Neautonomní systém

$$\mathbb{x}(t+1) = A\,\mathbb{x}(t) + \mathbb{b}(t)$$
$$\mathbb{x}(0) = \xi$$

$$\mathbb{x}(t) = Z(t)\left[\xi + \sum_{j=0}^{t-1} Z(j+1)^{-1}\,\mathbb{b}(j)\right] \quad ; \qquad \begin{array}{l} Z(t+1) = A\,Z(t) \\ Z(0) = I \end{array}$$

$$x(t+1) = 2x(t) + y(t) + t$$
$$y(t+1) = 2y(t) + 1$$

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad \mathbb{b}(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}, \quad Z(t) = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}^t = \begin{pmatrix} 2^t & t\,2^{t-1} \\ 0 & 2^t \end{pmatrix}, \quad Z(t)^{-1} = \frac{1}{4^t}\begin{pmatrix} 2^t & -t\,2^{t-1} \\ 0 & 2^t \end{pmatrix}$$

↑ Jordanův tvar

$$Z(j+1)^{-1}\,\mathbb{b}(j) = \frac{1}{4^{j+1}}\begin{pmatrix} 2^{j+1} & -(j+1)\,2^j \\ 0 & 2^{j+1} \end{pmatrix}\begin{pmatrix} j \\ 1 \end{pmatrix} = \frac{1}{2^{j+2}}\begin{pmatrix} 2 & -(j+1) \\ 0 & 2 \end{pmatrix}\begin{pmatrix} j \\ 1 \end{pmatrix} = \frac{1}{2^{j+2}}\begin{pmatrix} j-1 \\ 2 \end{pmatrix}$$

První složka:

$$\sum_{j=0}^{t-1} \frac{j-1}{2^{j+2}} = \frac{1}{4}\left(\sum_{j=0}^{t-1}\frac{j}{2^j} - \sum_{j=0}^{t-1}\frac{1}{2^j}\right)$$

platí: $$\sum_{j=0}^{t-1} j x^j = x\sum_{j=0}^{t-1} j x^{j-1} = x\left(\sum_{j=0}^{t-1} x^j\right)' = x\left(\frac{1-x^t}{1-x}\right)' = x\,\frac{-t x^{t-1}(1-x) + (1-x^t)}{(1-x)^2}$$

tedy $$\sum_{j=0}^{t-1} j\left(\tfrac{1}{2}\right)^j = \frac{1}{2}\,\frac{-t\left(\frac{1}{2}\right)^{t-1}\frac{1}{2} + 1 - \left(\frac{1}{2}\right)^t}{\frac{1}{4}} = 2\left(1 - (t+1)\left(\tfrac{1}{2}\right)^t\right) = 2 - (t+1)\frac{1}{2^{t-1}}$$

celkem: $$\sum_{j=0}^{t-1}\frac{j-1}{2^{j+2}} = \frac{1}{4}\left(2 - (t+1)\frac{1}{2^{t-1}} - \frac{1-\left(\frac{1}{2}\right)^t}{1-\frac{1}{2}}\right) = \frac{1}{4}\left(2 - (t+1)\frac{1}{2^{t-1}} - 2 + \frac{1}{2^{t-1}}\right) =$$
$$= -\frac{1}{4}\,\frac{t}{2^{t-1}} = -\frac{t}{2^{t+1}}$$

Druhá složka:

$$\sum_{j=0}^{t-1}\frac{2}{2^{j+2}} = \frac{1}{2}\sum_{j=0}^{t-1}\left(\tfrac{1}{2}\right)^j = \frac{1}{2}\,\frac{1-\left(\frac{1}{2}\right)^t}{1-\frac{1}{2}} = \frac{1}{2}\,\frac{2^t-1}{2^t}\,\frac{2}{1} = \frac{1}{2^{t+1}}\left(2^{t+1}-2\right)$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2^t & t\,2^{t-1} \\ 0 & 2^t \end{pmatrix}\left[\begin{pmatrix} \xi \\ \eta \end{pmatrix} + \frac{1}{2^{t+1}}\begin{pmatrix} -t \\ 2^{t+1}-2 \end{pmatrix}\right] = \begin{pmatrix} 2^t\left(\xi + \frac{1}{2}t\eta\right) \\ 2^t\eta \end{pmatrix} + \begin{pmatrix} -\frac{t}{2} + \frac{t}{4}\left(2^{t+1}-2\right) \\ 2^t - 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 2^t\left(\xi + \frac{1}{2}t\eta\right) + t\left(2^{t-1}-1\right) \\ 2^t\eta + 2^t - 1 \end{pmatrix}, \qquad \begin{array}{l} x(t) = 2^t\xi + t\,2^{t-1}(\eta+1) - t \\ y(t) = 2^t(\eta+1) - 1 \end{array}$$