé,* $\mathbf{A}_{k,k}$ *reálna symetrická matica.* $\mathcal{U}'\mathbf{A}\mathcal{U} \sim W_p(r, \mathbf{\Sigma})$ *práve vtedy ak* $\forall\, \mathbf{l} \in \mathcal{R}^p\ {}_{\mathbf{l}}\mathbf{Y}'\mathbf{A}\ {}_{\mathbf{l}}\mathbf{Y} \sim \sigma_{\mathbf{l}}^2 \chi_r^2$, $(\sigma_{\mathbf{l}}^2 = \mathbf{l}'\mathbf{\Sigma}\mathbf{l},\ {}_{\mathbf{l}}\mathbf{Y} = \mathcal{U}\mathbf{l})$. *V tomto prípade* $r = h(\mathbf{A}) = tr(\mathbf{A})$.

Dôkaz. Z lemy 2.4 vyplýva, že ak $\mathcal{U}'\mathbf{A}\mathcal{U} \sim W_p(r, \mathbf{\Sigma})$, tak $\forall\, \mathbf{l} \in \mathcal{R}^p\ {}_{\mathbf{l}}\mathbf{Y}'\mathbf{A}\ {}_{\mathbf{l}}\mathbf{Y} \sim \sigma_{\mathbf{l}}^2 \chi_r^2$. Samozrejme z (2.1) ${}_{\mathbf{l}}\mathbf{Y} \sim N_k(\mathbf{0}, \sigma_{\mathbf{l}}^2 \mathbf{I}_{k,k})$, čiže $\frac{{}_{\mathbf{l}}\mathbf{Y}}{\sigma_{\mathbf{l}}} \sim N_k(\mathbf{0}, \mathbf{I}_{k,k})$. Teda podľa vety 1.8 $\left(\frac{{}_{\mathbf{l}}\mathbf{Y}}{\sigma_{\mathbf{l}}}\right)' \mathbf{A} \frac{{}_{\mathbf{l}}\mathbf{Y}}{\sigma_{\mathbf{l}}} \sim \chi_r^2 \Leftrightarrow \mathbf{A}^2 = \mathbf{A}$ a v tom prípade $r = h(\mathbf{A}) = tr(\mathbf{A})$.

Naopak ak $\forall\, \mathbf{l} \in \mathcal{R}^p\ {}_{\mathbf{l}}\mathbf{Y}'\mathbf{A}\ {}_{\mathbf{l}}\mathbf{Y} = \mathbf{l}'\mathcal{U}'\mathbf{A}\mathcal{U}\mathbf{l} \sim \sigma_{\mathbf{l}}^2 \chi_r^2$, čo je podľa vety 1.8 ekvivalentné tomu, že $\mathbf{A}^2 = \mathbf{A}$, pričom v tom prípade $h(\mathbf{A}) = tr(\mathbf{A})$. $\mathbf{A}$ je reálna symetrická matica, idempotentná a $h(\mathbf{A}) = r$. Teda $\mathbf{A}$ je pozitívne semidefinitná a preto existuje ortonormálny systém vektorov $\mathbf{b}_1, ..., \mathbf{b}_k \in \mathcal{R}^k$, že $\mathbf{A} = \sum_{j=1}^r \lambda_j \mathbf{b}_j \mathbf{b}_j'$, $\mathbf{I} = \sum_{j=1}^k \mathbf{b}_j \mathbf{b}_j'$ (reálne čísla $\lambda_1 \geqq \lambda_2 \geqq ... \geqq \lambda_r > 0$ sú vlastné čisla matice $\mathbf{A}$ a $\mathbf{b}_1, ..., \mathbf{b}_r$ im prislúchajúce charakteristické vektory). Z rovnosti $\mathbf{A}^2 = \mathbf{A}$ dostávame

$$\sum_{j=1}^r \lambda_j \mathbf{b}_j \mathbf{b}_j' \sum_{s=1}^r \lambda_s \mathbf{b}_s \mathbf{b}_s' = \sum_{t=1}^r \lambda_t \mathbf{b}_t \mathbf{b}_t',$$

