Obyčejné diferenciální rovnice s počátečními podmínkami

$x_1(t) \;\dots\; x_N(t)$ $\qquad \bar{x}(t_0) = \bar{x}_0$ poč. podmínka

systém ODR

$$\frac{d\bar{x}}{dt} = \bar{F}(t, \bar{x})$$

rozepsáno na

$$\frac{dx_1}{dt} = F_1(t, x_1, \dots, x_N)$$

$$\vdots$$

$$\frac{dx_N}{dt} = F_N(t, x_1, \dots, x_N)$$

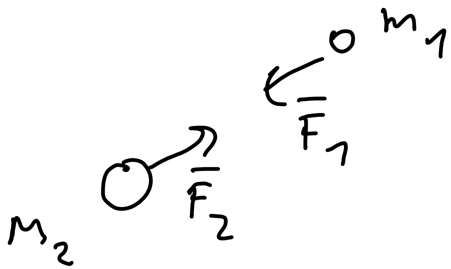
Př. tlumený harm. oscilátor

$$m\ddot{x} + \gamma\dot{x} + kx = F\cos\omega t \qquad x(t_0) = x_0 \quad v(t_0) = v_0$$

$$\dot{x} = v$$

$$\dot{v} = \frac{1}{m}\left(F\cos\omega t - kx - \gamma v\right)$$

Př.



m_1: x_1, y_1, v_{x1}, v_{y1}

m_2: x_2, y_2, v_{x2}, v_{y2}

$$F = \frac{G m_1 m_2}{\underbrace{(x_1-x_2)^2+(y_1-y_2)^2}_{r^2}}$$

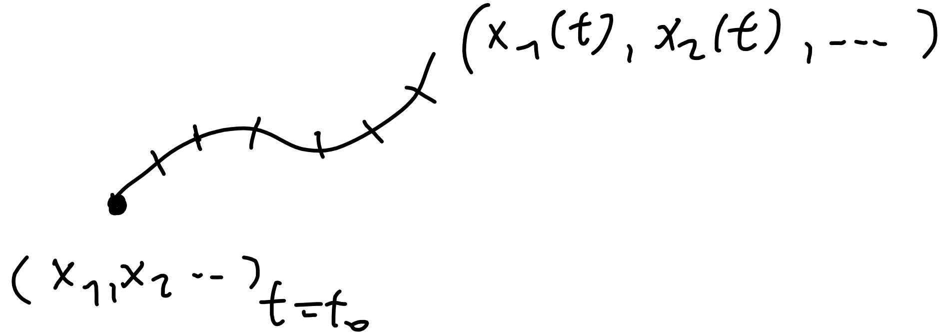
$$\frac{dx_1}{dt} = v_{x1} \qquad \frac{dx_2}{dt} = v_{x2}$$

$$\frac{dy_1}{dt} = v_{y1} \qquad \frac{dy_2}{dt} = v_{y2}$$

$$\frac{dv_{x1}}{dt} = F \, \frac{x_2-x_1}{r} \, \frac{1}{m} \qquad \frac{dv_{x2}}{dt} = F \, \frac{x_1-x_2}{r} \, \frac{1}{m}$$

$$\frac{dv_{y1}}{dt} = F \, \frac{y_2-y_1}{r} \, \frac{1}{m} \qquad \frac{dv_{y2}}{dt} = F \, \frac{y_1-y_2}{r} \, \frac{1}{m}$$

metody řešení

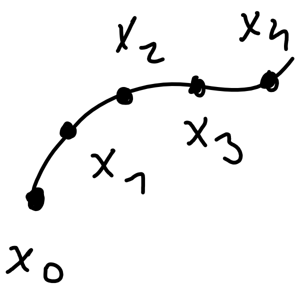


1) jednokrokové

Eulerova, Runge-Kutta

2) vícekrokové

Eulerova metoda (1768)



$$x_{n+1} = x_n + f(t_n, x_n)\, h$$

$$t_{n+1} = t_n + h$$

alternativní odvození $x(t)$ řešení s poč. podm. t_n, x_n

Taylor: $$x(t_n+h) = \underbrace{x(t_n)}_{x_n} + \underbrace{x'(t_n)}_{f(t_n,x_n)}\, h + O(h^2)$$

