Contents

CHAPTER 1

Lie groups

1. Lie groups

Definition 1.1. A *Lie group* G is a group, which is at the same time a smooth manifold in such a way that

- the multiplication $\mu : G \times G \to G$ is smooth,
- the inverse $\nu : G \to G$ is smooth.

By a *homomorphism of Lie groups* we understand a smooth group homomorphisms.

Notation. We denote by e the unit and write a^{-1} instead of $\nu(a)$. We will be using the left and right translations $\lambda_a, \rho_a : G \to G$ defined by

$$\lambda_a(b) = ab \qquad \rho_a(b) = ba$$

Theorem 1.2. *The smoothness of the inverse follows from the smoothness of the multiplication.*

Proof. The defining equation for the inverse is $\mu(a, \nu(a)) = e$. By the implicit function theorem it is enough to verify that the derivative of $\mu(a, -)$ at a^{-1} is invertible. This follows from the fact that $\mu(a, -) = \lambda_a$ has an inverse $\lambda_{a^{-1}}$. $\square$

Remark. Every Lie group is a topological group, i.e. a group and a topological group such that the multiplication and the inverse are continuous. The fifth Hilbert problem states that every topological group G that is at the same time a (topological) manifold admits a smooth structure for which G becomes a Lie group. This was proved in 1952 (in fact the structure is even analytic). If time permits we will get to the implication $C^2 \Rightarrow C^\infty$.

Let M, N be smooth manifolds. Then the projections $p : M \times N \to M$ and $q : M \times N \to N$ provide the canonical isomorphism

$$(p_*, q_*) : T_{(x,y)}(M \times N) \xrightarrow{\cong} T_xM \times T_yN.$$

The inverse isomorphism is obtained from the inclusions

$$\begin{aligned} i_y : M &\to M \times N & \qquad j_x : N &\to M \times N \\ a &\mapsto (a, y) & b &\mapsto (x, b) \end{aligned}$$

Under the above identification the pair $(X, Y) \in T_xM \times T_yN$ corresponds to $(i_y)_*X + (j_x)_*Y \in T_{(x,y)}(M \times N)$.

Lemma 1.3. *The following formulae hold for $A, B \in T_eG$:*

$$\mu_*(A, B) = A + B, \qquad \nu_*A = -A.$$

Proof. These are just simple calculations

$$\mu_*(A, B) = \mu((i_e)_*A + (j_e)_*B) = (\mu i_e)_*A + (\mu j_e)_*B = A + B$$

and by differentiating $e = \mu(a, \nu(a))$ in the direction $A \in T_eG$ we get

$$0 = \mu_*(A, \nu_*A) = A + \nu_*A$$

$\square$

Examples 1.4. The classical groups:

- The general linear group $\mathrm{GL}(n,\mathbb{R})$ - the group of invertible matrices (a_{ij}). Since $\mathrm{GL}(n,\mathbb{R}) \subseteq \mathbb{R}^{n\times n}$ can be described as $\mathrm{GL}(n,\mathbb{R}) = \det^{-1}(\mathbb{R} - \{0\})$ it is an open subset and hence a manifold. Multiplication is clearly smooth (even algebraic).
- The general linear group $\mathrm{GL}(n,\mathbb{C})$ with coefficients in $\mathbb{C}$. We think of $\mathrm{GL}(n,\mathbb{C})$ as a subgroup of $\mathrm{GL}(2n,\mathbb{R})$ via the identification $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$. The embedding becomes

$$A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

On the other hand $\mathrm{GL}(n,\mathbb{C}) \subseteq \mathbb{C}^{n\times n}$ is again open and hence a manifold.
- The special linear groups

$$\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid \det A = 1\}$$
$$\mathrm{SL}(n,\mathbb{C}) = \{A \in \mathrm{GL}(n,\mathbb{C}) \mid \det A = 1\}$$

are certainly closed submanifolds and also subgroups. Later we will prove

Theorem. *Every closed subgroup of a Lie group is a submanifold and with the submanifold smooth structure a Lie group (i.e. a Lie subgroup).*

- Let $\beta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bilinear form represented by a matrix $B = (b_{ij})$. A linear map $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ is said to preserve β if

$$\beta(\alpha x, \alpha y) = \beta(x, y) \quad \Longleftrightarrow \quad A^T B A = B$$

Such linear automorphisms clearly form a closed subgroup of $\mathrm{GL}(n,\mathbb{R})$.
  - Specifically for $\beta = \langle\ ,\ \rangle$, the scalar product, we have $B = E$, the identity matrix and we obtain the orthogonal group

$$\mathrm{O}(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid A^T A = E\}$$

  and also the special orthogonal group

$$\mathrm{SO}(n,\mathbb{R}) = \mathrm{O}(n,\mathbb{R}) \cap \mathrm{SL}(n,\mathbb{R})$$

  - Consider on $\mathbb{R}^{2n}$ the (nondegenerate antisymmetric) bilinear form

$$\sum_{i=1}^{n} (x_i y_{n+i} - y_i x_{n+i})$$

  with its matrix $J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$. The group of linear automorphisms preserving this form is called the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$. Analogously we obtain $\mathrm{Sp}(2n,\mathbb{C})$.
- The unitary group $\mathrm{U}(n) = \{A \in \mathrm{GL}(n,\mathbb{C}) \mid \bar{A}^T A = E\}$ and the special unitary group $\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n,\mathbb{C})$. There is also a complex orthogonal group which is different from the unitary group. One of the main qualitative differences is that $\mathrm{O}(n,\mathbb{C})$ is a complex manifold and a complex Lie group (reason being that the defining equation $A^T A = E$ is holomorphic unlike that for the unitary group - it contains complex conjugation).
- The spin group $\mathrm{Spin}(n)$. We will say more about it later. It is related to $\mathrm{SO}(n,\mathbb{R})$ by a short exact sequence of groups

$$1 \to \mathbb{Z}/2 \to \mathrm{Spin}(n) \to \mathrm{SO}(n,\mathbb{R}) \to 1.$$

- $\mathrm{Sp}(n) = \{A \in \mathrm{GL}(n,\mathbb{H}) \mid \bar{A}^T A = E\}$, the group of linear automorphisms of the quaternionic space $\mathbb{H}^n$ preserving the scalar product. Also $\mathrm{Sp}(n) = \mathrm{Sp}(2n,\mathbb{C}) \cap \mathrm{U}(2n)$.

2. Lie algebras

Definition 2.1. A vector space L over $\mathbb{R}$ is called a *Lie algebra* if there is given a bilinear map $[\ ,\ ] : L \times L \to L$ satisfying

- the antisymmetry: $[X, X] = 0$,
- the Jacobi identity: $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.

From bilinearity we obtain

$$0 = [X+Y, X+Y] = [X,X] + [X,Y] + [Y,X] + [Y,Y] = [X,Y] + [Y,X]$$

implying $[Y,X] = -[X,Y]$.

Example 2.2. The vector fields on a smooth manifold M with the bracket $[X,Y]$:

$$X = \sum_i X_i \tfrac{\partial}{\partial x^i}, Y = \sum_i Y_i \tfrac{\partial}{\partial x^i} \quad \Longrightarrow \quad [X,Y] = \sum_{i,j} \left(X_j \tfrac{\partial Y_i}{\partial x^j} - Y_j \tfrac{\partial X_i}{\partial x^j} \right) \tfrac{\partial}{\partial x^i}$$

Let L be a finite dimensional Lie algebra and $e_1, \dots, e_n$ its basis. Then $[e_i, e_j] = \sum_k c_{ij}^k e_k$. The numbers c_{ij}^k are called the structure constants of L with respect to the basis. They satisfy the following identities:

- $c_{ji}^k = -c_{ij}^k$,
- $\sum_k (c_{ij}^k c_{kl}^m + c_{jl}^k c_{ki}^m + c_{li}^k c_{kj}^m) = 0$.

Conversely, by giving the basis $e_1, \dots, e_n$ and the structure constants c_{ij}^k satisfying the above equalities we obtain a Lie algebra L. The complete classification of Lie algebras is not yet known.

Example 2.3. Let V be a vector space and denote $L = \hom(V,V)$. On L we define a bracket

$$[f,g] = f \circ g - g \circ f.$$

In this way we obtatin a Lie algebra $\mathfrak{gl}(V)$.

For a Lie group G we define $\mathfrak{g} = \mathrm{Lie}(G) = T_e G$ as a vector space. Now we proceed to introduce a bracket on $\mathfrak{g}$.

Definition 2.4. A vector field $X : G \to TG$ is called *left-invariant* if $(\lambda_a)_* \circ X = X \circ \lambda_a$ for any $a \in G$.

$$\begin{tikzcd}
TG \arrow[r, "(\lambda_a)_*"] & TG \\
G \arrow[u, "X"] \arrow[r, "\lambda_a"'] & G \arrow[u, "X"']
\end{tikzcd}$$

In other words X is λ_a-related with itself which we denote by $X \sim_{\lambda_a} X$.

Remark. The f-relatedness of vector fields X and Y has the following characterization via the flow lines, easily verified by differentiating both sides.

$$f(\mathrm{Fl}_t^X(x)) = \mathrm{Fl}_t^Y(f(x))$$

In other words f transfers the flow lines of X into the flow lines of Y. We will use this property quite often.

Remark. Let $A \in T_e G$ be an arbitrary vector. It defines a vector field $\lambda_A : G \to TG$ by the formula $\lambda_A(a) = (\lambda_a)_* A$. This vector field is clearly left-invariant as

$$\lambda_A(ab) = (\lambda_{ab})_* A = (\lambda_a \lambda_b)_* A = (\lambda_a)_*((\lambda_b)_* A) = (\lambda_a)_*(\lambda_A(b))$$

It remains to verify its smoothness. Since $(\lambda_a)_* A = \mu_*(0_a, A)$ this is achieved by the following diagram

$$\begin{tikzcd}
TG \times TG \arrow[r, "T\mu"] & TG \\
G \arrow[u, "{(0,A)}"] \arrow[ur, bend right, "\lambda_A"']
\end{tikzcd}$$

with $(0,A)$ being the map with components the zero section 0 and the constant map sending everything onto A.

Theorem 2.5. *Let X, Y be left invariant vector fields. Then $X+Y$, kX and $[X,Y]$ are again left-invariant.*

PROOF. Since X and Y are λ_a related with X and Y respectively, the same is true for their sum, multiples and bracket. $\square$

DEFINITION 2.6. The vector space $\mathfrak{g} = \mathrm{Lie}(G) = T_eG$ together with the bracket $[A, B] = [\lambda_A, \lambda_B]_e$ is called the *Lie algebra* of the Lie group G.

REMARK. For every finite dimensional Lie algebra $\mathfrak{g}$ there exists a Lie group G for which $\mathrm{Lie}(G) = \mathfrak{g}$.