čiže

$$\lambda_1^2 \mathbf{b}_1 \mathbf{b}_1' + \lambda_2^2 \mathbf{b}_2 \mathbf{b}_2' + ... \lambda_r^2 \mathbf{b}_r \mathbf{b}_r' = \lambda_1 \mathbf{b}_1 \mathbf{b}_1' + \lambda_2 \mathbf{b}_2 \mathbf{b}_2' + ... \lambda_r \mathbf{b}_r \mathbf{b}_r',$$

z čoho vyplýva, že $\lambda_i^2 = \lambda_i,\ i = 1, 2, ..., r$, čiže $\lambda_1 = \lambda_2 = ... = \lambda_r = 1$ (lebo $\lambda_i > 0$). Môžeme písať $\mathbf{A} = \sum_{j=1}^r \mathbf{b}_j \mathbf{b}_j'$ a tiež $\mathcal{U}'\mathbf{A}\mathcal{U} = \sum_{j=1}^r \mathcal{U}'\mathbf{b}_j \mathbf{b}_j' \mathcal{U} = \sum_{j=1}^r \mathbf{V}_j \mathbf{V}_j'$, pričom podľa lemy 2.1 $\mathbf{V}_i \sim N_p(\mathbf{0}, \mathbf{\Sigma})$ a $V_1, V_2, ..., V_r$ sú nezávislé. Z definície preto $\mathcal{U}'\mathbf{A}\mathcal{U} \sim W_p(r, \mathbf{\Sigma})$. □

Veta 2.6. *Nech* $\mathbf{S} \sim W_p(k, \mathbf{\Sigma})$ *a* $\mathbf{B}_{p,q}$ *matica konštánt. Potom* $\mathbf{B}'\mathbf{S}\mathbf{B} \sim W_q(k, \mathbf{B}'\mathbf{\Sigma}\mathbf{B})$.

Dôkaz. $\mathbf{B}'\mathbf{S}\mathbf{B} = \mathbf{B}'\mathcal{U}'\mathcal{U}\mathbf{B}$, kde

$$\mathcal{U}\mathbf{B} = \begin{pmatrix} \mathbf{U}_1' \\ \mathbf{U}_2' \\ \vdots \\ \mathbf{U}_k' \end{pmatrix}_{k,p} \mathbf{B} = \begin{pmatrix} \mathbf{V}_1' \\ \mathbf{V}_2' \\ \vdots \\ \mathbf{V}_k' \end{pmatrix},$$

$\mathbf{U}_i \sim N_p(\mathbf{0}, \mathbf{\Sigma})$ sú nezávislé. Preto

$$\begin{pmatrix} \mathbf{V}_1' \\ \mathbf{V}_2' \\ \vdots \\ \mathbf{V}_k' \end{pmatrix} = \begin{pmatrix} \mathbf{U}_1'\mathbf{B} \\ \mathbf{U}_2'\mathbf{B} \\ \vdots \\ \mathbf{U}_k'\mathbf{B} \end{pmatrix}$$

má riadky nezávislé, $cov(\mathbf{B}'\mathbf{U}_i, \mathbf{B}'\mathbf{U}_j) = \mathbf{B}'cov(\mathbf{U}_i, \mathbf{U}_j)\mathbf{B} = \mathbf{0}$ a $\mathbf{B}'\mathbf{U}_i \sim N_q(\mathbf{0}, \mathbf{B}'\mathbf{\Sigma}\mathbf{B})$. Platí $\mathbf{B}'\mathbf{S}\mathbf{B} = \sum_{i=1}^k \mathbf{V}_i \mathbf{V}_i' \sim W_q(k, \mathbf{B}'\mathbf{\Sigma}\mathbf{B})$ (priamo z definície). □

Dôsledok 2.7.
(a) Diagonálne submatice matice $\mathbf{S}$ *majú tiež Wishartovo rozdelenie, lebo ak*

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{pmatrix},$$

kde $\mathbf{S}_{11}$ *je rozmeru* $l \times l$*, tak*

$$(\,\mathbf{I}_{l,l} \quad \mathbf{0}\,)\,\mathbf{S} \begin{pmatrix} \mathbf{I}_{l,l} \\ \mathbf{0} \end{pmatrix} = \mathbf{S}_{11}.$$

(b) ak $\mathbf{S} \sim W_p(k, \mathbf{I})$ *a ak pre* $\mathbf{B}_{p,q}$ *platí* $\mathbf{B}'\mathbf{B} = \mathbf{I}$*, potom* $\mathbf{B}'\mathbf{S}\mathbf{B} \sim W_q(k, \mathbf{I})$*.*

Veta 2.8. *Nech* $\mathbf{S} \sim W_p(k, \mathbf{\Sigma})$ *a* $\mathbf{a} \in \mathcal{R}^p$ *je taký vektor konštánt, že* $\mathbf{a}'\mathbf{\Sigma}\mathbf{a} \neq 0$*. Potom* $\dfrac{\mathbf{a}'\mathbf{S}\mathbf{a}}{\mathbf{a}'\mathbf{\Sigma}\mathbf{a}} \sim \chi^2_k$*.*

