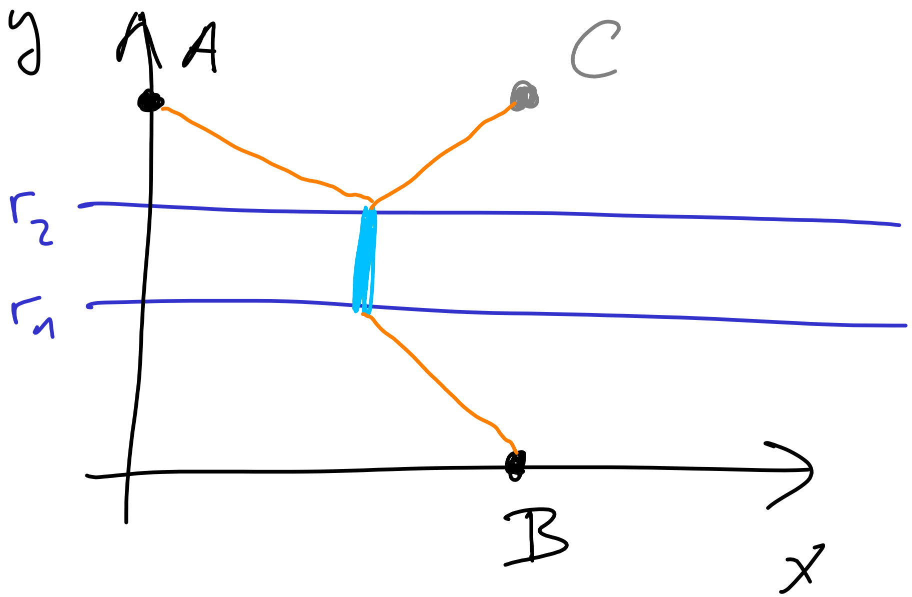
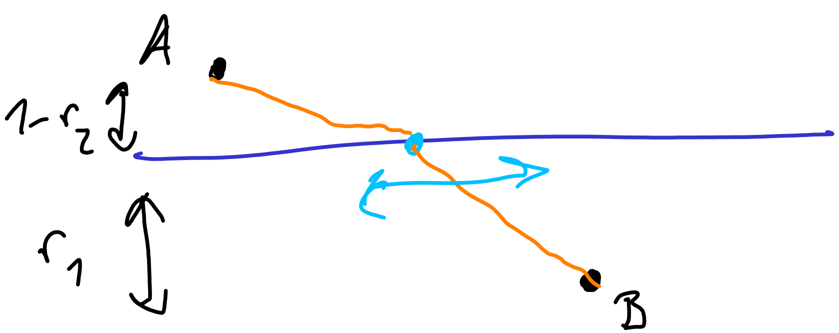
2 Hledání extrémů funkce jedné proměnné

2.1 Poloha mostu •

Města A, B a C jsou oddělena řekou, jejíž koryto je vymezeno přímkami $y = r_1$ a $y = r_2$, $0 < r_1 < r_2 < 1$. Města leží na souřadnicích $P_A = (0, 1)$, $P_B = (1, 0)$ a $P_C = (1, 1)$. Ve kterém místě je třeba postavit most přes řeku, aby celková délka silnic spojujících most s městy byla co nejkratší? Most bude kolmý na koryto řeky.



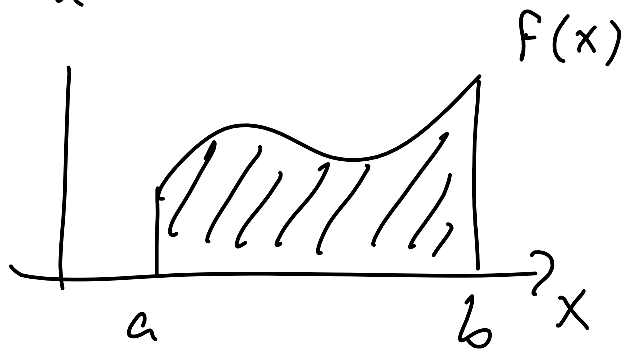
$$f(x) = \sqrt{x^2 + (1-r_2)^2} + \sqrt{(1-x)^2 + r_1^2} + \sqrt{(1-x)^2 + (1-r_2)^2}$$



