Parciální diferenciální rovnice

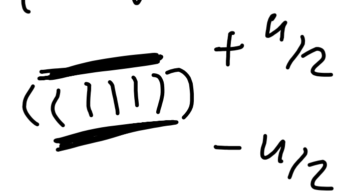
okrajové

počáteční úlohy

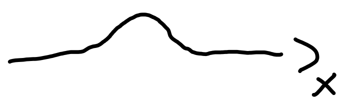
Př. $\nabla^2 \phi = -\frac{\rho}{\varepsilon_0}$ Poissonova

$\nabla^2 \phi = 0$ Laplaceova

$\varphi(x, y, z)$



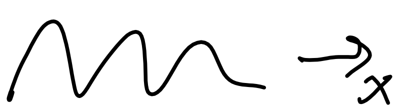
$u(x, t=0) \xrightarrow{\text{PDR}} u(x, t)$

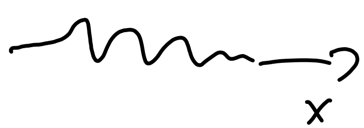


$D \nabla^2 n = \frac{\partial n}{\partial t}$ difúzní

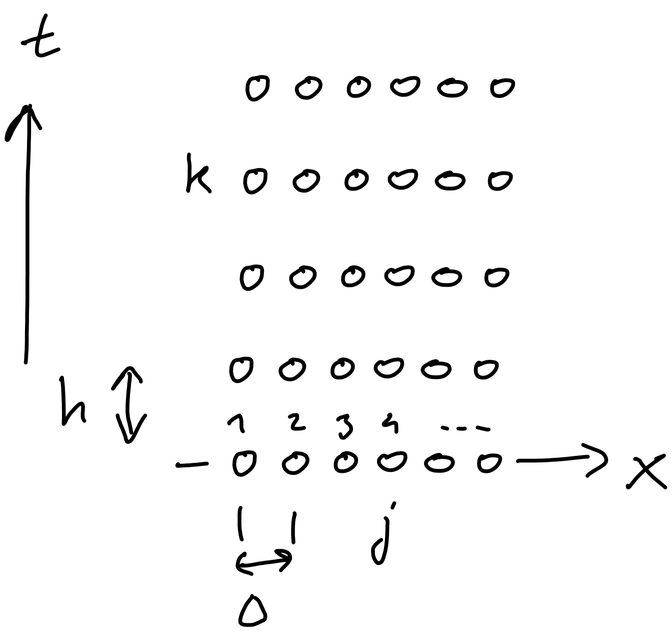
$\alpha \nabla^2 T = \frac{\partial T}{\partial t}$ r. provedení tepla

$-\frac{\hbar^2}{2m} \nabla^2 \psi + U\psi = i\hbar \frac{\partial \psi}{\partial t}$ Schr. rov.





$\nabla^2 u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$ vlnová rovnice



$u(x,t)$ $\quad t = hk \quad x = \Delta j$

diskrétní pokrytí (x,t) $\quad u_j^k$

konečné diference

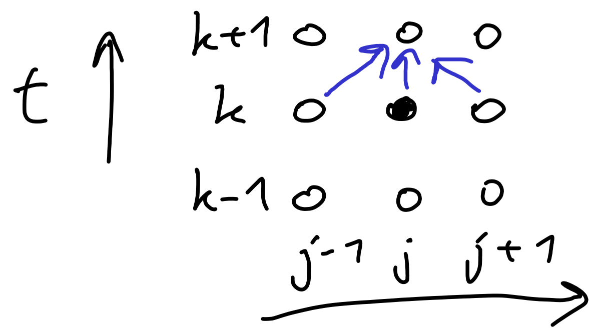
$$x=\Delta j \quad \begin{cases} \dfrac{\partial u}{\partial t} \approx \dfrac{u_j^{k+1} - u_j^k}{h} + \mathcal{O}(h) \\[2ex] \dfrac{\partial^2 u}{\partial t^2} \approx \dfrac{u_j^{k+1} - 2u_j^k + u_j^{k-1}}{h^2} + \mathcal{O}(h^2) \end{cases}$$

$$t=hk \quad \left\{ \dfrac{\partial^2 u}{\partial x^2} = \dfrac{u_{j-1}^k - 2u_j^k + u_{j+1}^k}{\Delta^2} + \mathcal{O}(\Delta^2) \right.$$

dosadíme do PDR $\longrightarrow$ soustava lin. rovnic

1) difuzní rovnice $\quad \dfrac{\partial u}{\partial t} = c\,\dfrac{\partial^2 u}{\partial x^2}$

