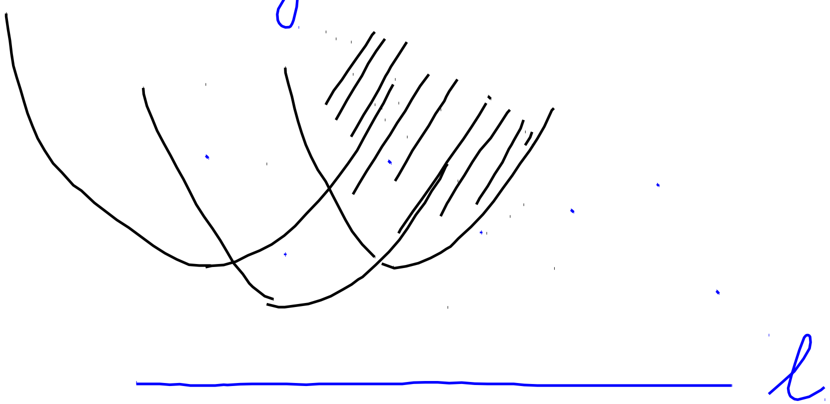
Voronoi diagrams

$P = \{p_1, \dots, p_n\}$

Voronoi diagram $\quad V(p_i) = \{ q \in \mathbb{R}^2, \ dist(q, p_i) \le dist(q, P) \}$

Method — sweep line

binary tree T ...... formed by arcs of parabolas given by the sweep line and a point from P

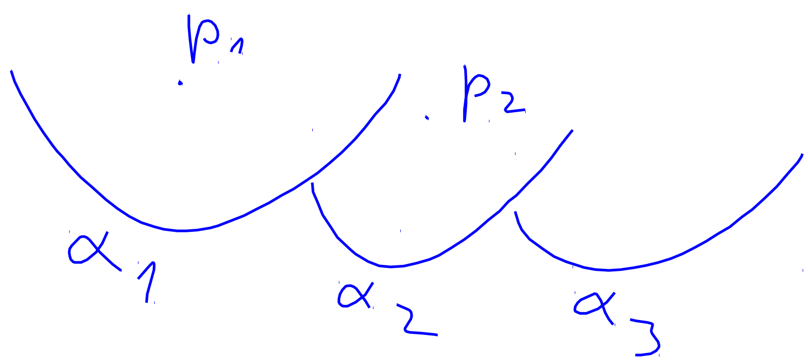


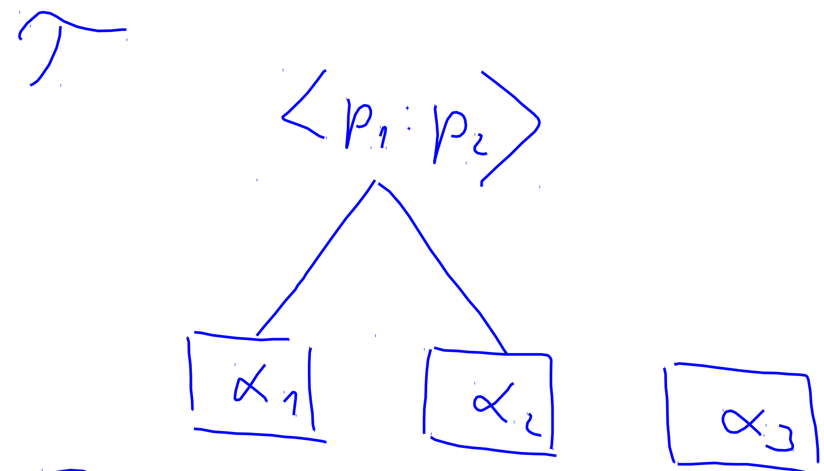
$\alpha(p, \ell)$ .... parabolas

$\alpha^+(p, \ell)$

$$\text{boundary of} \left(\bigcup_{p_i \in \ell^+} \alpha^+(p_i, \ell) \right) = \text{beach line}$$