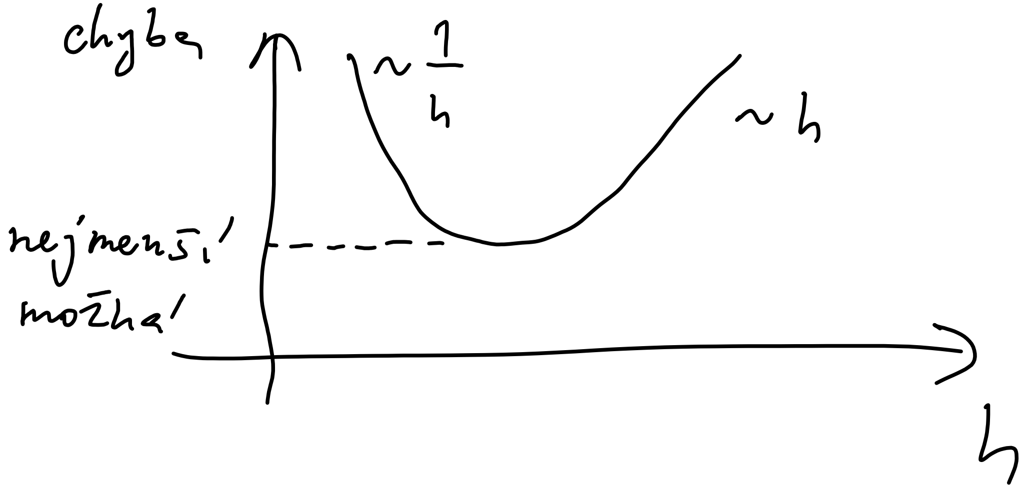
lokální chyba - chyba 1 kroku Euler $O(h^2)$

globální chyba $\frac{t_{fin} - t_0}{h}$ kroků $\times$ lokální chyba $\sim$ Euler $O(h)$

celková chyba u Eulerovy metody

$$\text{chyba} \sim \underbrace{A h}_{\text{globální chyba}} + \frac{t_{fin} - t_0}{h}\, \varepsilon$$

ε ← zaokrouhl. chyba 1 kroku



Př. $\frac{dx}{dt} = x \qquad x(0)=1 \qquad x(t) = e^t$

Eulerova metoda s krokem h $\qquad x_{n+1} = x_n + x_n h = (1+h)\,x_n$

$$\to \quad x_n = (1+h)^n$$

pro $t=1$ $\qquad x_{\text{fin}} = (1+h)^{\frac{1}{h}} \qquad$ přesné řešení e^1

$$= e^{\frac{1}{h}\underbrace{\ln(1+h)}_{h - \frac{h^2}{2} + \frac{h^3}{3} - \dots}}$$

$$= e^1 \underbrace{e^{-\frac{h}{2} + \frac{h^2}{3} - \dots}}_{1 + \text{relat. chyba}} \qquad e^{-\frac{h}{2}} \approx 1 - \frac{h}{2}$$

rel. chyba

cvičení

quad_singular1

$$\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} = \int_0^{\pi/2}\left(\frac{1}{\sqrt{\sin x}} - \frac{1}{\sqrt{x}}\right)dx + \underbrace{\int_0^{\pi/2} \frac{1}{\sqrt{x}}\,dx}_{\sqrt{2\pi}} = \sqrt{2}\,K\left(\tfrac{1}{2}\right)$$

eliptický integrál

quad_singular2

$$\int_0^{\pi/2} \ln\sin x\,dx = \int_0^{\pi/2} (\ln\sin x - \ln x)\,dx + \underbrace{\int_0^{\pi/2} \ln x\,dx}_{\frac{\pi}{2}\left(\ln\frac{\pi}{2}-1\right)} = -\frac{\pi}{2}\ln 2$$

$$\ln\frac{\sin x}{x} = \ln\frac{x - \frac{1}{6}x^3 + \dots}{x} \approx -\frac{1}{6}x^2$$

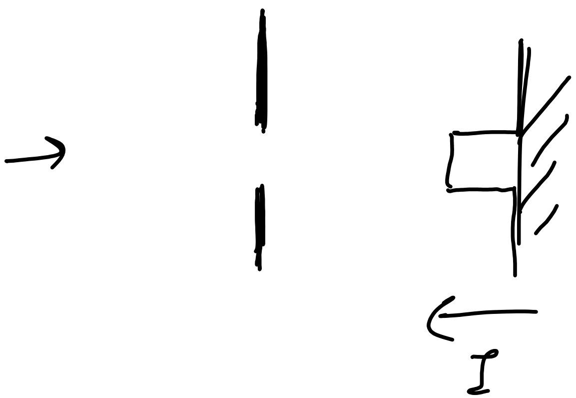
quad_singular3

$$\int_0^{\infty} e^{-x}\,dx = 1 \qquad \int_0^{A} e^{-x}\,dx = 1 - e^{-A}$$

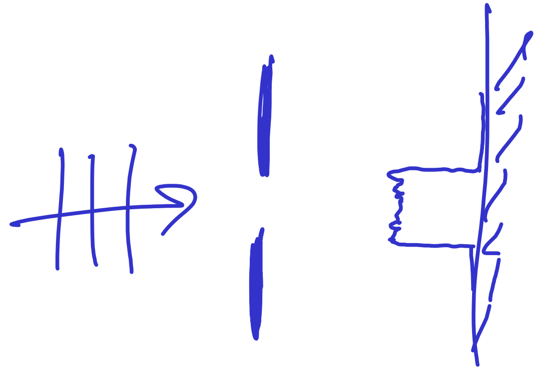
$x = \tan\vartheta$ $\quad \displaystyle\int_0^{\pi/2} e^{-\tan\vartheta}\,\frac{1}{\cos^2\vartheta}\,d\vartheta$

$$\int_1^{\infty} \frac{1}{x^2}\,dx = 1 \qquad \int_1^{A} \frac{1}{x^2}\,dx = 1 - \frac{1}{A}$$