$$x \cancel{x} = \frac{1-r_2}{1-r_2+r_1}$$

Numerická kvadratura

1) úvod



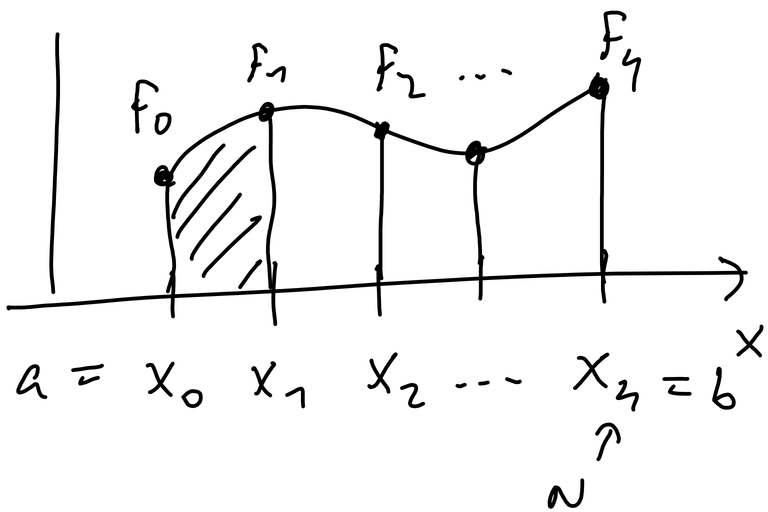
integrační pravidlo

$$I = \int_a^b f(x)\,dx \approx \sum_{j=0}^{N} w_j\, f(x_j)$$

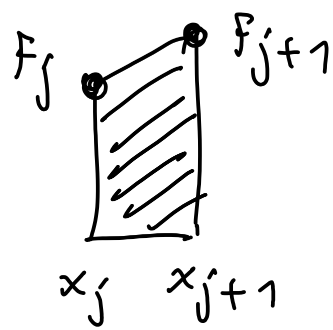
body (uzly) $x_0, x_1, \dots, x_N$

váhy $w_0, w_1, \dots, w_N$

klasická integrační pravidla



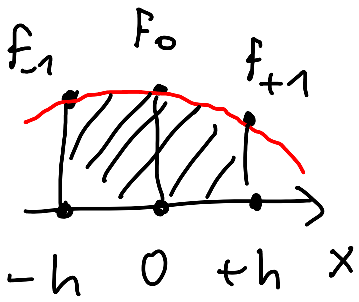
lichoběžníkové



$$(x_{j+1} - x_j)\,\frac{f_j + f_{j+1}}{2}$$

$$I = \frac{b-a}{N}\left(\frac{1}{2}f_0 + f_1 + f_2 + \dots + f_{N-1} + \frac{1}{2}f_N\right)$$

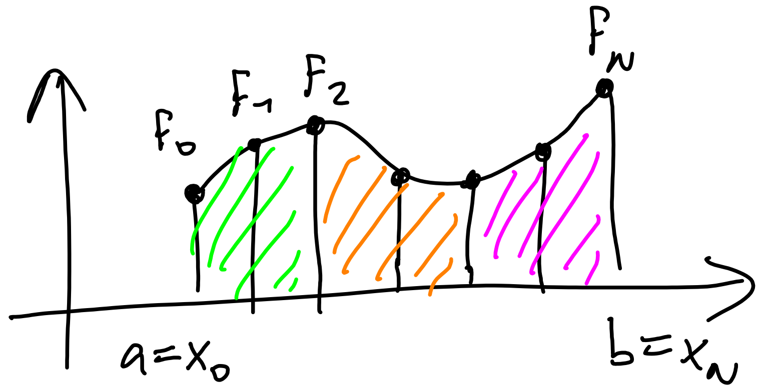
Simpsonovo pravidlo



$$f(x) = f_0 + \frac{f_1 - f_{-1}}{2h} x + \frac{1}{2} \frac{f_1 - 2f_0 + f_{-1}}{h^2} x^2$$