We would like to explain now why this is a reasonable object of study. We have seen that the first derivative at e does not see anything from the structure of the Lie group. The second derivative does but in order to make sense of the second derivative one has to fix the coordinate charts (which we will do later and for them the second derivative will be described exactly by the Lie bracket). Without a fixed choice of the charts the second derivative only makes sense when the first derivative vanishes at that point which is not the case for the product. The way out is to "subtract from μ the sum of the two coordinates" by considering

$$\begin{aligned}[\ ,\ ] : G \times G &\longrightarrow G \\ (a, b) &\mapsto aba^{-1}b^{-1}\end{aligned}$$

We will see shortly that the first derivative of the commutator vanishes at e and the essential part of the second derivative is exactly the Lie bracket.

NOTATION. Let X, Y be two vector fields on a manifold M. Then we denote

$$(\mathrm{Fl}^X_t)^* Y(x) = (\mathrm{Fl}^X_{-t})_* Y(\mathrm{Fl}^X_t(x)) \in T_xM$$

the pullback of Y along the flow Fl^X_t of X. For each $x \in M$ it is defined for t small.

LEMMA 2.7. $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_0} (\mathrm{Fl}^X_t)^* Y(x) = (\mathrm{Fl}^X_{t_0})^*[X, Y](x)$.

PROOF. First assume that $t_0 = 0$ and let $f : M \to \mathbb{R}$ be a smooth function. We differentiate f in the direction of the left hand side:

$$\begin{aligned}\Big(\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} (\mathrm{Fl}^X_t)^* Y(x)\Big) f &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \Big((\mathrm{Fl}^X_t)^* Y(x) f\Big) \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \Big((\mathrm{Fl}^X_{-t})_* Y(\mathrm{Fl}^X_t(x)) f\Big) \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \Big(Y(\mathrm{Fl}^X_t(x))(f \circ \mathrm{Fl}^X_{-t})\Big) \\ &= Y(x)(-Xf) + \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} (Yf)(\mathrm{Fl}^X_t(x)) \\ &= -(YXf)(x) + (XYf)(x) = \Big([X, Y](x)\Big) f\end{aligned}$$

For a general t_0 we have $(\mathrm{Fl}^X_t)^* Y(x) = (\mathrm{Fl}^X_{t_0})^*(\mathrm{Fl}^X_{t-t_0})^* Y(x)$. Since $(\mathrm{Fl}^X_{t_0})^*$ is a linear map we can interchange with $\frac{\mathrm{d}}{\mathrm{d}t}$. $\square$

COROLLARY 2.8. *The following conditions are equivalent:*

- $[X, Y] = 0$,
- $(\mathrm{Fl}^X_t)^* Y = Y$, *i.e.* Y *is* Fl^X_t*-related with itself for all* t,
- $\mathrm{Fl}^X_t \mathrm{Fl}^Y_s(x) = \mathrm{Fl}^Y_s \mathrm{Fl}^X_t(x)$, *i.e. the flow lines commute.*

In general we have $\mathrm{Fl}^Y_{-s} \mathrm{Fl}^X_{-t} \mathrm{Fl}^Y_s \mathrm{Fl}^X_t(x) = x + st[X, Y](x) + o(s, t)^2$.

PROOF. Differentiating twice we get

$$\tfrac{\partial}{\partial t}\big|_{t=0} \tfrac{\partial}{\partial s}\big|_{s=0} \mathrm{Fl}^Y_{-s} \mathrm{Fl}^X_{-t} \mathrm{Fl}^Y_s \mathrm{Fl}^X_t(x) = \tfrac{\partial}{\partial t}\big|_{t=0} \Big(-Y(x) + (\mathrm{Fl}^X_t)^* Y(x)\Big) = [X, Y](x)$$

The remaining derivatives of order at most two are clearly zero. $\square$

EXAMPLE 2.9. Let $M = G$, a Lie group. What does $[A, B]$ for $A, B \in \mathfrak{g}$ express? Let us consider the following integral curves

- $\varphi(t)$ the flow line of λ_A with $\varphi(0) = e$,

- $\psi(t)$ the flow line of λ_B with $\psi(0) = e$.

A flow line of λ_A through a general $a \in G$ is easily $a \cdot \varphi : t \mapsto a\varphi(t)$. This follows from the λ_a-relatedness of λ_A with itself:

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(a\varphi(t)) = (\lambda_a)_* \tfrac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = (\lambda_a)_*\lambda_A(\varphi(t)) = \lambda_A(a\varphi(t))$$

In other words $\mathrm{Fl}_t^{\lambda_A}(a) = a\varphi(t)$. This implies that $\varphi(t_1 + t_2) = \varphi(t_1)\varphi(t_2)$ and it is a homomorphism of groups. We now compute

$$\mathrm{Fl}_{-s}^{\lambda_B} \mathrm{Fl}_{-t}^{\lambda_A} \mathrm{Fl}_s^{\lambda_B} \mathrm{Fl}_t^{\lambda_A}(e) = \varphi(t)\psi(s)\varphi(-t)\psi(-s) = \varphi(t)\psi(s)\varphi(t)^{-1}\psi(s)^{-1}.$$

In other words the group theoretic commutator $[\varphi(t), \psi(s)]$ has a Taylor polynomial

$$[\varphi(t), \psi(s)] = e + [A, B]st + o(s, t)^2$$

This can also be rewritten as $\mathrm{d}^2[\ ,\ ]_{(e,e)}((A, 0), (0, B)) = [A, B]$. The Lie bracket thus measures the non-commutativity of the Lie group. More precisely $[A, B] = 0$ if and only if all the elements $\varphi(t)$ commute with all $\psi(s)$. We will see later that the connection between commutativity of G and vanishing of the bracket works perfectly for connected Lie groups.

Definition 2.10. Let L, L' be two Lie algebras. A linear map $\varphi : L \to L'$ is called a *homomorphism of Lie algebras* if $\varphi[A, B]_L = [\varphi A, \varphi B]_{L'}$.

Theorem 2.11. *Let $f : G \to H$ be a (smooth) homomorphism of Lie groups. Then its derivative $f_* : \mathfrak{g} \to \mathfrak{h}$ at e is a homomorhpism of Lie algebras.*

Proof. Let us rewrite $f(ab) = f(a)f(b)$ using the left translations as

$$f \circ \lambda_a = \lambda_{f(a)} \circ f$$

Differentiating in the direction $A \in \mathfrak{g}$ we obtain $f_*(\lambda_a)_*A = (\lambda_{f(a)})_*f_*A$ or

$$f_*\lambda_A(a) = \lambda_{f_*A}(f(a))$$

which means that λ_A is f-related to λ_{f_*A}. Since the bracket respects relatedness, $[\lambda_A, \lambda_B]$ must be f-related to $[\lambda_{f_*A}, \lambda_{f_*B}]$. Evaluating at e yields the result. □

Definition 2.12. A smooth map $f : G \to H$ between Lie groups is a *local isomorphism* if it is both a homomorphism and a local diffeomorphism at e (i.e. the derivative $f_{*e} : \mathfrak{g} \to \mathfrak{h}$ is an isomorphism).

Two Lie groups G, H are called *locally isomorphic* if there exist neighbourhoods $U \ni e$ and $V \ni e$, in G and H respectively, together with a diffeomorphism $f : U \to V$ which satisfies:

- $f(ab) = f(a)f(b)$ whenever $a, b, ab \in U$,
- $f^{-1}(ab) = f^{-1}(a)f^{-1}(b)$ whenever $a, b, ab \in V$.

Clearly if there exists a local isomorphism $f : G \to H$ then G and H are locally isomorphic.

Theorem 2.13. *Locally isomorphic groups have isomorphic Lie algebras.* □

Example 2.14. The additive groups $\mathbb{R}$ and $\mathbb{T} = \mathrm{SU}(1)$ (the group of complex units in $\mathbb{C}$) are locally isomorphic. We think of the first as the group of translations of the line while the second is the group of rotations of the circle (or $\mathbb{C}$ for that matter). This is because there exists a local isomorphism $\mathbb{R} \to \mathbb{T}$ sending $t \mapsto e^{2\pi it}$.

Definition 2.15. Let L, L' be Lie algebras. On their product $L \times L'$ we consider the bracket

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2]_L, [Y_1, Y_2]_{L'}).$$

We call $L \times L'$ together with this bracket the *product of Lie algebras* L and L'.

Theorem 2.16. $\mathrm{Lie}(G \times H) \cong \mathrm{Lie}(G) \times \mathrm{Lie}(H)$.

Proof. The projections $p : G \times H \to G$ and $q : G \times H \to H$ are homomorphisms and hence they induce homomorphisms of the Lie algebras in question. This means

$$p_*[(X_1, Y_1), (X_2, Y_2)] = [p_*(X_1, Y_1), p_*(X_2, Y_2)] = [X_1, X_2]$$

and similarly for q. The canonical isomorphism $(p_*, q_*) : \mathrm{Lie}(G\times H) \to \mathfrak{g}\times\mathfrak{h}$ is then an isomorphism of Lie algebras. □

Remark. With the above Lie algebra structure $L \times L'$ forms a product in the category of Lie algebras. The previous proof is then just a demonstration of the fact that Lie is a functor and preserve products (which is obvious from the fact that this happens already at the level of tangent vector spaces at e).

What happens if we change sides? Denoting ρ_A the right-invariant vector field with value A at e the next theorem asserts that the Lie bracket defined via the right-invariant vector fields agrees with the usual one up to the minus sign.

Theorem 2.17. *For $A, B \in \mathfrak{g}$ the following holds: $[\rho_A, \rho_B]_e = -[\lambda_A, \lambda_B]_e$.*

Proof. Consider the opposite group G^* with multiplication $a * b = ba$. The inverse $\nu : G^* \to G$ is a group homomorphism and

$$[A, B]^* = [\lambda_A^*, \lambda_B^*]_e = [\rho_A, \rho_B]_e$$

Thus $-[\rho_A, \rho_B]_e = \nu_*[A, B]^* = [\nu_* A, \nu_* B] = [-A, -B] = [\lambda_A, \lambda_B]_e$. □

Corollary 2.18. *For a commutative group G the bracket on its Lie algebra is identically zero.*

3. Subgroups and subalgebras

Definition 3.1. A *Lie subalgebra* $L' \subseteq L$ is a vector subspace closed under $[\ ,\ ]$.

