4.2 Epidemie moru

epidemiologické modely

susceptible

infected

SIR ← recovered

SIS

SI

z ... zdraví

n ... nemocní

$$\frac{dz}{dt} = -\beta n z \qquad \beta = 0{,}0177\ /\text{měsíc}$$

$$\frac{dn}{dt} = +\beta n z - \alpha n \qquad \alpha = 2.82\ /\text{měsíc}$$

$$\frac{d}{dt}\begin{pmatrix} z \\ n \end{pmatrix} = \begin{pmatrix} -\beta n z \\ +\beta n z - \alpha n \end{pmatrix}$$

$$z(0) = 254$$

$$n(0) = 7$$

fun(t, x) $\qquad x = \begin{pmatrix} z \\ n \end{pmatrix}$

$z = x[0]$

$n = x[1]$ $\xrightarrow{\text{fun}}$ $[-\beta n z, \beta n z - \alpha n]$

Vlastní problém

$A\bar{x} = \lambda \bar{x}$ A matice $N \times N$ $\rightarrow$ N vlastních čísel λ_n

N vlastních vektorů $\bar{x}_n$

reálná symetrická A:

$\rightarrow$ reálné vl. hodnoty & vl. vektory, ortogonalita $\bar{x}_m \cdot \bar{x}_n = \delta_{mn}$

matice přechodu $T = \begin{pmatrix} | & | & & | \\ \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_N \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$ $T^T T = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$ $T^{-1} = T^T$

$\bar{x} = T\bar{x}'$

původní báze ($\bar{x}$), báze vl. vektorů ($\bar{x}'$)

$$\Lambda = T^{-1} A T = T^T A T = \begin{pmatrix} \lambda_1 & & & \cancel{0} \\ & \lambda_2 & & \\ & & \ddots & \\ \cancel{0} & & & \lambda_N \end{pmatrix}$$

- Hermiteovský vlastní problém

$$H\bar\psi = E\bar\psi \quad \to \quad \text{reálné } E \;, \quad \text{komplexní } \bar\psi = \bar u + i\bar v$$

$\uparrow$

hermiteovská $\quad H^{+} = H^{*T} = H = A + iB \qquad A^T = A$

$(N\times N) \qquad H^{*T} = A^T - iB^T \qquad B^T = -B$

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} \bar u \\ \bar v \end{pmatrix} = E \begin{pmatrix} \bar u \\ \bar v \end{pmatrix}$$

$$H\bar\psi = E\bar\psi$$

$$(A+iB)(\bar u + i\bar v) = E(\bar u + i\bar v)$$

$$A\bar u - B\bar v = E\bar u$$

$$\cancel{i}(A\bar v + B\bar u) = \cancel{i}E\bar v$$

$$\Big\downarrow{\scriptstyle T}$$

$$\begin{pmatrix} A^T & B^T \\ -B^T & A^T \end{pmatrix}$$

$\to \; 2N\times 2N \qquad E$ stejné $\;, \quad \bar\psi = \bar u + i\bar v$

$$\bar\psi = i(\bar u + i\bar v) = -\bar v + i\bar u$$

- zobecněný vlastní problém

$A\bar{x} = \lambda B\bar{x}$ $\qquad$ A, B symetrické matice, B pozitivně definitní

1) $B^{-1}A\bar{x} = \lambda\bar{x}$

obecně

$$(B^{-1}A)^T = A^T(B^{-1})^T = AB^{-1} \neq B^{-1}A$$

2) Choleského rozklad $\qquad B = LL^T$

$$A\bar{x} = \lambda LL^T\bar{x} \rightarrow L^{-1}A\bar{x} = \lambda L^T\bar{x} \rightarrow \underbrace{[L^{-1}A(L^{-1})^T]}_{C}\underbrace{L^T\bar{x}}_{\bar{y}} = \lambda \underbrace{L^T\bar{x}}_{\bar{y}}$$