$$\int_{-h}^{h} f(x)\,dx = 2h f_0 + \frac{1}{2} \frac{f_1 - 2f_0 + f_{-1}}{h^2} \frac{h^3}{3} 2$$

$$= f_{-1} \frac{h}{3} + f_0 \frac{4h}{3} + f_1 \frac{h}{3} = \frac{h}{3} \left(f_{-1} + 4f_0 + f_1 \right)$$



$$h = \frac{b-a}{N}$$

$$I \approx \frac{b-a}{N} \frac{1}{3} \left(f_0 + 4f_1 + 2f_2 \; - - - - \; + 4f_{N-1} + f_N \right)$$

$\frac{h}{3} \times$

1 4 1

1 4 1

1 4 1

$\frac{h}{3}$ 1 4 2 4 2 4 1

Rombergova integrace

Eulerova – Maclaurinova formule

$$\int_{x_0}^{x_N} f(x)\,dx = h\left(\tfrac{1}{2}f_0 + f_1 + f_2 + \dots + f_{N-1} + \tfrac{1}{2}f_N\right) - \frac{B_2 h^2}{2}\left(f_N' - f_0'\right) - \dots - \frac{B_{2k} h^{2k}}{(2k)!}\left(f_N^{(2k-1)} - f_0^{(2k-1)}\right) - \dots$$

zbytek $\quad A h^2 + B h^4 + C h^6 + \dots$

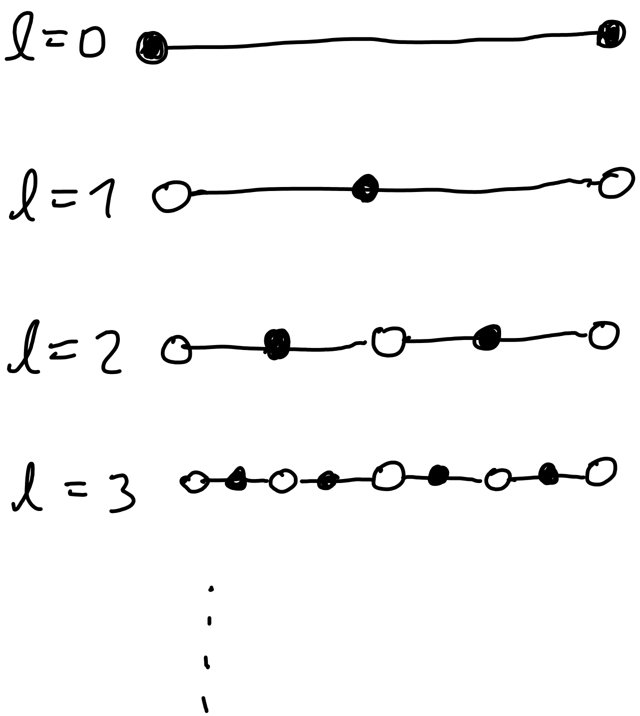
vstupují vlastnosti $f(x)$ na krajích

lichoběžníkové pravidlo s dělením N $\quad T(N) \quad A h^2 + \dots \quad h = \frac{b-a}{N}$