Theorem 3.2. *If $H \subseteq G$ is both a submanifold and a subgroup then $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra.*

Proof. In the diagram

$$\begin{tikzcd} H \times H \arrow[r, dashed, "\mu"] \arrow[d, hook] & H \arrow[d, hook, "\iota"] \\ G \times G \arrow[r, "\mu"'] & G \end{tikzcd}$$

the map μ (which exists since H is a subgroup) is smooth since H is a submanifold. Hence H is a Lie group and the inclusion $\iota : H \to G$ is a homomorphism. Thus its derivative $\iota_* : \mathfrak{h} \to \mathfrak{g}$ is a homomorphism of Lie algebras (saying that the bracket of $\mathfrak{h}$ is a restriction of the bracket on $\mathfrak{g}$) and its image is therefore a subalgebra. □

Example 3.3. Consider $\mathbb{R}^2$. Then every line $\{(x, kx) \mid x \in \mathbb{R}\}$ (for $k \in \mathbb{R}$) is a subgroup (and a submanifold). Now consider the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Again we get subgroups for any $k \in \mathbb{R}$. For $k \in \mathbb{Q}$ this subgroup is a submanifold but not for irrational k when this subgroup is dense.

Definition 3.4. A subset $A \subseteq M$ of a smooth manifold M is called an *initial submanifold* (of dimension k) if for each $x \in A$ there exists a chart

$$\varphi : U \xrightarrow{\cong} \mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^{m-k}$$

such that $\varphi^{-1}(\mathbb{R}^k \times \{0\})$ is exactly the path component of $U \cap A$ containing x.

Theorem 3.5. *Every initial submanifold is the image of an (essentially unique) injective immersion i satisfying the following universal property:*

$$\begin{tikzcd} & A \arrow[d, hook, "i"] \\ N \arrow[ur, dashed, "g"] \arrow[r, "f"'] & M \end{tikzcd}$$

For every smooth map $f : N \to M$ with the property $f(N) \subseteq i(A)$ the unique map $g : N \to A$ satisfying $ig = f$ is also smooth.

Proof. Let $\varphi : U \longrightarrow \mathbb{R}^m$ be a chart on N from the definition of an initial submanifold. Declare its restriction

$$C_x(U \cap A) \xrightarrow{\cong} \mathbb{R}^k \times \{0\}$$

to the path component of $U \cap A$ containing x to be a chart for A. This does endow A with a smooth structure. It differs from the subspace topology (which is inevitable) but the inclusion is clearly an injective immersion.

We verify the universal property for inclusions of initial submanifolds. Let $y \in N$ with $f(y) = x$ and V a path connected neighbourhood of y which maps into U. Since its image is also path connected it must be contained in $U \cap A$. Thus g in the chart provided by ψ is just a restriction of f and hence smooth.

Suppose now that $i' : A' \hookrightarrow M$ is another injective immersion with the same image as i. Then there exists a factorization

$$\begin{tikzcd} A' \arrow[rr, "h"] \arrow[dr, hook, "i'"'] & & A \arrow[dl, hook', "i"] \\ & M & \end{tikzcd}$$

with h an immerison and a bijection at the same time. Since its inverse is also an immersion by the same argument h must be in fact a diffeomorphism. □

Remark. It is also true that any injective immersion i satisfying the above universal property is in fact an inclusion of an initial submanifold but we will not need this fact.

Remark. We have not proved that A has a countable basis for its topology. In fact A might well have an uncountable number of components. However each of the components of A is second countable.

Definition 3.6. A *Lie subgroup* $H \subseteq G$ is an initial submanifold which is at the same time a subgroup.

Theorem 3.7. *A Lie subgroup $H \subseteq G$ with its canonical smooth structure (and multiplication) is a Lie group. Moreover $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra.*

Proof. The whole proof is contained in the diagram

$$\begin{tikzcd} H \times H \arrow[r, dashed, "\mu"] \arrow[d, hook] & H \arrow[d, hook, "\iota"] \\ G \times G \arrow[r, "\mu"'] & G \end{tikzcd}$$

□

Our new definition includes the wild subgroups of the torus $\mathbb{T}^2$. In fact we are able to construct a Lie subgroup for any Lie subalgebra of $\mathfrak{g}$. To motivate our construction observe that for a Lie subgroup $H \subseteq G$ and $a \in H$ we have $T_a H = (\lambda_a)_* \mathfrak{h}$ and H is an integral submanifold of the left invariant distribution determined by $\mathfrak{h}$.

More generally for a linear subspace $P \subseteq \mathfrak{g}$ of dimension k the left translations $(\lambda_a)_* P =: \lambda_P(a) \subseteq T_a G$ form a k-dimensional distribution λ_P on G. This distribution is smooth: if $A_1, \ldots, A_k$ is a basis of P then $\lambda_{A_1}(a), \ldots, \lambda_{A_k}(a)$ is a basis of $\lambda_P(a)$.

A distribution $\mathcal{S}$ on M is called involutive if for every two vector fields $X, Y \in \mathcal{S}$ their bracket $[X, Y]$ also lies in $\mathcal{S}$.

Theorem 3.8 (Frobenius theorem). *If $\mathcal{S}$ is involutive then for every $x \in M$ there exists a local coordinate system $y^1, \ldots, y^m$ in a neighbourhood U of x such that the vector fields $\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^k}$ form a basis of the distribution $\mathcal{S}$ on U. In particular $\mathcal{S}$ is integrable.*

PROOF. Let $X_1, \dots, X_k$ be local vector fields which, near x, span the distribution $\mathcal{S}$ and let us choose a coordinate system around x in which $X_i(x) = \frac{\partial}{\partial x^i}$. We then define a map

$$\begin{aligned} \varphi : \mathbb{R}^m \supseteq U &\longrightarrow M \\ (t^1, \dots, t^m) &\longmapsto \mathrm{Fl}_{t^1}^{X_1} \cdots \mathrm{Fl}_{t^k}^{X_k}(0, \dots, 0, t^{k+1}, \dots, t^m) \end{aligned}$$

The partial derivatives at the origin clearly consist of the vectors $\frac{\partial}{\partial x^i}$ and thus φ is a local diffeomorphism.

Let us compute the partial derivative with respect to t^i for $i \le k$ at a general point.

$$\frac{\partial \varphi}{\partial t^i} = \left(\mathrm{Fl}_{t^1}^{X_1}\right)_* \cdots \left(\mathrm{Fl}_{t^{i-1}}^{X_{i-1}}\right)_* X_i \left(\mathrm{Fl}_{t^{i+1}}^{X_{i+1}} \cdots \mathrm{Fl}_{t^m}^{X_m}(x)\right)$$

To conclude the proof it is therefore enough to show that for any Y belonging to $\mathcal{S}$ the pullbacks $(\mathrm{Fl}_t^Y)^* X_i$ also belong to $\mathcal{S}$. Denote this pullback by

$$Y_i(t) = (\mathrm{Fl}_t^Y)^* X_i(x)$$

and write $[Y, X_i] = \sum a_{ij} X_j$. By Lemma 2.7 the paths $Y_i(t)$ satisfy the following system of differential equations

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Y_i(t) = (\mathrm{Fl}_t^Y)^*[Y, X_i] = \sum a_{ij}(\mathrm{Fl}_t^Y(x)) Y_j(t)$$

We have $Y_i(0) = X_i(x) \in \mathcal{S}(x)$ and since the system is linear we must have $Y_i(t) \in \mathcal{S}(x)$ for all t. Namely applying any linear form α to this system we see that $\alpha(Y_i(t))$ satisfy the very same linear system of differential equations. Using the uniqueness and the existence of the zero solution we see that $\alpha(Y_i(0)) = 0$ for all i implies $\alpha(Y_i(t)) = 0$ for all i and t. $\square$

By an integral submanifold we will now understand a connected initial submanifold $A \subseteq M$ for which $T_x A = \mathcal{S}_x$ for all $x \in A$. A maximal integral submanifold is one that is not contained in any bigger.

THEOREM 3.9. *If $\mathcal{S}$ is involutive then to every point $x \in M$ there exists a unique maximal integral submanifold going through that point.*

PROOF. We will obtain this initial submanifold as the set A of all points $y \in M$ which can be joined with x by a path $\gamma : I \to M$ tangent to the distribution $\mathcal{S}$, i.e. with the properties

- $\gamma(0) = x$, $\gamma(1) = y$,
- $\dot\gamma = \frac{\mathrm{d}}{\mathrm{d}t}\gamma \in \mathcal{S}$.

We need to verify that A is indeed an initial submanifold, maximality should be obvious. In a coordinate chart $\varphi_j : U_j \to \mathbb{R}^m$ from the Frobenius theorem $U_j \cap A$ is clearly the disjoint union

$$\bigsqcup_{(c_{k+1}, \dots, c_m) \in C_j} \mathbb{R}^k \times \{(c_{k+1}, \dots, c_m)\}$$

It is enough to show that each C_j is at most countable since every countable subset of $\mathbb{R}^{m-k}$ is totally disconnected (in between any two distinct x, y in a countable set $X \subseteq \mathbb{R}$ there lies some $z \notin X$). First we prove an auxiliary fact:

Let B be an integral submanifold which is second countable. Then B intersects each U_j in at most a countable number of leaves $\mathbb{R}^k \times \{(c_{k+1}, \dots, c_m)\}$: if, by contradiction, the number was uncountable then choosing a point from B in each leaf we would find an uncountable discrete subset of B.

In particular every leaf of φ_j intersects at most countable number of leaves of φ_k. Now start with $A_0 = \{x\}$ and at each step "leaf complete" A_i to obtain A_{i+1}. Then $A = \bigcup A_i$ and it is second countable, hence intersects only a countable number of leaves of each φ_j. $\square$

Let us return to a linear subspace $P \subseteq \mathfrak{g}$ and the distribution λ_P on G.

LEMMA 3.10. *λ_P is involutive if and only if P is a Lie subalgebra.*

PROOF. Since $[X, fY + gZ] = f[X, Y] + (Xf)Y + g[X, Z] + (Xg)Z$ it is enough to check the brackets of vector fields of the form λ_A with $A \in P$. But $[\lambda_A, \lambda_B] = \lambda_{[A,B]}$. $\square$

THEOREM 3.11. *Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Lie subalgebra. Then the maximal integral submanifold H passing through e is a Lie subgroup.*

PROOF. Let $a \in H$. Since $(\lambda_{a^{-1}})_* \lambda_{\mathfrak{h}} = \lambda_{\mathfrak{h}}$, the map $\lambda_{a^{-1}}$ preserves integral submanifolds. As $\lambda_{a^{-1}}(a) = e$ and both $a, e \in H$ we must have $\lambda_{a^{-1}}(H) = H$ and thus $a^{-1}b \in H$ for all $a, b \in H$. □

Now we tackle the uniqueness issue. First a lemma.

LEMMA 3.12. *Let $f : G \to H$ be a homomorphism of Lie groups whose derivative at identity is surjective. Then the image of f is a union of components of H.*

PROOF. The image is certainly a subgroup which is open. Since any open subgroup is necessarily also closed (its complement being a union of cosets which are open) the assertion follows. □

REMARK. Later we will use a simple variation of this lemma: Let U be a connected neighbourhood of e in a Lie group G. Then the subgroup generated by U is exactly the connected component G_e of G containing e. Here G_e is a subgroup since the pointwise product of a path from e to a and a path from e to b is a path from e to ab.