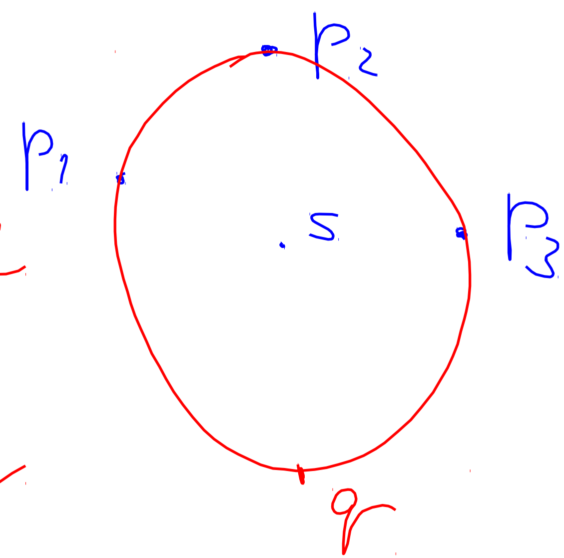
T ... is determined by the ordering of the arcs from the beach line





Q queue formed by

- site events ... points from P
- circle event



q is a circle event under the assumption that the arcs for p_1, p_2, p_3 are subsequent arcs in the beach line.

Lemma 1 New arc arises in the beach line only when the sweep line crosses a site event.

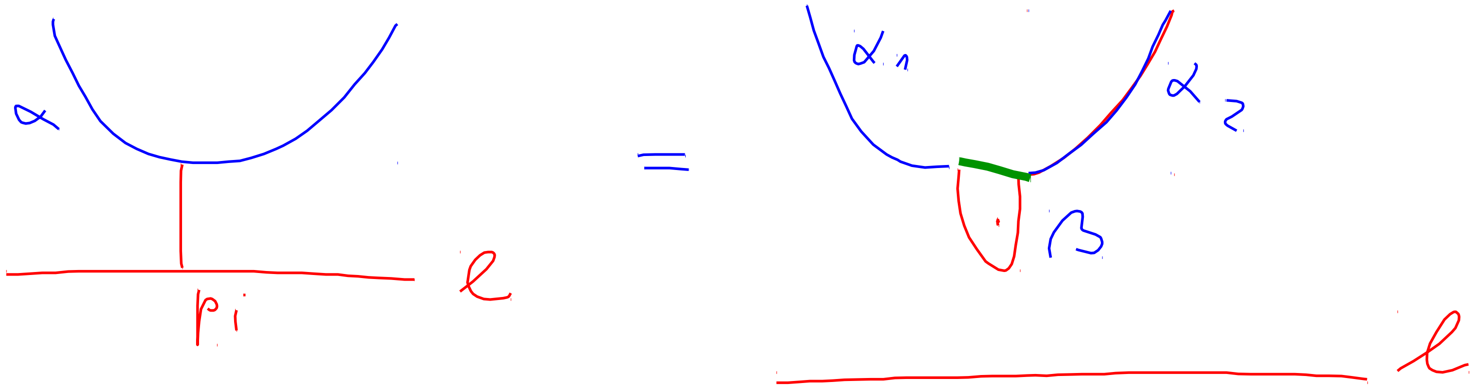
Lemma 2 An arcs disappears from the beach line only when the sweep line crosses the corresponding circle event.

Algorithm At the beginning we put all points from P into the queue Q. T is empty.

When ℓ (sweep line) crosses the first point in Q, T becomes a tree with one leaf.

- When ℓ crosses another site event p_i,
  - p_i is removed from Q
  - we find using T an arc in T which lies over p_i.