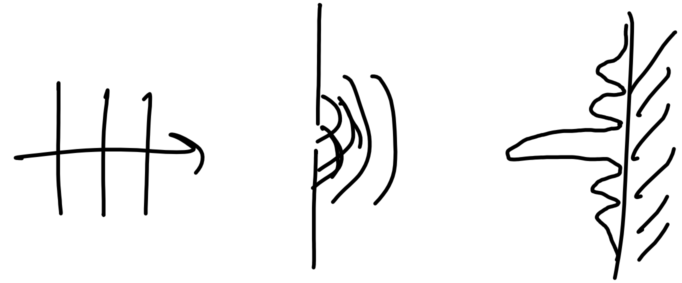
Fresnelova difrakce na hraně



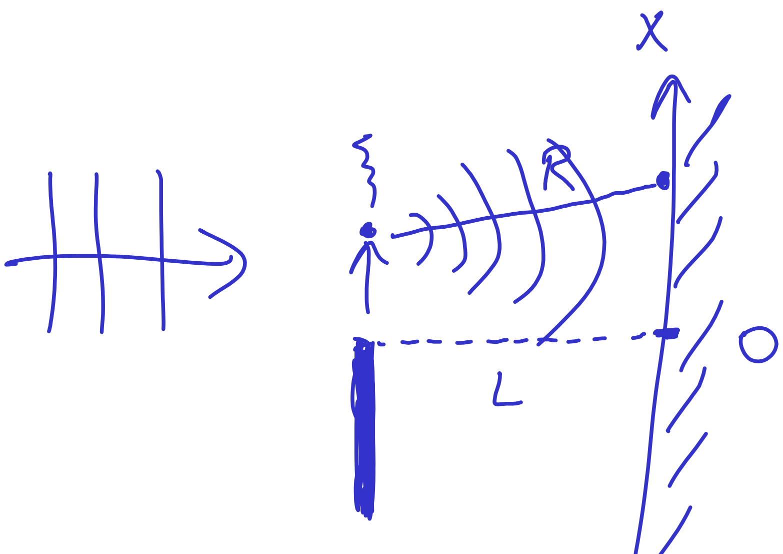
geom. optika



Fresnelova difr.



Fraunhoferova difrakce



$I(x)$ = intenzita superponovaných vln

z bodu ξ vlnky $\dfrac{e^{ikR}}{\sqrt{R}}$

$$R = \sqrt{L^2 + (x-\xi)^2}$$

amplituda $\psi(x) = \int_0^\infty d\xi\, e^{ikR}/\sqrt{R}$

intenzita $I = |\psi|^2$

$$R = \sqrt{L^2 + (x-\xi)^2} = L\sqrt{1 + \left(\frac{x-\xi}{L}\right)^2} \approx L\left[1 + \frac{1}{2}\left(\frac{x-\xi}{L}\right)^2\right]$$

Fresnelova aproximace

$$\psi(x) \approx \frac{1}{L} e^{ikL} \int_0^\infty e^{ik\frac{(x-\xi)^2}{2L}} d\xi$$

$$\psi(x) \sim \int_0^\infty \left[\cos\frac{k(x-\xi)^2}{2L} + i\sin\frac{k(x-\xi)^2}{2L}\right] d\xi =$$

$$= \underbrace{\int_0^x \cos\frac{k(x-\xi)^2}{2L} d\xi}_{\int_0^x \cos\frac{kt^2}{2L} dt} + \underbrace{\int_x^\infty \cos\frac{k(x-\xi)^2}{2L} d\xi}_{\int_0^\infty \cos\frac{kt^2}{2L} dt} + \text{sin členy}$$

$t = x - \xi$

$dt = -d\xi$

Fresnelovy integrály $\quad C(w) = \int_0^w \cos\frac{\pi u^2}{2} du \qquad S(w) = \int_0^w \sin\frac{\pi u^2}{2} du$

$$u = \frac{\xi}{a} \qquad a = \sqrt{\frac{\pi L}{k}} = \sqrt{\frac{L\lambda}{2}}$$

$$\psi(x) \sim \mathcal{C}\left(\frac{x}{a}\right) + i\,\mathcal{S}\left(\frac{x}{a}\right) + \underbrace{\mathcal{C}(\infty)}_{1/2} + i\,\underbrace{\mathcal{S}(\infty)}_{1/2}$$

$$I(x) = \frac{I_0}{2}\left\{\left[\mathcal{C}\left(\frac{x}{a}\right) + \frac{1}{2}\right]^2 + \left[\mathcal{S}\left(\frac{x}{a}\right) + \frac{1}{2}\right]^2\right\}$$