THEOREM 3.13. *Let $H \subseteq G$ be a connected Lie subgroup. Then H is the maximal integral submanifold of $\lambda_{\mathfrak{h}}$. In particular two connected Lie subgroups are equal if and only if they have the same Lie algebra.*

PROOF. Let H_0 be the maximal integral submanifold of $\lambda_{\mathfrak{h}}$ passing through e. Since both H is also an integral submanifold it must be contained in H_0 and the inclusion $H \hookrightarrow H_0$ is both injective and surjective by the previous lemma (the derivative at e is the identity on $\mathfrak{h}$) and thus $H = H_0$. □

4. Homomorphisms of Lie groups and algebras

LEMMA 4.1. *A group homomorphism $f : G \to H$ which is smooth near e is smooth everywhere.*

PROOF. This is a classical homogeneity argument. Denoting by U the neighbourhood of e where f is smooth pick any $a \in G$ and consider the diagram

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\lambda_a"'] & H \arrow[d, "\lambda_{f(a)}"] \\ aU \arrow[r, "f"'] & H \end{tikzcd}$$

in which aU is a neighbourhood of a and thus f is smooth everywhere. □

The essential idea of this section is to construct homomorphisms through their graphs. Let us consider $\varphi : \mathfrak{g} \to \mathfrak{h}$, a linear map between Lie algebras. The graph of φ is the subset $\mathrm{Graph}(\varphi) = \{(A, \varphi(A)) \mid A \in \mathfrak{g}\}$.

LEMMA 4.2. $\mathrm{Graph}(\varphi)$ *is a Lie algebra if and only if ϕ is a homomorphism of Lie algebras.*

PROOF. By the definition of the bracket in the product

$$[(A, \varphi(A)), (B, \varphi(B))] = ([A, B], [\varphi(A), \varphi(B)])$$

which lies in $\mathrm{Graph}(\varphi)$ if and only if $[\varphi(A), \varphi(B)] = \varphi[A, B]$. □

Let $\varphi : \mathfrak{g} \to \mathfrak{h}$ be now a homomorphism of Lie algebras, $\mathrm{Graph}(\varphi) \subseteq \mathfrak{g} \times \mathfrak{h}$ its graph, a Lie subalgebra. There exists a unique connected Lie subgroup $F \subseteq G \times H$ with $\mathrm{Lie}(F) = \mathrm{Graph}(\varphi)$. Assuming that the composition $F \hookrightarrow G \times H \to G$ is a diffeomorphism F will be a graph of a homomorphism $f : G \to H$ with $f_* = \varphi$. In general however this projection is only a local diffeomorphism: its derivative at e is the isomorphism $\mathrm{Graph}(\varphi) \to \mathfrak{g}$ and at other points this follows from the diagram

$$\begin{tikzcd} T_eF \arrow[r, "f_*", "\cong"'] \arrow[d, "(\lambda_a)_*"', "\cong"] & T_eG \arrow[d, "\cong"', "(\lambda_{f(a)})_*"] \\ T_aF \arrow[r, "f_*"'] & T_{f(a)}G \end{tikzcd}$$

DEFINITION 4.3. A continuous map $f : X \to Y$ is a *covering* if for each $y \in Y$ there exists its neighbourhood U such that

$$\begin{tikzcd} f^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "f"'] & & \bigsqcup_{c\in C} U \arrow[dl, "\sum_{c\in C}\mathrm{id}"] \\ & U & \end{tikzcd}$$

LEMMA 4.4. *Every local isomorphism of Lie groups is a covering.*

PROOF. Let $f : G \to H$ be the local isomorphism, $U \ni a$, $V \ni b$ open neighbourhoods for which $f|_U : U \xrightarrow{\cong} V$ with inverse g. Then we will show that

$$f^{-1}(V) = \bigsqcup_{k\in\ker f} k \cdot U$$

Therefore let $x \in f^{-1}(V)$. Then $x = (x \cdot g(f(x))^{-1}) \cdot g(f(x))$ is the decomposition. Also $kx = k'x'$ implies that $x(x')^{-1} = k^{-1}k' \in \ker f$ and thus $f(x) = f(x')$. Since f in injective on U, $x = x'$ and necessarily $k = k'$.

The proof is finished by recalling that the image of f is a union of components (so that for any b the a above exists). $\square$

THEOREM 4.5. *Let X be a path connected and locally simply connected topological space. Then X is simply connected if and only if every connected covering of X is a global homeomorphism.*

Before going into the proof we draw a corollary:

THEOREM 4.6. *Let G be a simply connected Lie group, H any Lie group. Then for every homomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$ of Lie algebras there exists a unique homomorphism of Lie groups $f : G \to H$ with the property $f_* = \varphi$. For connected G the uniqueness part is still valid.*

PROOF. The above constructed homomorphism $F \to G$ is a covering and according to the previous theorem a diffeomorphism. Thus F is the graph of f. $\square$

COROLLARY 4.7. *Two simply connected Lie groups G and H are isomorphic if and only if their Lie algebras are isomorphic.* $\square$

The assumption of simple connectivity is essential: the canonical projection map $\mathbb{R} \to \mathbb{R}/\mathbb{Z} = \mathbb{T}$ is a homomorphism but there is no non-trivial homomorphism in the opposite direction despite the fact $\mathrm{Lie}\,\mathbb{R} = \mathrm{Lie}\,\mathbb{T}$.

PROOF OF THEOREM 4.5. Let us construct the universal covering of X. Set

$$\tilde{X} = \{[\gamma] \mid \gamma : (I, 0) \to (X, x)\}$$

where $[\gamma]$ denotes the class with respect to homotopies preserving *both* endpoints. The projection $p : \tilde{X} \to X$ sends $[\gamma] \mapsto \gamma(1)$. Then clearly

- $p^{-1}(x) \cong \pi_1(X, x)$.
- p is a covering: Let U be a simply connected neighbourhood of x'. Then

$$p^{-1}(U) \cong \bigsqcup_{\substack{[\gamma] \\ \gamma(0)=x \\ \gamma(1)=x'}} [\gamma] * \underbrace{\{[\delta] \mid \delta : (I, 0) \to (U, y)\}}_{\substack{\text{in bijection with } U \text{ by} \\ \text{simple connectivity}}}$$

This bijection defines a topology on $\tilde{X}$ for which p is a covering. Therefore $\tilde{X}$ is a smooth manifold if X was to start with (again we leave out the proof that $\tilde{X}$ is second countable).

REMARK. We have shown that $\pi_1(X,x)$ is at most countable since $p^{-1}(x)$ is discrete and X second countable.

- p is universal: let $q : Y \to X$ be a covering with connected Y and let $y \in q^{-1}(x)$. Then there exists a unique $f : \tilde{X} \to Y$ satisfying $qf = p$ and $f(\tilde{x}) = y$ where $\tilde{x} = [x] \in \tilde{X}$ is the class of the constant path

$$\begin{tikzcd} (\tilde{X},\tilde{x}) \arrow[rr, dashed, "f", "\exists!"'] \arrow[dr, "p"'] & & (Y,y) \arrow[dl, "q"] \\ & (X,x) & \end{tikzcd}$$

  This is about the path lifting property: the path $\gamma : (I,0) \to (X,x)$ has a unique continuous lift to $(\tilde{X},\tilde{x})$, namely $t \mapsto [\gamma|_{[0,t]}]$. Denote the unique lift to (Y,y) by $\tilde{\gamma}$. Since the lifts must be preserved f must send $[\gamma] \mapsto \tilde{\gamma}(1)$.
- If $\pi_1(X,x) = \{e\}$ then $\tilde{X} \to X$ is a homeomorphism: it is a local homeomorphism from the definition of a covering and surjective from the path connectedness of X. We will prove injectivity. Let $p[\gamma] = p[\delta]$, i.e.

$$\gamma, \delta : (I,0,1) \to (X,x,x')$$

  The concatenation $\gamma * \delta^{-1}$ is a loop in X, hence contractible to a point which gives $[\gamma] = [\delta]$.

$\square$

5. The exponential map

DEFINITION 5.1. A *one-parameter subgroup* in G is a homomorphism $\gamma : \mathbb{R} \to G$.

THEOREM 5.2. *For every $A \in \mathfrak{g}$ there exists a unique one-parameter subgroup $\gamma_A : \mathbb{R} \to G$ such that $\dot{\gamma}_A(0) = A$.*

PROOF. $\mathbb{R}$ is simply connected and $\operatorname{Lie}\mathbb{R} = \mathbb{R}$ with the trivial bracket and thus a homomorphism $\mathbb{R} \to \mathfrak{g}$ of Lie algebras is the same thing as a linear map. $\square$

The one-parameter subgroup γ_A is an integral curve of λ_A and more generally for every $a \in G$ the curve $t \mapsto a \cdot \gamma_A(t)$ is:

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} a\gamma_A(t) = \tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} a\gamma_A(t_0)\gamma_A(t-t_0) = (\lambda_{a\gamma_A(t_0)})_* A = \lambda_A(a \cdot \gamma_A(t_0))$$

THEOREM 5.3. *The flow of the left-invariant vector field λ_A is*

$$\mathrm{Fl}_t^{\lambda_A}(a) = a\gamma_A(t) = \rho_{\gamma_A(t)}(a)$$

Moreover λ_A is complete (the integral curves are defined for all $t \in \mathbb{R}$).

DEFINITION 5.4. The map $\exp : \mathfrak{g} \to G$ sending $A \mapsto \gamma_A(1)$ is called the *exponential map* of the Lie group G.

EXAMPLE 5.5. For $G = (\mathbb{R}^+, \cdot)$ the associated Lie algebra is $\operatorname{Lie} G = \mathbb{R}$, the left-invariant vector field $\lambda_A(a) = (\lambda_a)_* A = aA$. The equation for the flow is

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\gamma_A = \gamma_A A$$

and its solution is clearly $\gamma_A(t) = e^{tA}$. Hence $\exp(A) = e^A$.

EXAMPLE 5.6. More generally for $G = \mathrm{GL}(n,\mathbb{R})$ the exponential map is

$$\exp : \mathfrak{gl}(n,\mathbb{R}) \longrightarrow \mathrm{GL}(n,\mathbb{R})$$

$$A \longmapsto e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

THEOREM 5.7. *It holds* $\exp(tA) = \gamma_A(t)$.

PROOF. $\gamma_A(t) = \mathrm{Fl}^{\lambda_A}_{t\cdot 1}(e) = \mathrm{Fl}^{t\cdot\lambda_A}_1(e) = \mathrm{Fl}^{\lambda_{tA}}_1(e) = \exp(tA)$. □

THEOREM 5.8. *The map* $\exp : \mathfrak{g} \to G$ *is smooth and a diffeomorphism on a neighbourhood of* 0.