Dôkaz. Podľa vety 2.6 platí, že $\mathbf{a}'\mathbf{S}\mathbf{a} \sim W_1(k, \mathbf{a}'\mathbf{\Sigma}\mathbf{a})$, čo znamená podľa poznámky (ii) pod definíciou 2.3, že $\dfrac{\mathbf{a}'\mathbf{S}\mathbf{a}}{\mathbf{a}'\mathbf{\Sigma}\mathbf{a}} \sim \chi^2_k$. □

Veta 2.9. *Nech* $\mathbf{U}_1, ..., \mathbf{U}_n$ *je náhodný výber z* $N_p(\mathbf{0}, \mathbf{\Sigma})$ *(teda* $\mathcal{U}'\mathcal{U} \sim W_p(n, \mathbf{\Sigma})$*),* $\mathbf{C}_{n,n}$ *je symetrická matica. Platí*

$$\mathcal{U}'\mathbf{C}\mathcal{U} \sim W_p(r, \mathbf{\Sigma}) \Leftrightarrow \mathbf{C}^2 = \mathbf{C}.$$

V takomto prípade $r = tr(\mathbf{C})$*.*

Dôkaz. Podľa lemy 2.5 je $\mathcal{U}'\mathbf{C}\mathcal{U} \sim W_p(r, \mathbf{\Sigma}) \Leftrightarrow \forall\, \mathbf{l} \in \mathcal{R}^p \;\; {}_{\mathbf{l}}\mathbf{Y}'\mathbf{C}\; {}_{\mathbf{l}}\mathbf{Y} \sim \sigma^2_{\mathbf{l}}\chi^2_r$, ($\sigma^2_{\mathbf{l}} = \mathbf{l}'\mathbf{\Sigma}\mathbf{l}$, ${}_{\mathbf{l}}\mathbf{Y} = \mathcal{U}\mathbf{l}$). V tomto prípade $r = h(\mathbf{C}) = tr(\mathbf{C})$. Pretože podľa (2.1) je $\dfrac{{}_{\mathbf{l}}\mathbf{Y}}{\sqrt{\mathbf{l}'\mathbf{\Sigma}\mathbf{l}}} \sim N_p(\mathbf{0}, \mathbf{I})$, je podľa vety 1.7 ${}_{\mathbf{l}}\mathbf{Y}'\mathbf{C}\; {}_{\mathbf{l}}\mathbf{Y} \sim \sigma^2_{\mathbf{l}}\chi^2_r \Leftrightarrow \dfrac{{}_{\mathbf{l}}\mathbf{Y}'}{\sqrt{\mathbf{l}'\mathbf{\Sigma}\mathbf{l}}}\mathbf{C}\dfrac{{}_{\mathbf{l}}\mathbf{Y}}{\sqrt{\mathbf{l}'\mathbf{\Sigma}\mathbf{l}}} \sim \chi^2_r \Leftrightarrow \mathbf{C}^2 = \mathbf{C}$. V tomto prípade $r = h(\mathbf{C})$. □

Lema 2.10. *Nech* $\mathbf{S}_1 \sim W_p(n_1, \mathbf{\Sigma})$*,* $\mathbf{S}_2 \sim W_p(n_2, \mathbf{\Sigma})$*.* $\mathbf{S}_1$ *a* $\mathbf{S}_2$ *sú nezávislé. Potom* $\mathbf{S}_1 + \mathbf{S}_2 \sim W_p(n_1 + n_2, \mathbf{\Sigma})$*.*

Dôkaz. $\mathbf{S}_1 = \mathcal{U}_1'\mathcal{U}_1$, $\mathbf{S}_2 = \mathcal{U}_2'\mathcal{U}_2$, kde $\mathcal{U}_1' = (\mathbf{U}_1 \vdots ... \vdots \mathbf{U}_{n_1})$, $\mathcal{U}_2' = (\mathbf{U}_{n_1+1} \vdots ... \vdots \mathbf{U}_{n_1+n_2})$ a $\mathbf{U}_i \sim N_p(\mathbf{0}, \mathbf{\Sigma}), i = 1, 2, ..., n_1 + n_2$ sú nezávislé. Preto ak označíme $\mathcal{U}' = (\mathcal{U}_1' \vdots \mathcal{U}_2')_{p, n_1+n_2}$, tak $\mathbf{S}_1 + \mathbf{S}_2 = (\mathcal{U}_1'\mathcal{U}_1 + \mathcal{U}_2'\mathcal{U}_2) = \mathcal{U}'\mathcal{U} \sim W_p(n_1 + n_2, \mathbf{\Sigma})$. □