— || — $\quad 2N \quad T(2N) \quad A\left(\frac{h}{2}\right)^2 + \dots$

$$\frac{4\,T(2N) - T(N)}{4 - 1} = \text{integrál s chybou} \sim h^4$$

dělení $h = (b-a)/2^l$



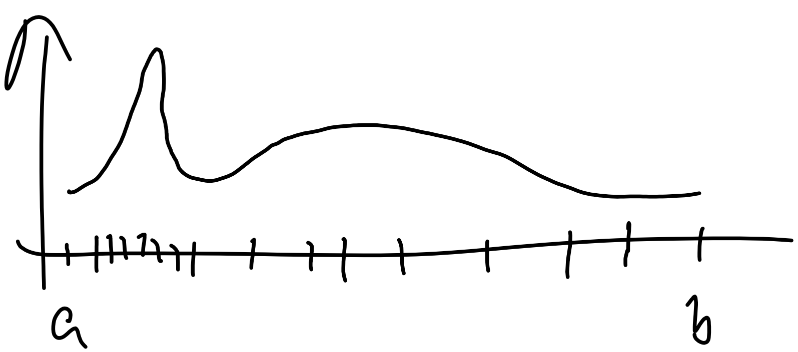
lichob. pravidlo

$$T_{lk} = \frac{2^{2k}\, T_{l,k-1} - T_{l-1,k-1}}{2^{2k} - 1}$$

$$\begin{array}{lccccccc}
 & T_{00} & \searrow & & & & & \\
 & T_{10} & \rightarrow & T_{11} & \searrow & & & \\
 & T_{20} & \rightarrow & T_{21} & \rightarrow & T_{22} & \searrow & \\
 & T_{30} & \rightarrow & T_{31} & \rightarrow & T_{32} & \rightarrow & T_{33} \\
\text{chyba} & h^2 & & h^4 & & h^6 & & h^8
\end{array}$$

	k=0	k=1	k=2	k=3	k=4
l=0	0.785398163397448				
1	0.948059448968520	1.002279877492210			
2	0.987115800972775	1.000134584974194	0.999991565472993		
3	0.996785171886170	1.000008295523968	0.999999876227286	1.000000008144021	
4	0.999196680485072	1.000000516684707	0.999999998095422	1.000000000029837	0.999999999998017

Adaptivní integrace



 $\rightarrow$ integrál & chyba

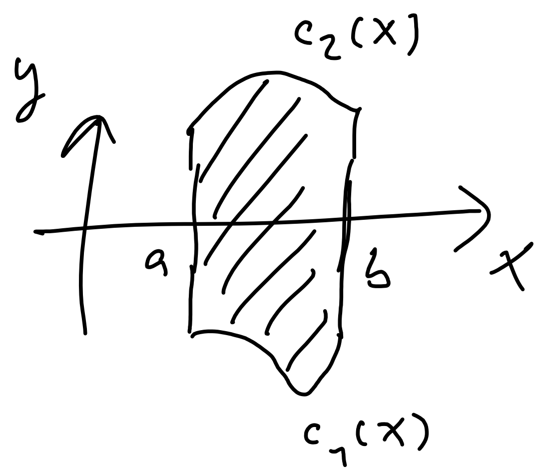
integrál $\quad T_{22}$

chyba $\quad |T_{21} - T_{11}|$

$$\text{celková chyba} \approx \sum_{\text{podint.}} \text{chyba z podintervalu}$$

Vícerozměrné integrály

1) $$\iint\limits_{\text{oblast}} f(x,y)\,dx\,dy$$

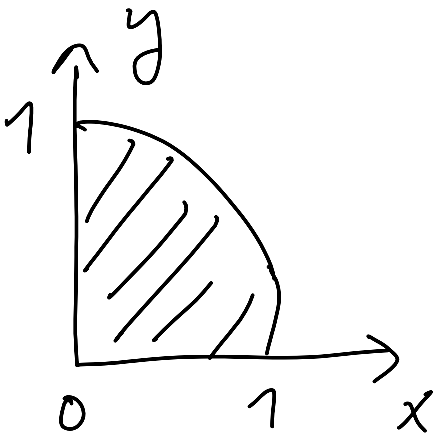


$$= \int_a^b dx \underbrace{\int_{c_1(x)}^{c_2(x)} dy\; f(x,y)}_{F(x)} = \int_a^b dx\; F(x)$$