- we divide the arc α into 2 arcs α_1, α_2 and a new arc β appears between them



- new breakpoint determined a new edge in the V-diagram
- we have to cancel circle events for the original arc α
- we have to compute circle events for α_1, α_2

The sweep line crosses a circle event (arc B, q)

- remove q from Q
- remove B from T, rebalance T
- remove potential circle events corresponding to the arcs left to B, right to B.
- compute new circle events for these arcs and put them into the queue

We do these actions while Q is empty.
In this case T is not empty. The breakpoints in the beach line in this situation determine the edges of V-diagram which are halflines
We can find a square such that all point from P and all remaining breakpoints lie in this square

Time complexity

Making Q at the beginning needs $O(n \log n)$ time

number of arc $\leq 2n-1$

Site events .... actions need a constant time or
$O(\log n)$ time

they are only n

Circle events - if realized they are at most $2n-1$
- actions less than $O(\log n)$

All together $O(n \log n)$

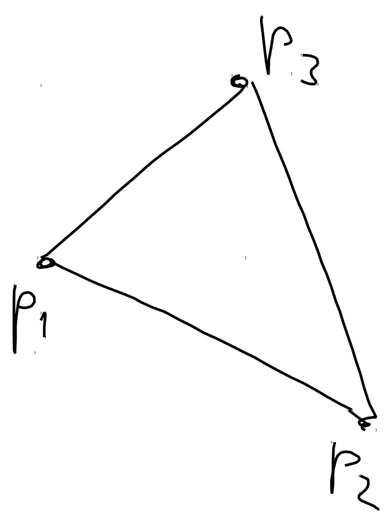
Size of Q and T $O(n)$

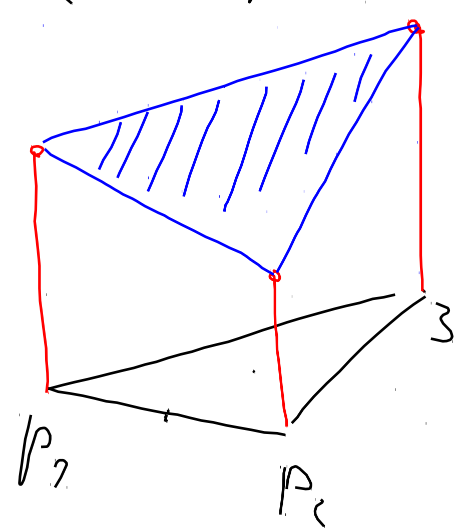
Delaunay triangulation

Motivation We know the values of a function

$$f: \mathbb{R}^2 \to \mathbb{R}$$

only in finitely many numbers. We want to approximate f at least on the convex hull of these points. The easiest way how to do it on a triangle $p_1p_2p_3$ is to use linear approximation.