PROOF. The vector field λ_A depends smoothly on A and thus also exp. We compute the derivative of exp by considering a curve $t \mapsto tA$ in $\mathfrak{g}$. Its image under exp is $t \mapsto \exp(tA) = \gamma_A(t)$ whose derivative at 0 is $\dot\gamma_A(0) = A$. We conclude that $\exp_* = \mathrm{id} : \mathfrak{g} \to \mathfrak{g}$. □

THEOREM 5.9. *For every homomorphism of Lie groups the following diagram commutes.*

$$\begin{tikzcd} G \arrow[r, "f"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "f_*"'] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

PROOF. $f(\gamma_A(t))$ is a one-parameter subgroup with initial speed f_*A and thus equal to $\gamma_{f_*A}(t)$. Evaluating at $t = 1$ yields the result. □

LEMMA 5.10. *Let* $f : G \to H$ *be a homomorphism of Lie groups with* G *connected and let* $K \subseteq H$ *be a Lie subgroup. Then* $f(G) \subseteq K$ *if and only if* $f_*(\mathfrak{g}) \subseteq \mathfrak{k}$.

PROOF. Suppose that $f_*(\mathfrak{g}) \subseteq \mathfrak{k}$. Then $f(\exp(\mathfrak{g})) = \exp(f_*(\mathfrak{g})) \subseteq \exp(\mathfrak{k}) \subseteq K$. Since $\exp(\mathfrak{g})$ is a neighbourhood of e in G, $f^{-1}(K)$ is an open subgroup of G. As G is connected $f^{-1}(K)$ must equal G. □

THEOREM 5.11. *Let* $\varphi : \mathbb{R} \to G$ *be a continuous group homomorphism. Then* φ *is smooth.*

PROOF. In a neighbourhood of $0 \in \mathbb{R}$ we can write uniquely $\varphi(t) = \exp(A(t))$ with $X(t)$ a continuous path in $\mathfrak{g}$ starting at 0. We would like to show that $X(t)$ is linear. Let $\varphi[-t_0, t_0] \subseteq \exp U$ where U is a ball centered at 0 and such that exp is a diffeomorphism on $2U$. Let $n \in \mathbb{N}$. We will show that $kX\left(\frac{t_0}{n}\right) = X\left(k\frac{t_0}{n}\right)$ for $0 \le k \le n$ by induction on k. For $k = 0$ or $k = 1$ this is clear. Assuming the statement true for k write

$$(k+1)X\left(\tfrac{t_0}{n}\right) = kX\left(\tfrac{t_0}{n}\right) + X\left(\tfrac{t_0}{n}\right) \in 2U$$

Since

$$\begin{aligned} \exp\left((k+1)X\left(\tfrac{t_0}{n}\right)\right) &= \left(\exp X\left(\tfrac{t_0}{n}\right)\right)^{k+1} = \varphi\left(\tfrac{t_0}{n}\right)^{k+1} \\ &= \varphi\left((k+1)\tfrac{t_0}{n}\right) = \exp\left(X\left((k+1)\tfrac{t_0}{n}\right)\right) \end{aligned}$$

and exp is injective on $2U$ this finishes the induction step. As a particular case $nX\left(\frac{t_0}{n}\right) = X(t_0)$ and thus $X\left(\frac{k}{n}t_0\right) = \frac{k}{n}X(t_0)$ which easily holds also for all integers k with $|k| \le n$. From continuity $X(rt_0) = rX(t_0)$ for all $r \in [-1, 1]$. Since $\varphi|_{[-t_0,t_0]}$ is now linear and hence smooth, it is smooth everywhere by the usual argument (homogeneity). □

THEOREM 5.12. *Let* G, H *be Lie groups and* $f : G \to H$ *a continuous group homomorphism between them. Then* f *is smooth.*

PROOF. Pick a basis $A_1, \ldots, A_m$ in $\mathfrak{g}$ and define a map $\varphi : \mathbb{R}^m \to G$ by

$$(t_1, \ldots, t_m) \mapsto \exp(t_1A_1) \cdots \exp(t_mA_m)$$

Clearly φ is a diffeomorphism near 0. It is called a coordinate chart of a second kind (the first kind is exp itself). The composition $f\varphi$ is the map

$$(t_1, \ldots, t_m) \mapsto f(\exp(t_1A_1)) \cdots f(\exp(t_mA_m))$$

which is smooth: each continuous one-parameter subgroup $f(\exp(t_iA_i))$ is smooth by the previous theorem and so is their product. Again we can globalize by homogeneity. □

THEOREM 5.13 (The closed subgroup theorem). *Let $H \subseteq G$ be a subgroup (in the algebraic sense) which is also closed as a subspace of a Lie group G. Then H is a submanifold and thus a Lie subgroup.*

PROOF. We divide the proof into a few steps:

- Define
$$\mathfrak{h} = \{\dot\gamma(0) \mid \gamma : (\mathbb{R}, 0) \to (G, e) \text{ a smooth curve}\}$$
Then $\mathfrak{h}$ is a linear subspace since $\dot\gamma_1(0) + \dot\gamma_2(0) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(\gamma_1(t)\cdot\gamma_2(t))$ and $k\dot\gamma(0) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\gamma(kt)$.
- Let $A_n \in \mathfrak{g}$ be a sequence converging to A and let $t_n > 0$ converge to $0 \in \mathbb{R}$. We claim that if $\exp(t_n An) \in H$ then $\exp(tA) \in H$ for all $t \in \mathbb{R}$. We may suppose that $t > 0$. Choose $m_n \in \mathbb{N}$ in such a way that $|t - m_n t_n|$ is minimal. Then $|t - m_n t_n| \to 0$ and consequently $m_n t_n A_n \to tA$. But $\exp(m_n t_n A_n) = \exp(t_n A_n)^{m_n} \in H$ and since H is closed it follows that $\exp(tA) \in H$ too.
- We show that $\mathfrak{h} = \{A \in \mathfrak{g} \mid \exp(tA) \in H\ \forall t \in \mathbb{R}\}$. The inclusion $\supseteq$ follows from the definition of $\mathfrak{h}$. For the reverse inclusion let $A \in \mathfrak{g}$ be $\dot\gamma(0)$ for some curve $\gamma : \mathbb{R} \to H$. For t small we write $\gamma(r) = \exp(A(t))$. Then
$$A = \dot\gamma(0) = \exp_*(\dot A(0)) = \dot A(0) = \lim_{n\to\infty} \frac{A(\frac{1}{n})}{\frac{1}{n}}$$
Setting $A_n = nA\left(\frac{1}{n}\right) \to A$ and $t_n = \frac{1}{n}$ we have
$$\exp(t_n A_n) = \exp\left(A\left(\tfrac{1}{n}\right)\right) = \gamma\left(\tfrac{1}{n}\right) \in H.$$
and by the previous point $\exp(tA) \in H$ for all $t \in \mathbb{R}$.
- Let $\mathfrak{k} \subseteq \mathfrak{g}$ be a linear subspace complementary to $\mathfrak{h}$. We claim that there exists a neighbourhood $0 \in W \subseteq \mathfrak{k}$ such that $\exp(W) \cap H = \{e\}$. By contradiction let $B_n \to 0$ be a sequence in $\mathfrak{k}$ such that $\exp(B_n) \in H$. With respect to some norm on $\mathfrak{k}$ consider $A_n = \frac{B_n}{|B_n|}$. By passing to a subsequence we may assume that A_n converges to some $A \in \mathfrak{k}$. Putting $t_n = |B_n|$ we have $\exp(t_n A_n) = \exp(B_n) \in H$ and thus $\exp(tA) \in H$ for all $t \in \mathbb{R}$. By the previous point $A \in \mathfrak{h}$, a contradiction to $A \in \mathfrak{k}$.
- Define $\varphi : \mathfrak{h} \times \mathfrak{k} \to G$ by $(A, B) \mapsto \exp A \cdot \exp B$. We will show that there exists a neighbourhood $0 \in V \subseteq \mathfrak{h}$ for which the restriction
$$\varphi : V \times W \xrightarrow{\cong} U \subseteq G$$
is a diffeomorphism onto its image U (which is easy) and such that
$$U \cap H = \varphi(V \times \{0\}).$$
Therefore let $x \in U \cap H$ be in the image, $x = \exp A \cdot \exp B$. As both $x, \exp A \in H$, also $\exp B \in H$. By the previous point $B = 0$.

Thus we found a chart at e which flattens out H. Charts at other points are obtained by translation. □

6. Homogeneous spaces

DEFINITION 6.1. By a *left action* of a Lie group G on a smooth manifold M we understand a smooth map $\ell : G \times M \to M$ satisfying $\ell_e = \mathrm{id}$ and $\ell_a \circ \ell_b = \ell_{ab}$ where we write $\ell_a = \ell(a, -)$. The algebraic content is a homomorphism $G \to \mathrm{Diff}(M)$.

The *right action* $r : M \times G \to M$ has to satisfy $r_e = \mathrm{id}$ and $r_a \circ r_b = r_{ba}$.

We will write $\ell_a(x) = a \cdot x$ and $r_a(x) = xa$.

REMARK. A right action of G is the same as a left action of the opposite group G^*.

Definition 6.2. The orbit of a point $x \in M$ is the subset $Gx = \{ax \mid a \in G\}$. We call the action *transitive* if there is only one orbit in M or equivalently if $Gx = M$ for every $x \in M$.

The *stabilizer subgroup* of a point $x \in M$ is the (closed) subgroup

$$S_x = \{a \in G \mid ax = x\}.$$

The action is called *free* if the stabilizer subgroup of each point is trivial, $S_x = \{e\}$ for every $x \in M$. The action is called *effective* if $\ell_a = \ell_b$ implies $a = b$, i.e. if the homomorphism $G \to \mathrm{Diff}(M)$ is injective.

Set theoretically the action yields a diagram

$$\begin{tikzcd} G \arrow[rr, "\ell(-,x)"] \arrow[dr, "p"'] & & M \\ & G/S_x \arrow[ur, dashed] & \end{tikzcd} \qquad aS_x \mapsto ax \qquad \text{an injective map}$$

and if the action is transitive then $G/S_x \to M$ is even a bijection. Naturally G/S_x is a topological space, a quotient of G:

$$U \subseteq G/S_x \text{ is open} \iff p^{-1}(U) \subseteq G \text{ is open.}$$

Theorem 6.3. *Let $H \subseteq G$ be a closed subgroup of a Lie group G. Then there exists a unique smooth structure on G/H for which $p : G \to G/H$ is a submersion.*

Proof. First we will demonstrate uniqueness in a more general context. The idea here is that surjective submersions are quotient objects:

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "f"'] & P \\ N \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

If f is a surjective submersion and g any smooth map which factors through f set-theoretically, i.e. such that $\ker f \subseteq \ker g$ (or more concretely $f(x) = f(x')$ implies $g(x) = g(x')$), then the unique map h satisfying $g = hf$ is smooth. This follows easily from the fact that f admits smooth local sections (and h is thus a composition of g with such a section).