- explicitní Eulerovo schéma



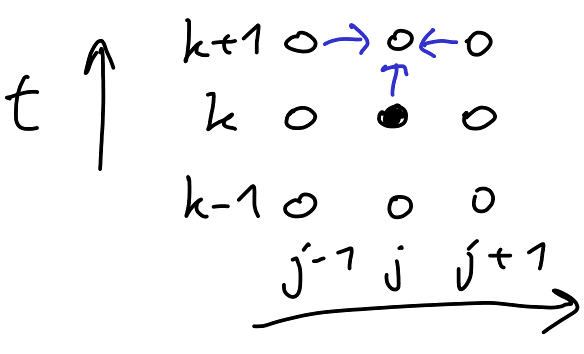
laplacián v čase kh: $\quad \dfrac{u_{j-1}^{k} - 2u_j^{k} + u_{j+1}^{k}}{\Delta^2}$

čas. derivace v čase kh v místě $j\Delta$: $\quad \dfrac{u_j^{k+1} - u_j^{k}}{h}$

$$u_j^{k+1} = u_j^{k} + \frac{hc}{\Delta^2}\left(u_{j-1}^{k} - 2u_j^{k} + u_{j+1}^{k}\right)$$

přesnost $\quad \mathcal{O}(h) + \mathcal{O}(\Delta^2) \qquad$ stabilní pro $h < \dfrac{\Delta^2}{2c}$

- implicitní Eulerovo schéma

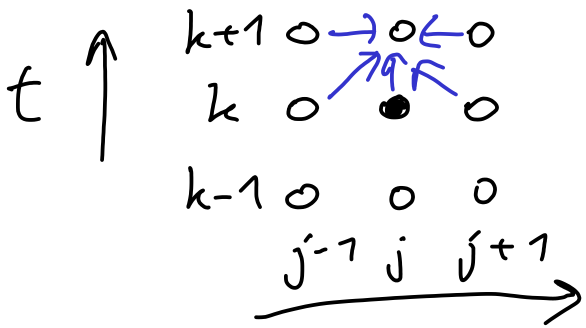


$$u_j^{k+1} = u_j^{k} + \frac{hc}{\Delta^2}\left(u_{j-1}^{k+1} - 2u_j^{k+1} + u_{j+1}^{k+1}\right)$$

soustava lin. rovnic pro čas $(k+1)h$

přesnost $\quad \mathcal{O}(h) + \mathcal{O}(\Delta^2) \qquad$ stabilní pro libov. h

- Crankovo-Nicholsonovo schéma



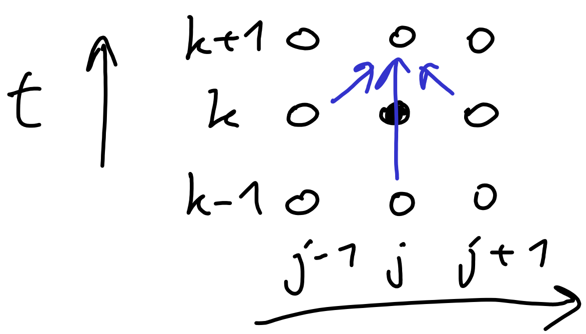
$$u_j^{k+1} = u_j^k + \frac{hc}{\Delta^2}\left(u_{j-1}^k - 2u_j^k + u_{j+1}^k\right)$$ expl. Euler

$$u_j^{k+1} = u_j^k + \frac{hc}{\Delta^2}\left(u_{j-1}^{k+1} - 2u_j^{k+1} + u_{j+1}^{k+1}\right)$$ impl. Euler

$$u_j^{k+1} = u_j^k + \frac{hc}{\Delta^2}\,\frac{1}{2}\left(u_{j-1}^k - 2u_j^k + u_{j+1}^k + u_{j-1}^{k+1} - 2u_j^{k+1} + u_{j+1}^{k+1}\right)$$ CN