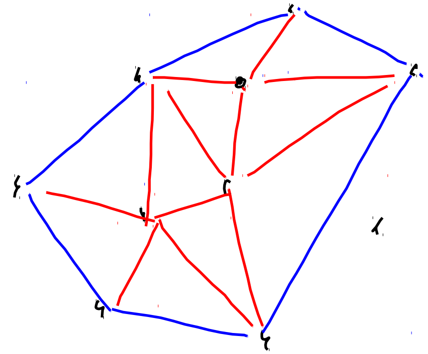


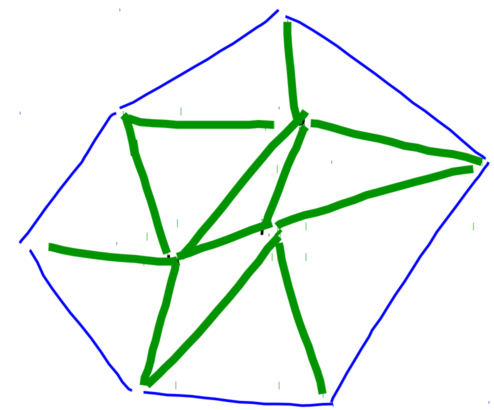
$$q \in \triangle p_1p_2p_3$$

$$q = t_1p_1 + t_2p_2 + t_3p_3$$

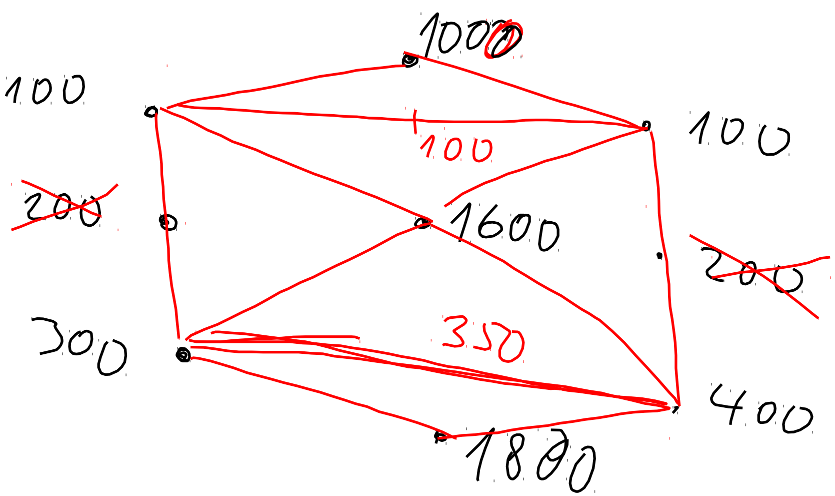
$$t_1 + t_2 + t_3 = 1$$

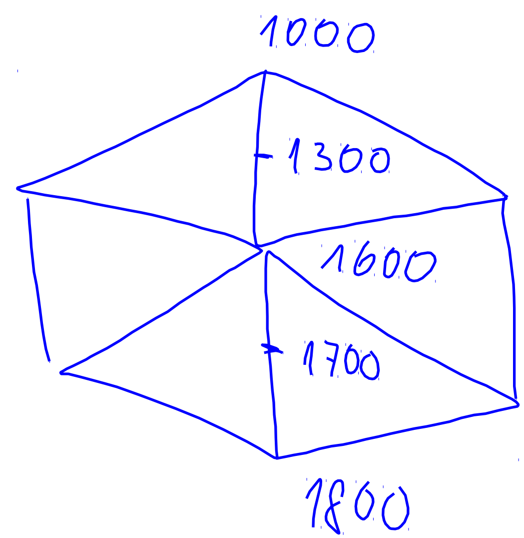
$$0 \le t_i \le 1$$





Question is: how to find "the best" triangulation





Number of triangles in the triangulation of the convex hull of a finite set is the same for all triangulations

P ... has n points

$CH(P)$ has k edges

$\Bigg\}$ Every triangulation has

$2n-2-k$ triangles ✓

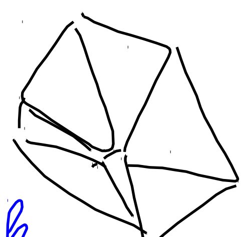
$3n-3-k$ edges

Proof: Let m be the number of triangles, h number of edges.

(1) $n - h + m + 1 = 2$

(2) $h = \dfrac{3m+k}{2}$

The triangulation forms a connected planar graph



We substitute from (2) into (1)

$n - \dfrac{3m+k}{2} + m + 1 = 2$

$\nearrow$

$2n - k - 2 = m$

T triangulation of the convex hull of a set P having m triangles

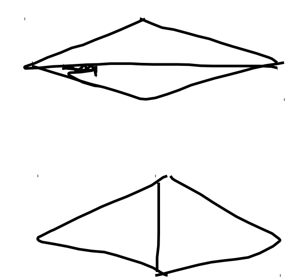
Consider angles of these triangles ordered

$$\alpha_1 \le \alpha_2 \le \ldots\ldots \le \alpha_{3m}$$

Let T has angles $\alpha_1 \le \ldots \le \alpha_{3m}$ and

T' has angles $\beta_1 \le \ldots \le \beta_{3m}$.

We define $T < T'$ if $(\alpha_1, \ldots \alpha_{3m})$ is smaller than $(\beta_1, \ldots \beta_{3m})$ in lexicographic ordering.