Veta 2.11. *Nech* $\mathbf{C}_{n,n} = \mathbf{C}'$ *je p.s.d. matica konštánt,* $\mathbf{U}_i \sim N_p(\mathbf{0}, \mathbf{\Sigma}), i = 1, 2, ..., n$ *nezávislé. Platí, že* $\mathcal{U}'_{p,n}\mathbf{C}\mathcal{U} \sim \sum_{i=1}^n \lambda_i W_p^{(i)}(1, \mathbf{\Sigma})$*, kde* $\lambda_1, ..., \lambda_n$ *sú vlastné čísla matice* $\mathbf{C}$ *a* $W_p^{(1)}(1, \mathbf{\Sigma}), ..., W_p^{(n)}(1, \mathbf{\Sigma})$ *sú nezávislé.*

Dôkaz. Môžeme písať $\mathbf{C} = \sum_{i=1}^n \lambda_i \mathbf{p}_i \mathbf{p}_i'$, $\mathbf{I} = \sum_{i=1}^n \mathbf{p}_i \mathbf{p}_i'$, pričom $\lambda_1 \geqq ... \geqq \lambda_n \geqq 0$ sú vlastné čísla matice $\mathbf{C}$ a $\mathbf{p}_1, ...\mathbf{p}_n$ ortonormálne vektory. Teda $\mathcal{U}'\mathbf{C}\mathcal{U} = \sum_{i=1}^n \lambda_i \mathcal{U}'\mathbf{p}_i\mathbf{p}_i'\mathcal{U} = \sum_{i=1}^n \lambda_i \mathbf{V}_i\mathbf{V}_i'$, kde $\mathbf{V}_i \sim N_p(\mathbf{0}, \mathbf{\Sigma})$ a sú nezávislé (lema 2.1). Z vety 2.9 vieme, že $\mathcal{U}'\mathbf{p}_i\mathbf{p}_i'\mathcal{U} = \mathbf{V}_i\mathbf{V}_i' \sim W_p^{(i)}(1, \mathbf{\Sigma})$. □

Lema 2.12. *Pre matice príslušných rozmerov platí*

$$vec\mathbf{ABC} = (\mathbf{C}' \otimes \mathbf{A})vec\mathbf{B}, \tag{2.5}$$

$$tr\mathbf{AB} = (vec\mathbf{B}')'vec\mathbf{A}.$$

Dôkaz. Lemu dokážte ako cvičenie.

Veta 2.13. *Nech* $\mathbf{U}_i \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $i = 1, 2, ..., n$, $\mathbf{U}_1, \mathbf{U}_2, ..., \mathbf{U}_n$ *sú nezávislé,* $\mathbf{C}_1, \mathbf{C}_2$ *symetrické a idempotentné.* $\mathcal{U}'\mathbf{C}_1\mathcal{U}$ *a* $\mathcal{U}'\mathbf{C}_2\mathcal{U}$ *sú nezávislé* $\Leftrightarrow \mathbf{C}_1\mathbf{C}_2 = \mathbf{0}$.

Dôkaz. ak $\mathcal{U}'\mathbf{C}_1$ a $\mathcal{U}'\mathbf{C}_2$ sú nezávislé, tak sú nezávislé aj $\mathcal{U}'\mathbf{C}_1\mathcal{U}$ a $\mathcal{U}'\mathbf{C}_2\mathcal{U}$. $\mathcal{U}'\mathbf{C}_1$ a $\mathcal{U}'\mathbf{C}_2$ sú nezávislé práve vtedy ak sú nezávislé $\mathbf{I}\mathcal{U}'\mathbf{C}_1$ a $\mathbf{I}\mathcal{U}'\mathbf{C}_2$ a to je práve vtedy ak sú nezávislé $vec(\mathbf{I}\mathcal{U}'\mathbf{C}_1)$ a $vec(\mathbf{I}\mathcal{U}'\mathbf{C}_2)$, čiže podľa lemy 2.12 ak sú nezávislé vektory $(\mathbf{C}_1' \otimes \mathbf{I})vec\mathcal{U}'$ a $(\mathbf{C}_2' \otimes \mathbf{I})vec\mathcal{U}'$, ktoré sú podľa (2.3) normálne rozdelené, pričom $vec\mathcal{U}' \sim N_{np}(\mathbf{0}, \mathbf{I}_{n,n} \otimes \boldsymbol{\Sigma}_{p,p})$. Pretože $(\mathbf{C}_1' \otimes \mathbf{I})(\mathbf{I} \otimes \boldsymbol{\Sigma})(\mathbf{C}_2 \otimes \mathbf{I}) = (\mathbf{C}_1'\mathbf{C}_2 \otimes \boldsymbol{\Sigma}) = \mathbf{0}$, sú $\mathcal{U}'\mathbf{C}_1$ a $\mathcal{U}'\mathbf{C}_2$ nezávislé. Teraz už ľahko dokončíme dôkaz. $\square$