The uniqueness now follows formally since in the diagram

$$\begin{tikzcd} & G \arrow[dl, "p"'] \arrow[dr, "p"] & \\ G/H \arrow[rr, bend left, "\mathrm{id}"] & & G/H \arrow[ll, bend left, "\mathrm{id}"] \end{tikzcd} \qquad \longleftarrow \text{possibly different smooth structures}$$

the unique factorization maps are the identity maps and the fact that they are both smooth means precisely that the two smooth structures coincide.

It remains to prove the existence. Let $\mathfrak{k} \subseteq \mathfrak{g}$ be a linear subspace complementary to $\mathfrak{h}$. There are neighbourhoods $0 \in V \subseteq \mathfrak{k}$, $0 \in W \subseteq \mathfrak{h}$ and $e \in U \subseteq G$ such that

$$\begin{aligned} \varphi : V \times W &\longrightarrow U \\ (A, B) &\longmapsto \exp A \cdot \exp B \end{aligned}$$

is a diffeomorphism and $U \cap H = \varphi(\{0\} \times W)$. Let $0 \in V' \subseteq V$ be such that

$$(\exp V')^{-1} \cdot (\exp V') \subseteq U$$

which is possible by continuity of the operations. Suppose now that $A_1, A_2 \in V'$ are such that $(\exp A_1) \cdot H = (\exp A_2) \cdot H$. Then $(\exp A_1)^{-1} \cdot \exp A_2 \in U \cap H$ and is equal to $\exp B$ for a unique $B \in W$. Multiplying back

$$\varphi(A_2, 0) = \varphi(A_1, B)$$

which implies $A_1 = A_2$ and $B = 0$. This says that the map

$$\begin{aligned} f : V' \times H &\longrightarrow G \\ (A, b) &\longmapsto (\exp A) \cdot b \end{aligned}$$

is injective. Since it is also a local diffeomorphism on $V' \times (\exp W)$ by translation it is so everywhere and f is in fact a diffeomorphism onto its image.

We have now identified a neighbourhood of $H \subseteq G$ with a product $V' \times H$ and in such a way that the cosets $a \cdot H$ lying in this "chart" are of the form $\{A\} \times H$. Thus the map

$$\psi : V' \cong V' \times \{e\} \hookrightarrow V' \times H \hookrightarrow G \xrightarrow{p} G/H$$

(sending A to $(\exp A) \cdot H$) embeds V' as a neighbourhood of the coset $eH \in G/H$. We therefore declare it a chart on G/H. In this way the map p becomes the projection $V' \times H \to V'$ and thus a submersion. To get a chart near arbitrary aH redefine f as

$$\begin{aligned} f_a : V' \times H &\longrightarrow G \\ (A, b) &\longmapsto a \cdot (\exp A) \cdot b \end{aligned}$$

and consequently $\psi_a(A) = a \cdot (\exp A) \cdot H$. The transition map $\psi_{a'a}$ between the resulting charts $\psi_{a'}$ and ψ_a is computed from

$$a \cdot (\exp(\psi_{a'a} A)) \cdot H = a' \cdot (\exp A) \cdot H$$

Multiplying by a^{-1} we obtain

$$\exp(\psi_{a'a} A) \in a^{-1} a' \cdot (\exp A) \cdot H$$

and thus $\psi_{a'a}$ is the composition

$$V' \xrightarrow{\exp} U \xrightarrow{\lambda_{a^{-1}a'}} U \xrightarrow{f^{-1}} V' \times H \longrightarrow V'$$

with all arrows smooth and $\lambda_{a^{-1}a'}$ only locally defined. □

Definition 6.4. The manifold G/H is called a *homogeneous space*.

Remark. In the lecture I mentioned AT THIS POINT what a bundle is and that $p : G \to G/H$ is an important example.

Theorem 6.5. *The orbit of each point is an immersed submanifold (i.e. image of an injective immersion).*

Proof. Consider the diagram

$$\begin{tikzcd} G \arrow[rr, "\ell(-,x)"] \arrow[dr, "p"'] & & M \\ & G/S_x \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

with the map f smooth by the previous theorem. We need to show that it is an immersion (on the other hand it is injective almost by the definition of S_x). Suppose first that for $A \in \mathfrak{g}$ its image $p_* A$ is sent to $0 \in T_x M$ by f_*. Then $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \exp(tA)x = 0$. On the other hand

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} \exp(tA)x &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} \exp(t_0 A) \exp((t - t_0)A)x \\ &= (\ell_{\exp(t_0 A)})_* \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} \exp((t - t_0)A)x}_{0} = 0 \end{aligned}$$

Thus $\exp(tA)x = x$ for all $t \in \mathbb{R}$ and $\exp(tA) \in S_x$ implying that $A \in \ker p_*$ and $p_* A = 0$. This finishes the proof that f is an immersion at eS_x. At other points this is guaranteed by the homogeneity:

$$
\begin{tikzcd}
eS_x \arrow[d, mapsto] & G/S_x \arrow[r, "f"] \arrow[d, "\ell_a"', "\cong"] & M \arrow[d, "\cong"', "\ell_a"] \\
aS_x & G/S_x \arrow[r, "f"'] & M
\end{tikzcd}
$$

□

Example 6.6. Fix $v \in \mathbb{R}^2$ and consider the following action of $\mathbb{R}$ on $\mathbb{R}^2$

$$
\begin{aligned}
\mathbb{R} \times \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\
(t, u) &\longmapsto u + tv
\end{aligned}
$$

Clearly the orbit of u is the line $u + \mathbb{R}v$. Passing to the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ we see that orbits need not be embedded submanifolds.

Remark. In general every orbit is an initial submanifold.

Corollary 6.7. *For a transitive action the map $f : G/S_x \to M$ is a diffeomorphism.*

Proof. From Sard's theorem it easily follows that smooth bijections exist only between manifolds of the same dimension. Hence the immersion f is in fact a local diffeomorphism. Being also bijective it is a diffeomorphism by the inverse function theorem. □

Examples 6.8. Examples of homogeneous spaces:

- $\mathbb{R}$ acts on the unit circle via rotations, i.e. t act as rotation by $2\pi \cdot t$. The stabilizer of any point is the subgroup $\mathbb{Z}$ and thus $\mathbb{R}/\mathbb{Z} \cong S^1$.
- Let V be a vector space. Then $\mathrm{GL}(V)$ acts transitively on $V - \{0\}$ and thus $V - \{0\} \cong \mathrm{GL}(V)/S_v$ where $v \in V - \{0\}$.
- The sphere S^{n-1} with the action of $O(n)$ is a homogeneous space, $S^{n-1} \cong O(n)/O(n-1)$ where $O(n-1)$ is thought of as a subgroup of $O(n)$ consisting of block matrices
$$O(n-1) \cong \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in O(n) \;\middle|\; A \in O(n-1) \right\}$$
- The n-dimensional affine space is acted upon by the group
$$GA(n) = \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(n+1) \;\middle|\; A \in \mathrm{GL}(n), v \in \mathbb{R}^n \right\}$$
of affine transformations, namely we identify a point $x \in \mathbb{R}^n$ with a vector $\left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right)$ in $\mathbb{R}^{n+1}$ and then
$$\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Ax + v \\ 1 \end{pmatrix}$$
The origin is preserved exactly by the subgroup
$$\mathrm{GL}(n) = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in GA(n) \;\middle|\; A \in \mathrm{GL}(n) \right\}$$
describing $\mathbb{R}^n$ as $GA(n)/\mathrm{GL}(n)$. Similarly with $\mathrm{GL}(n)$ replaced by $O(n)$ we arrive at $\mathbb{R}^n \cong Euc(n)/O(n)$ with $Euc(n)$ denoting the group of (not necessarily origin preserving) isometries of $\mathbb{R}^n$.
- The Stiefel manifold (of orthonormal k-frames in V)
$$S_k(V) = \{(v_1, \ldots, v_k) \mid \langle v_i, v_j \rangle = \delta_{ij}\}$$
has as examples $S_1(V)$, the unit sphere in V, $S_n(\mathbb{R}^n) = O(n)$. For general $S_k(\mathbb{R}^n)$ there is a natural action of $O(n)$ componentwise:
$$A(v_1, \ldots, v_k) = (Av_1, \ldots, Av_k)$$

The stabilizer of the k-tuple $(e_1, \dots, e_k)$ of the first k vectors of the standard basis is clearly

$$O(n-k) \cong \left\{ \begin{pmatrix} E & 0 \\ 0 & C \end{pmatrix} \in O(n) \;\middle|\; C \in O(n-k) \right\}$$

Thus $S_k(\mathbb{R}^n) \cong O(n)/O(n-k)$.

- The Grassmann manifold $G_k(V)$ of all k-dimensional subspaces of V is naturally a quotient of $S_k(V)$, namely by the means of the map

$$\begin{aligned} S_k(V) &\longrightarrow G_k(V) \\ (v_1, \dots, v_k) &\longmapsto \mathrm{span}(v_1, \dots, v_k) \end{aligned}$$

The $O(n)$-action on $S_k(\mathbb{R}^n)$ passes to $G_k(\mathbb{R}^n)$ with the stabilizer of $\mathbb{R}^k$ being

$$O(k) \times O(n-k) \cong \left\{ \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \in O(n) \;\middle|\; B \in O(k), C \in O(n-k) \right\}$$

and thus providing $G_k(\mathbb{R}^n) \cong O(n)/O(k) \times O(n-k)$.

- I have mentioned EXAMPLES of the homogeneous spaces of scalar products, complex structures etc.

Theorem 6.9. *Let $N \subseteq G$ be a closed normal subgroup. Then G/N with its canonical smooth structure is a Lie group.*

Proof. The left vertical arrow in

$$\begin{tikzcd} G \times G \arrow[r, "\mu"] \arrow[d, "p \times p"'] & G \arrow[d, "p"] \\ G/N \times G/N \arrow[r, dashed] & G/N \end{tikzcd}$$

is a surjective submersion therefore the dotted arrow (the multiplication in G/N) is smooth. □

We have already met an example. The additive group $\mathbb{R}$ admits a homomorphism to $\mathbb{T} = \mathrm{SU}(1)$ by $t \mapsto e^{2\pi i t}$. Clearly the kernel is $\mathbb{Z}$ and thus we obtained an induced isomorphism $\mathbb{R}/\mathbb{Z} \xrightarrow{\cong} \mathbb{T}$.

7. The adjoint representation

Definition 7.1. By a *representation* of G we understand a left action of G on a vector space V by linear maps (automorphisms), i.e. for which each $\ell_a : V \to V$ is linear. Equivalently $\rho : G \to \mathrm{GL}(V)$ is a (smooth) homomorphism of Lie groups.

Definition 7.2. A representation of a Lie algebra L on a vector space V is a homomorphism $\pi : L \to \mathfrak{gl}(V)$ of Lie algebras. More concretely π is a linear map for which $\pi[X,Y](v) = \pi X \circ \pi Y(v) - \pi Y \circ \pi X(v)$.