přesnost $\mathcal{O}(h^2) + \mathcal{O}(\Delta^2)$ stabilní vždy

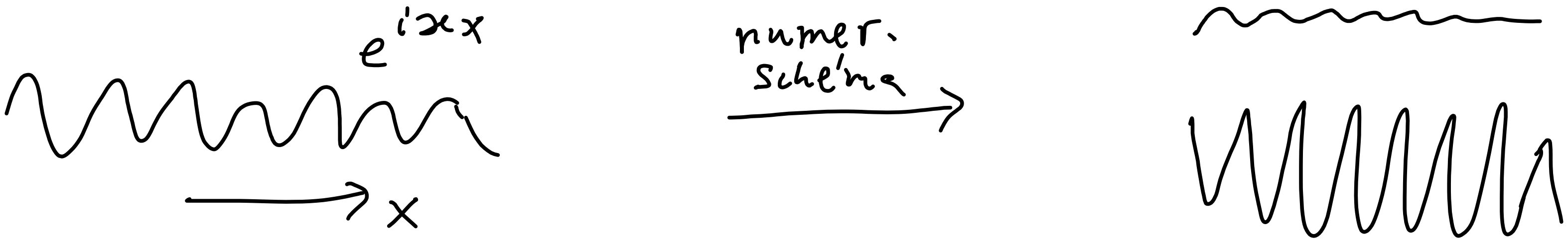
2) vlnová rovnice $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$



$$u_j^{k+1} = 2u_j^k - u_j^{k-1} + \frac{h^2c^2}{\Delta^2}\left(u_{j-1}^k - 2u_j^k + u_{j+1}^k\right)$$

přesnost $\mathcal{O}(h^2) + \mathcal{O}(\Delta^2)$ stabilní pro $h < \dfrac{\Delta}{c}$

• Vyšetřování stability podle J. von Neumanna



$|\xi| < 1$ st.

$|\xi| > 1$ nest.

$$u_j^k = \boxed{e^{i\varkappa j\Delta}\,\xi^k}$$ Ansatz dosadíme do num. sch. $\rightarrow \xi(\varkappa)$

stabilní, když $|\xi(\varkappa)| \leq 1$ pro všechna $\varkappa$ (VN kritérium stability)

Př. $$u_j^{k+1} = u_j^k + \frac{hc}{\Delta^2}\left(u_{j-1}^k - 2u_j^k + u_{j+1}^k\right)$$ expl. Euler

$$e^{i\varkappa j\Delta}\,\xi^{k+1} = e^{i\varkappa j\Delta}\,\xi^k + \frac{hc}{\Delta^2}\left(e^{i\varkappa(j-1)\Delta} - 2e^{i\varkappa j\Delta} + e^{i\varkappa(j+1)\Delta}\right)\xi^k$$

$$\xi(\varkappa) = 1 + \frac{hc}{\Delta^2}\left(e^{-i\varkappa\Delta} - 2 + e^{i\varkappa\Delta}\right) = 1 + \frac{hc}{\Delta^2}\,2\left(-1 + \cos\varkappa\Delta\right)$$

$|\xi(\varkappa)| \leq 1$ pro $\frac{2hc}{\Delta^2} \leq 1 \longrightarrow$ stabilní pro $h < \frac{\Delta^2}{2c}$

Př. $u_j^{k+1} = u_j^k + \frac{hc}{\Delta^2}\left(u_{j-1}^{k+1} - 2u_j^{k+1} + u_{j+1}^{k+1}\right)$ impl. Euler

$$e^{i\varkappa j\Delta}\xi^{k+1} = e^{i\varkappa j\Delta}\xi^k + \frac{hc}{\Delta^2}\left(e^{i\varkappa(j-1)\Delta} - 2e^{i\varkappa j\Delta} + e^{i\varkappa(j+1)\Delta}\right)\xi^{k+1}$$

$$\xi = 1 + \frac{hc}{\Delta^2}\left(e^{-i\varkappa\Delta} - 2 + e^{i\varkappa\Delta}\right)\xi = 1 + \frac{hc}{\Delta^2}2(-1+\cos\varkappa\Delta)\xi$$

$$\xi = \frac{1}{1 + \frac{2hc}{\Delta^2}(1-\cos\varkappa\Delta)} \leq 1$$ vždy stabilní

Př. $u_j^{k+1} = 2u_j^k - u_j^{k-1} + \frac{h^2c^2}{\Delta^2}\left(u_{j-1}^k - 2u_j^k + u_{j+1}^k\right)$ vlnová rovnice

$: e^{i\varkappa j\Delta}\xi^{k-1} \rightarrow \quad \xi^2 = 2\xi - 1 + \frac{h^2c^2}{\Delta^2}\xi\left(e^{-i\varkappa\Delta} - 2 + e^{i\varkappa\Delta}\right)$

$$\xi^2 - 2\underbrace{\left[1 + \frac{h^2c^2}{\Delta^2}(\cos\varkappa\Delta - 1)\right]}_{\alpha}\xi + 1 = 0$$

$\xi^2 - 2\alpha\xi + 1 = 0$

$\xi = \alpha \pm \sqrt{\alpha^2 - 1}$

$\alpha^2 < 1$ $\quad \xi_{1,2} = \alpha \pm i\sqrt{1-\alpha^2}$ stab.