$$(L^{-1})^T L^T = 1$$

$$C\bar{y} = \lambda\bar{y} \qquad C^T = L^{-1}A^T(L^{-1})^T = C$$

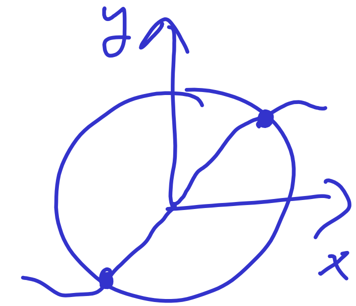
Soustavy nelineárních rovnic

$$\bar{f}(\bar{x}) = \bar{0} \qquad \begin{aligned} f_1(x_1, x_2 \dots x_N) &= 0 \\ f_2(x_1, x_2 \dots x_N) &= 0 \\ &\vdots \\ f_N(x_1, x_2 \dots x_N) &= 0 \end{aligned}$$

Př.

$$x^2 + y^2 = 1$$

$$\sin x - y = 0$$



$$x^2 + \sin^2 x = 1$$

$$x^2 = \cos^2 x$$

Newtonova – Raphsonova metoda

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \sigma(\Delta x^2) \quad \rightarrow \quad \bar{x} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Taylorův rozvoj kolem poč. odhadu $\bar{x}^{(n)}$

$$f_i(x_1 \dots x_N) = f_i(x_1^{(n)} \dots x_N^{(n)}) + \sum_{j=1}^{N} \left.\frac{\partial f_i}{\partial x_j}\right|_{\bar{x}^{(n)}} (x_j - x_j^{(n)}) + \sigma(\Delta \bar{x}^2)$$

$$\bar{f}(\bar{x}) = \bar{f}(\bar{x}^{(n)}) + J(\bar{x} - \bar{x}^{(n)}) + \sigma(\Delta \bar{x}^2) \qquad J_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\bar{x}^{(n)}}$$

$$\bar{f}(\bar{x}^{(n)}) + J(\bar{x}^{(n+1)} - \bar{x}^{(n)}) = 0 \quad \rightarrow \quad \bar{x}^{(n+1)} = \bar{x}^{(n)} - J^{-1}\bar{f}(\bar{x}^{(n)})$$

v bodě $\bar{x}^{(n)}$

Př.

$$x^2+y^2=1$$
$$\sin x - y = 0$$

$$f_1(x,y) = x^2+y^2-1$$
$$f_2(x,y) = \sin x - y$$

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & 2y \\ \cos x & -1 \end{pmatrix}$$

$$\bar{x}^{(n+1)} = \bar{x}^{(n)} - \begin{pmatrix} 2x & 2y \\ \cos x & -1 \end{pmatrix}^{-1}_{\bar{x}^{(n)}} \begin{pmatrix} x^2+y^2-1 \\ \sin x - y \end{pmatrix}_{\bar{x}^{(n)}}$$

Prokládání polynomem

$(x_i, y_i) \quad i = 1 \dots N$ $\qquad P_M(x) = \sum_{m=0}^{M} a_m x^m$

$S = \sum_{i=1}^{N} \left[y_i - P_M(x_i) \right]^2$ minimalizujeme vhodnou volbou $a_0 \dots a_M$

v minimu $\quad \frac{\partial S}{\partial a_j} = 0 \quad j = 0 \dots M$

$$\frac{\partial S}{\partial a_j} = \sum_{i=1}^{N} \frac{\partial}{\partial a_j} \left[y_i - \sum_{m=0}^{M} a_m x_i^m \right]^2 = \sum_{i=1}^{N} 2 \left[y_i - \sum a_m x_i^m \right] (- x_i^j) = 0$$

$$\sum_{m=0}^{M} a_m \sum_{i=1}^{N} x_i^{m+j} = \sum_{i=1}^{N} y_i x_i^j \qquad j = 0 \dots M$$

$$\begin{pmatrix} \sum_i 1 & \sum_i x_i & \dots \\ \sum_i x_i & \sum_i x_i^2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_M \end{pmatrix} = \begin{pmatrix} \sum_i y_i \\ \sum_i y_i x_i \\ \vdots \end{pmatrix}$$

$$\underset{A}{A} \; \underset{\text{coef}}{\bar{a}} = \underset{b}{\bar{b}}$$

1) 2015 322m

2) čas ve stoletích, počítáno od r. 1900