Definition 7.3. A linear subspace $W \subseteq V$ is called *invariant* with respect to a representation ρ if $\rho(a)(W) \subseteq W$ for all $a \in G$. Analogously it is called *invariant* with respect to a representation π if $\pi(X)(W) \subseteq W$ for all $X \in L$.

Theorem 7.4. *Let G be a connected Lie group and ρ its representation on V, $\rho_* : \mathfrak{g} \to V$ the induced representation of $\mathfrak{g}$. Then $W \subseteq V$ is invariant with respect to ρ if and only if it is invariant with respect to ρ_*.*

Proof. Consider the following subgroup of $\mathrm{GL}(V)$

$$\mathrm{GL}(V,W) = \{\varphi \in \mathrm{GL}(V) \mid \varphi(W) \subseteq W\}.$$

It is easy to show that

$$\mathfrak{gl}(V,W) = \mathrm{Lie}(\mathrm{GL}(V,W)) = \{\varphi \in \mathfrak{gl}(V) \mid \varphi(W) \subseteq W\}.$$

The statement then becomes a special case of Lemma 5.10. □

Let $\ell : G \times M \to M$ be a left action and $x \in M$ its fixed point (i.e. $S_x = G$). Then $\rho : G \to \mathrm{GL}(T_xM)$ given by $a \mapsto (\ell_a)_{*x}$ is smooth by

$$\begin{tikzcd} TG \times TM \arrow[r, "\ell_*"] & TM \\ G \times T_xM \arrow[u, hook, "0\times\mathrm{id}"] \arrow[ur, "\rho^\sharp"'] & \end{tikzcd}$$

and consequently a representation of G on T_xM. We apply these general considerations to the action of G on itself via conjugation (inner automorphisms):

$$(a, b) \longmapsto \mathrm{int}_a b = aba^{-1}$$

Now $e \in G$ is a fixed point and we define $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ as above

$$\mathrm{Ad}(a)B = (\mathrm{int}_a)_* B$$

Moreover $\mathrm{Ad}(a) \in \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g})$ since int_a is a homomorphism of Lie groups. We denote the induced representation by $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ (in fact $\mathrm{Der}(\mathfrak{g})$).

Theorem 7.5. *For each $A, B \in \mathfrak{g}$ it holds* $\mathrm{ad}(A)(B) = [A, B]$.

Proof. We compute

$$\begin{aligned} \mathrm{ad}(A)(B) &= \tfrac{\partial}{\partial s}\big|_{s=0} \mathrm{Ad}(\exp(sA))(B) \\ &= \tfrac{\partial}{\partial s}\big|_{s=0} \tfrac{\partial}{\partial t}\big|_{t=0} \mathrm{int}_{\exp(sA)} \exp(tB) \\ &= \tfrac{\partial}{\partial s}\big|_{s=0} \tfrac{\partial}{\partial t}\big|_{t=0} \exp(sA)\exp(tB)\exp(-sA) \\ &= \tfrac{\partial}{\partial s}\big|_{s=0} \tfrac{\partial}{\partial t}\big|_{t=0} \mathrm{Fl}^{\lambda_A}_{-s} \mathrm{Fl}^{\lambda_B}_{t} \mathrm{Fl}^{\lambda_A}_{s}(e) \\ &= \tfrac{\partial}{\partial s}\big|_{s=0} (\mathrm{Fl}^{\lambda_A}_{-s})_* \lambda_B(\mathrm{Fl}^{\lambda_A}_{s}(e)) = [\lambda_A, \lambda_B]_e = [A, B] \end{aligned}$$

□

Theorem 7.6. *If $H \subseteq G$ is a normal subgroup then $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal, i.e. a linear subspace such that $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$ (meaning $[A, B] \in \mathfrak{h}$ for all $A \in \mathfrak{g}$ and $B \in \mathfrak{h}$).*

Proof. Since $aHa^{-1} \subseteq H$ or $\mathrm{int}_a H \subseteq H$ we differentiate to get $\mathrm{Ad}(a)(\mathfrak{h}) \subseteq \mathfrak{h}$ and finally $\mathrm{ad}(\mathfrak{g})\mathfrak{h} \subseteq \mathfrak{h}$. □

Theorem 7.7. *Let H be a connected Lie subgroup of a connected Lie group G such that $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal. Then H is a normal subgroup.*

Proof. We have $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}, \mathfrak{h})$. Since G is connected $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}, \mathfrak{h})$. It is enough to show that $\mathrm{int}_a(\exp tB) \in H$ for all $B \in \mathfrak{h}$ since the subgroup generated by such elements is the whole group H. But $\mathrm{int}_a(\exp tB) = \exp(\mathrm{Ad}(a)(tB)) \in H$ since $\mathrm{Ad}(a)(tB) \in \mathfrak{h}$. □

Theorem 7.8. *Let $\varphi : G \to H$ be a homomorphism of Lie groups. Then its kernel is a closed normal subgroup $K \subseteq G$ and its Lie algebra $\mathfrak{k}$ is the kernel of $\varphi_* : \mathfrak{g} \to \mathfrak{h}$.*

Proof. $A \in \mathfrak{k}$ iff $\exp tA \in K$ iff $\exp(t \cdot \varphi_* A) = \varphi(\exp tA) = e$ iff $\varphi_* A = 0$. □

Definition 7.9. The *centre* C of a Lie group G is the set

$$C = \{a \in G \mid ab = ba \ \forall b \in G\}$$

In other words, C is the kernel of $\mathrm{int} : G \to \mathrm{Aut}(G)$.

Theorem 7.10. *The centre of a connected Lie group G is the kernel of the adjoint representation* Ad.

Proof. $a \in C$ iff $\mathrm{int}_a(G) = e$ iff $\mathrm{Ad}(a)\mathfrak{g} = 0$ iff $\mathrm{Ad}(a) = 0$. □

Definition 7.11. The *centre* of a Lie group L is the ideal

$$Z = \{X \in L \mid [X, Y] = 0 \ \forall Y \in L\}$$

In other words, Z is the kernel of $\mathrm{ad} : L \to \mathfrak{gl}(L)$.

THEOREM 7.12. *For a connected Lie group G, the centre Z of $\mathfrak{g}$ the Lie algebra of the centre C of G.*

PROOF. Since $C = \ker(\mathrm{Ad})$, its Lie algebra $\mathrm{Lie}(C) = \ker(\mathrm{ad})$. $\square$

REMARK. If the centre of L is zero then L can be embedded into $\mathfrak{gl}(L)$ via the representation ad.

THEOREM 7.13 (Ado). *Every Lie algebra can be embedded into $\mathfrak{gl}(V)$ for some finite-dimensional vector space V.*

COROLLARY 7.14. *Every Lie algebra is induced from some Lie group.*

PROOF. By Ado's theorem $L \subseteq \mathfrak{gl}(n)$. Since $\mathfrak{gl}(n) = \mathrm{Lie}(\mathrm{GL}(n))$ one can find a Lie subgroup of $\mathrm{GL}(n)$ corresponding to L. $\square$

8. Fundamental vector fields

Consider a left action $\ell : G \times M \to M$. To every vector $A \in \mathfrak{g}$ we associate a vector field ℓ_A on M by $\ell_A(x) = (\ell(-,x))_* A$. As usual ℓ_A is smooth and is called the *fundamental vector field* on M corresponding to $A \in \mathfrak{g}$. Analogously we define fundamental vector fields for right actions.

THEOREM 8.1. *In the case of a left action of G on M it holds $[\ell_A, \ell_B] = \ell_{-[A,B]}$. For the right action we obtain $[r_A, r_B] = r_{[A,B]}$.*

PROOF. On $M \times G$ consider the vector field $(0, \lambda_A)(x,a) = (0_x, \lambda_A(a))$.

$$r_{*(x,a)}(0, \lambda_A) = (r(x,-))_{*a}\lambda_A = (r(xa,-))_{*e}A = r_A(xa)$$

says that $(0, \lambda_A)$ is r-related to r_A. As the same is true for B we obtain for the brackets that $[(0,\lambda_A),(0,\lambda_B)]$ is r-related to $[r_A, r_B]$. But

$$[(0,\lambda_A),(0,\lambda_B)] = ([0,0],[\lambda_A,\lambda_B]) = (0, \lambda_{[A,B]})$$

which is r-related to $r_{[A,B]}$. Thus $[r_A, r_B] = r_{[A,B]}$. $\square$

The last theorem can be expressed by saying that $r : \mathfrak{g} \to \mathfrak{X}M$, $A \mapsto r_A$ is a homomorphism of Lie algebras. The left action gives an antihomomorphism.

DEFINITION 8.2. By a *right infinitesimal action* of a Lie group G on a manifold M we understand a homomorphism $R : \mathfrak{g} \to \mathfrak{X}M$ of Lie groups. A right infinitesimal action is called *complete* if R_A is a complete vector field for each $A \in \mathfrak{g}$. Analogously a *left infinitesimal action* is an antihomomorphism.

EXAMPLE 8.3. The fundamental vector fields are complete: $r(x, \exp tA) = x \exp tA$ is an integral curve through x defined for all $t \in \mathbb{R}$.

REMARK. A left action is a homomorphism of Lie groups $G \to \mathrm{Diff}(M)$ (with infinite dimensional target). The induced Lie algebra homomorphism is $\mathfrak{g} \to \mathrm{Lie}(\mathrm{Diff}(M))$, the latter being $\mathfrak{X}M$ but with the opposite bracket. As for finite dimensional Lie groups we can "integrate" a homomorphism of Lie groups but here under additional requirements - the completeness.

THEOREM 8.4. *For a complete right infinitesimal action $R : \mathfrak{g} \to \mathfrak{X}M$ of a simply connected Lie group G on M there exists a unique right action $r : M \times G \to M$ of G on M such that R_A is its fundamental vector field r_A.*

REMARKS.

- The simple connectivity is necessary: for the action of $G = \mathbb{R}$ on itself by translations the infinitesimal action $r_t = t$ passes to an infinitesimal action of the quotient $\mathbb{R}/\mathbb{Z}$ on $\mathbb{R}$ for which no action exists.
- The theorem holds locally without the assumptions of completeness and simple connectivity.
- The usual translation between left and right yields an analogous statement for left actions.

PROOF. Let first r be an action of G on M. Let S_x denote the following submanifold

$$S_x = \{(xa, a) \mid a \in G\} \subseteq M \times G$$

The tangent space of S_x is

$$TS_x = \{(r_A(xa), \lambda_A(a)) \mid a \in G, A \in \mathfrak{g}\}$$

Thus S_x is an integral submanifold of the distribution $\langle (r_A, \lambda_A) \mid A \in \mathfrak{g} \rangle$.