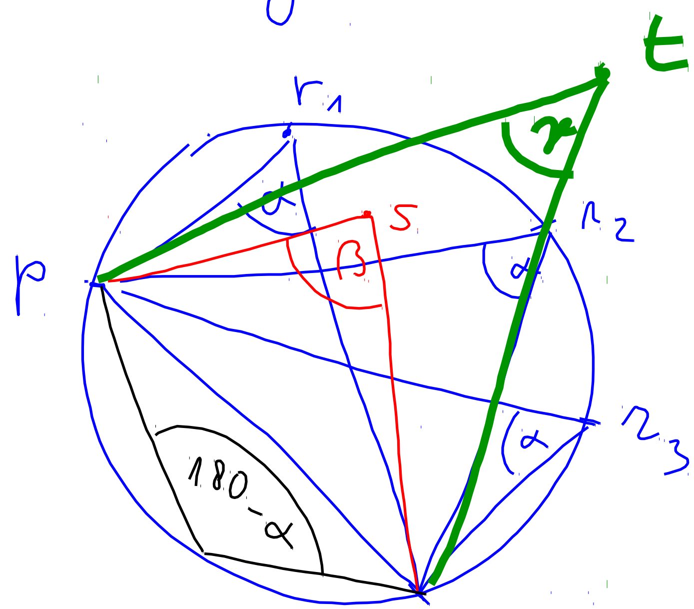
$\exists\, i \in \{1, 2, \ldots 3m\}$ such that

$\alpha_j = \beta_j$ for $j < i$

$\alpha_i < \beta_i$

Optimal triangulation is the triangulation maximal in this ordering.

High school geometry

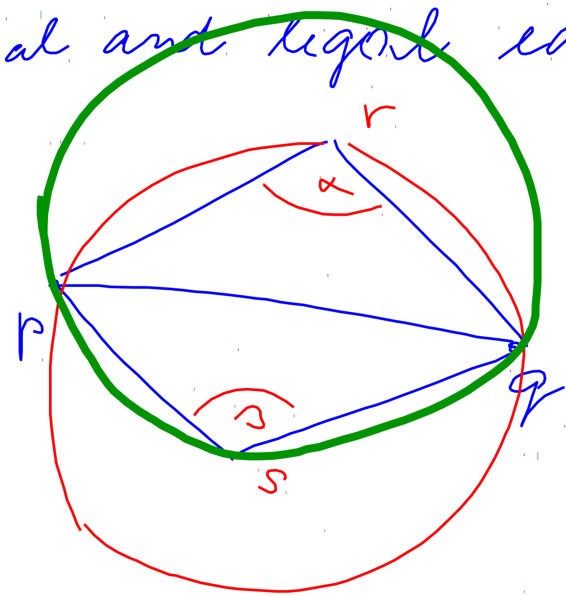


$\beta > \alpha > \gamma$

Consequence
If we have
4-gon so the vertices
lie on one circle $\Longleftrightarrow$ the sum of opposite angles $= 180°$.

Illegal and legal triangulations

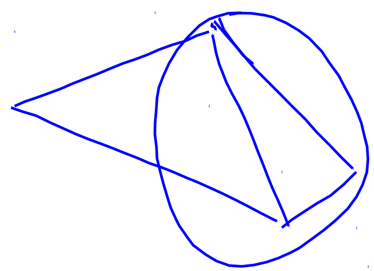
Illegal and legal edge pq of the triangulation

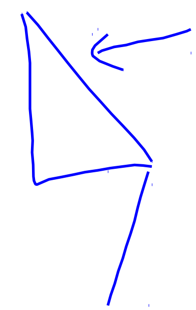


$\alpha + \beta > 180°$

The edge pq is called illegal.

Edges which are not illegal are legal.





Triangulation with only legal edges is called legal triangulation

Flip .... the way how to remove ilegal edges.

If pq lies in the triangulation T and T' is the triangulation after the flip, then

$$T < T'$$

The proof - figure 10.5

If we have optimal triangulation, this legal triangulation.