$|\xi_{1,2}| = \sqrt{\alpha^2 + (1-\alpha^2)} = 1$

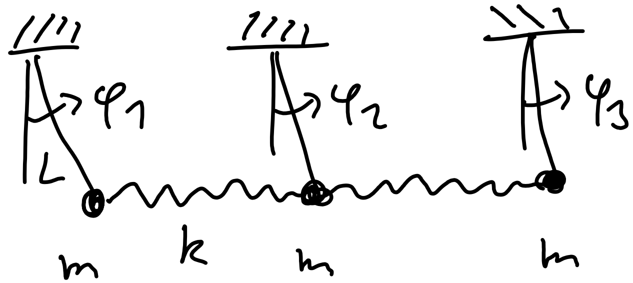
$\alpha^2 > 1$ $\quad \alpha + \sqrt{\alpha^2-1} > 1$ nestab.

nebo

$\alpha - \sqrt{\alpha^2-1} < -1$

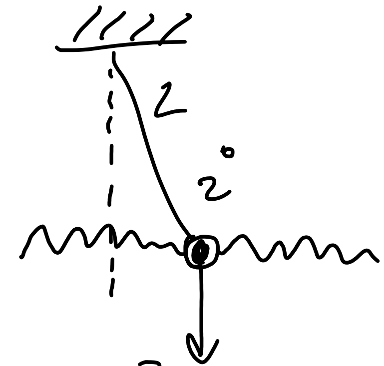
$\alpha^2 \leq 1 \rightarrow \frac{h^2c^2}{\Delta^2} \leq 1 \rightarrow h \leq \frac{\Delta}{c}$ stabilní

Cvičení



pohybové rovnice pro malé výchylky

$$J\ddot{\varphi} = M$$



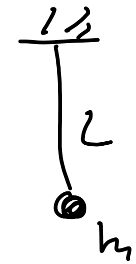
$$mL^2\ddot{\varphi}_i = -mgL\varphi_i + L\left[kL(\varphi_{i+1}-\varphi_i) - kL(\varphi_i-\varphi_{i-1})\right]$$

$$\ddot{\varphi}_i = -\frac{g}{L}\varphi_i + \frac{k}{m}\left(\varphi_{i-1} - 2\varphi_i + \varphi_{i+1}\right)$$ pohyb. rovnice pro i-té kyvadlo

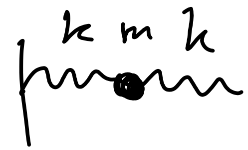
$$\varphi_i = \phi_i \cos\omega t \quad \rightarrow \quad \omega^2\phi_i = \underbrace{\frac{g}{L}}_{\omega_k^2}\phi_i + \underbrace{\frac{2k}{m}}_{\omega_p^2}\left(\phi_i - \frac{1}{2}\phi_{i-1} - \frac{1}{2}\phi_{i+1}\right)$$

$$\begin{pmatrix} \ddots & \ddots & \ddots & \\ & -\frac{1}{2}\omega_p^2 & \omega_k^2+\omega_p^2 & -\frac{1}{2}\omega_p^2 \\ & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ \phi_{i-1} \\ \phi_i \\ \phi_{i+1} \\ \vdots \end{pmatrix} = \omega^2 \begin{pmatrix} \vdots \\ \phi_{i-1} \\ \phi_i \\ \phi_{i+1} \\ \vdots \end{pmatrix}$$

$$
\begin{pmatrix}
\ddots & \ddots & \ddots & & \\
\ddots & \omega_k^2+\omega_p^2 & -\frac{\omega_p^2}{2} & 0 & \\
 & -\frac{\omega_p^2}{2} & \omega_k^2+\omega_p^2 & -\frac{\omega_p^2}{2} & \\
 & 0 & -\frac{\omega_p^2}{2} & \omega_k^2+\omega_p^2 & \ddots \\
 & & \ddots & \ddots & \ddots
\end{pmatrix}
$$



$$\omega_k = \sqrt{\frac{g}{l}}$$



$$\omega_p = \sqrt{\frac{2k}{m}}$$

k diagonalizaci