2) spec. souřadnice

$$\iint\limits_{\substack{\bigcirc \\ R=1}} f(x,y)\,dx\,dy = \int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{+\sqrt{1+x^2}} dy\; f(x,y) = \int_0^1 dr\, r \int_0^{2\pi} d\varphi\; f$$

3) Monte-Carlo integrace



$$\iint 1 \, dx \, dy = \frac{\pi}{4}$$

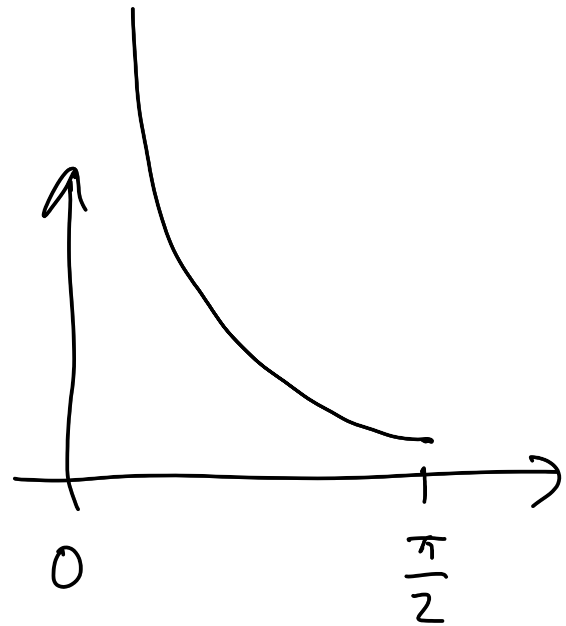
konvergence

chyba $\sim \frac{1}{\sqrt{N}}$

Nevlastní integrály

1) singularita

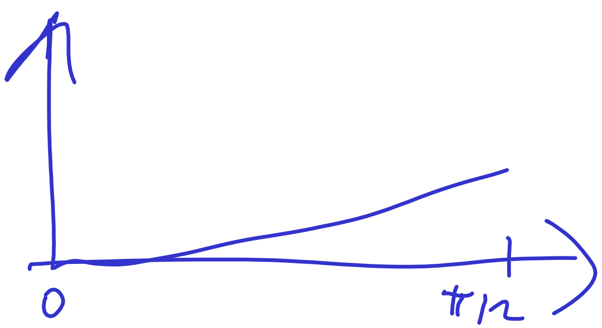
$$I = \int_0^{\pi/2} \frac{1}{\sqrt{\sin x}}\,dx$$



integrand v okolí $x=0$ $\sim \frac{1}{\sqrt{x}}$

$$I = \int_0^{\pi/2} \left(\frac{1}{\sqrt{\sin x}} - \frac{1}{\sqrt{x}} \right) dx + \int_0^{\pi/2} \frac{1}{\sqrt{x}}\,dx$$

analyticky

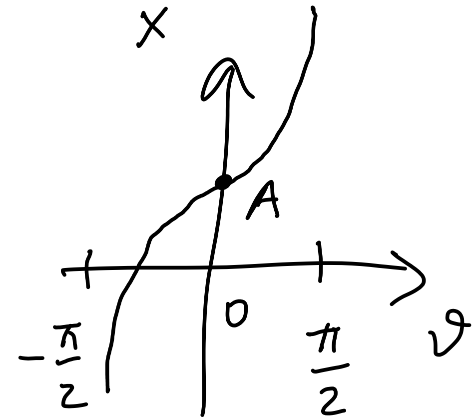


2) nekonečné meze

$$\int_{-\infty}^{\infty} f(x)\,dx$$

$$x = A + B\tan\vartheta$$

$$dx = B\,\frac{1}{\cos^2\vartheta}\,d\vartheta$$



$$\rightarrow \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} f(A + B\tan\vartheta)\,(\,?\,)\,d\vartheta$$