Let us now start the actual proof of the theorem by considering the distribution

$$D = \langle (R_A, \lambda_A) \mid A \in \mathfrak{g} \rangle$$

Then D is involutive since

$$[(R_A, \lambda_A), (R_B, \lambda_B)] = ([R_A, R_B], [\lambda_A, \lambda_B]) = (R_{[A,B]}, \lambda_{[A,B]}).$$

Let S_x be the maximal integral submanifold of D through $(x, e) \in M \times G$. We claim now that $p_x : S_x \hookrightarrow M \times G \to G$ is a diffeomorphism.

First we show that it is a covering. Fix $a \in G$ and consider an arbitrary $(y, a) \in M \times G$. The computation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} \underbrace{(\mathrm{Fl}_t^{R_A}(y), a \exp tA)}_{\gamma(t)} = (R_A(\mathrm{Fl}_{t_0}^{R_A}(y)), \lambda_A(a \exp t_0 A)) \in D$$

shows that $\gamma(t)$ is tangent to the distribution D. Let $U \subseteq \mathfrak{g}$ be an open ball centered at 0 on which exp is a diffeomorphism. If $(y, a) \in S_x$ then also $(\mathrm{Fl}_1^{R_A}(y), a \exp A) \in S_x$ for all $A \in U$ and such points form an open neighbourhood on which p_x is a diffeomorphism onto $a \exp U$. If $(z, b) \in S_x$ is arbitrary with $b \in a \exp U$ then $b = a \exp A$ and thus the above subset considered for $(\mathrm{Fl}_1^{R_{-A}}(z), b \exp(-A))$ contains (z, b). This finishes the proof that p_x is a covering and in fact a diffeomorphism as G is simply connected.

We define for $x \in M$ and $a \in G$ the action by the requirement

$$(xa, a) \in S_x$$

By the previous part there is a unique choice for xa. We need to show that r is smooth but first let us prove the axioms of an action. Clearly $xe = x$ as S_x is an integral manifold through (x, e). Consider now a left action of G on $M \times G$ by $a(y, b) = (y, ab)$. The distribution D is invariant under this action (as $(\mathrm{id}, \lambda_a)_*(R_A, \lambda_A) = (R_A, \lambda_A)$) and thus also its maximal integral submanifolds. The requirement for our action r can be then rewritten as

$$S_x = aS_{xa} = a(bS_{(xa)b}) = (ab)S_{(xa)b}$$

As also $S_x = (ab)S_{x(ab)}$ the maximal integral submanifolds $S_{(xa)b}$ and $S_{x(ab)}$ must also be equal proving $(xa)b = x(ab)$.

A word about smoothness... □

DEFINITION 8.5. Consider two actions r and r' of a Lie group G on manifolds M and M'. A map $f : M \to M'$ is called *equivariant* if $f(xa) = f(x)a$.

THEOREM 8.6. *If $f : M \to M'$ is equivariant then r_A is f-related to r'_A.*

PROOF. The requirement from the definition is $f \circ r(x, -) = r'(f(x), -)$. Applying the derivatives of both sides to A we get $f_* r_A = r'_A f$. □

THEOREM 8.7. *Let $f : M \to M'$ be a smooth map such that r_A is f-related to r'_A. If G is connected then f is equivariant.*

PROOF. Consider the set $H \subseteq G$ of all $a \in G$ for which $f(xa) = f(x)a$ for all $x \in M$. Then H is clearly a subgroup and thus we only need that it contains a neighbourhood of e. But $f(x \exp tA) = f(\mathrm{Fl}_t^{r_A}(x)) = \mathrm{Fl}_t^{r'_A}(f(x)) = f(x) \exp tA$, hence $\exp \mathfrak{g} \subseteq H$ and H is open and therefore equal to G. □

9. Locally isomorphic Lie groups

Let G be a connected Lie group. Recall that the universal covering of G is

$$\begin{tikzcd} \tilde{G} \arrow[r, equal] \arrow[d, "p"'] & \{[\gamma] \mid \gamma : (I,0) \to (G,e)\} \arrow[d, maps to] \\ G & \gamma(1) \end{tikzcd}$$

with $[\gamma]$ the homotopy class of γ relative to the boundary. $\tilde{G}$ is simply connected: firstly $\pi_1\tilde{G} \to \pi_1 G$ is injective (this works for any covering) since we can lift homotopies and constant paths lift to constant paths. The image consists exactly of the classes of loops that lift to loops. For $\tilde{G}$ if $\gamma : I \to G$ lifts to a loop its endpoints must be equal $\tilde{e} = [\gamma]$ and the image is therefore trivial.

We give $\tilde{G}$ a structure of a Lie group: let $\gamma, \gamma' : (I,0) \to (G,e)$ be two paths. Define their product to be the path

$$(\gamma \cdot \gamma')(t) = \gamma(t)\gamma'(t)$$

which easily passes to homotopy classes rel ∂I.

Theorem 9.1. *The above multiplication on $\tilde{G}$ describes a structure of a Lie group for which the projection $p : \tilde{G} \to G$ is a local isomorphism (i.e. a homomorphism and a local diffeomorphism).*

Proof. The unit and inverses are also pointwise. The diagram

$$\begin{tikzcd} \tilde{G} \times \tilde{G} \arrow[r, dashed] \arrow[d] & \tilde{G} \arrow[d, "\text{local diffeomorphism}"] \\ G \times G \arrow[r, "\text{smooth}"'] & G \end{tikzcd}$$

shows that the dotted arrow (the multiplication in $\tilde{G}$) is smooth. (This is cheating, one needs to compute $(\gamma * \delta) \cdot (\gamma' * \delta') = (\gamma \cdot \gamma') * (\delta \cdot \delta')$ and if both δ and δ' were small then so is $\delta \cdot \delta'$. Add more DETAILS.) $\square$

Remark. I would like to CHANGE the proceeding along this way: we know that $\pi_1 G \subset \tilde{G}$ is the kernel of $p : \tilde{G} \to G$ and as such is a discrete normal subgroup. It is therefore central (this was before an exercise). We show that the multiplication coming from $\tilde{G}$ is the same as the concatenation (and in fact the multiplication $\pi_1 G \times \tilde{G} \to \tilde{G}$ may also be equivalently defined using concatenation). The theorem may be deduced from lifting homomorphisms of Lie algebras to Lie groups. The map $\tilde{G} \to G'$ is then automatically a (surjective) homomorphism and thus a quotient by a subgroup $\Gamma \subseteq \pi_1 G$.

There is an action of $\pi_1 G$ on $\tilde{G}$, $\pi_1 G \times \tilde{G} \to \tilde{G}$ given by

$$([\alpha], [\gamma]) \longmapsto [\alpha] \cdot [\gamma] = [\alpha * \gamma]$$

which respects the projection $p : \tilde{G} \to G$. Let $\Gamma \subseteq \pi_1 G$ be a subgroup and consider

$$p_\Gamma : \tilde{G}/\Gamma \to G$$

where $\tilde{G}/\Gamma$ is the space of orbits of the restriction of the action to Γ. Locally

$$\begin{tikzcd} \pi_1 G \times U \arrow[r, hook] \arrow[d] & \tilde{G} \arrow[d, "p"] \\ U \arrow[r, hook] & G \end{tikzcd}$$

and the action of Γ is by left multiplication in $\pi_1 G$. Thus the projection p_Γ from $\tilde{G}/\Gamma$ to G is locally of the form

$$(\pi_1 G/\Gamma) \times U \to U$$

and in particular is a covering of G.

THEOREM 9.2. *Let G be a connected Lie group. Then the mapping*

$$\{\textit{subgroups } \Gamma \subseteq \pi_1 G\} \longrightarrow \left\{\begin{matrix}\textit{local isomorphisms } \rho : G' \to G \\ \textit{with } G' \textit{ any connected Lie group}\end{matrix}\right\} /\textit{iso}$$
$$\Gamma \longmapsto (p_\Gamma : \tilde{G}/\Gamma \to G)$$

is a bijection with inverse $\rho \longmapsto \operatorname{im}(\pi_1\rho : \pi_1 G' \to \pi_1 G)$.

PROOF. The image of $\pi_1 p_\Gamma$ consists of those loops that lift to loops in $\tilde{G}/\Gamma$. These are precisely those in Γ. In the opposite direction any ρ fits into the diagram

$$\begin{tikzcd}
\tilde{G} \arrow[rr] \arrow[dr] \arrow[ddr, "p"'] & & G' \arrow[ddl, "\rho"] \\
& \tilde{G}/\Gamma \arrow[ur, dashed, "\cong"] \arrow[d, "p_\Gamma"] & \\
& G &
\end{tikzcd}$$

with $\Gamma = \operatorname{im}(\pi_1\rho)$. The top arrow exists by universality of $\tilde{G}$. NEED THAT IT IS A HOMOMORPHISM. The dotted arrow exists since loops in Γ lift to loops in G'. It is an isomorphism of Lie groups. $\square$

REMARK. We will show in the tutorial that $\pi_1 G \hookrightarrow \tilde{G}$ is a homomorphism and the action of $\pi_1 G$ on $\tilde{G}$ is by left translations, i.e. $\tilde{G}/\Gamma$ is a quotient of $\tilde{G}$ by (a central subgroup) Γ.

EXAMPLE 9.3 (The universal covering of a commutative connected Lie group G). Since $\operatorname{Lie} G = \mathbb{R}^n$ with zero bracket it is also the Lie algebra of the simply connected Lie group $\mathbb{R}^n$ (with vector addition) and thus $\tilde{G} = \mathbb{R}^n$. Therefore $G \cong \mathbb{R}^n/\Gamma$ where Γ is some discrete subgroup of $\mathbb{R}^n$. We will show now that $\Gamma = \mathbb{Z}^k \subseteq \mathbb{R}^n$ in some coordinates on $\mathbb{R}^n$.

First reduction is to the case $n = k$, namely we have $\operatorname{span}\Gamma = \mathbb{R}^k \subseteq \mathbb{R}^n$ and Γ is still discrete in $\mathbb{R}^k$. We must show that $\Gamma = \mathbb{Z}^k$ in some coordinates on $\mathbb{R}^k$.

We start an induction by $k = 1$ which we proved in the tutorial. For the induction step we may assume that $\Gamma \subseteq \mathbb{R} \times \mathbb{R}^k = \mathbb{R}^{k+1}$ is such that the intersection $\Gamma \cap \mathbb{R} \neq 0$ with the first coordinate axis is nonzero. Since it is also discrete it is generated by some a_0. In $\mathbb{R}^k = \mathbb{R}^{k+1}/\mathbb{R}$ consider its subgroup $\Gamma/\langle a_0\rangle$. We show by contradiction that it is discrete. Namely assume the existence of a sequence $\alpha_n = (\beta_n, \gamma_n) \in \Gamma$ with $\gamma_n \to 0$ in $\mathbb{R}^k$. By adding a suitable multiple of a_0 to each α_n we may assume that $\beta_n \in [-a_0/2, a_0/2]$ and by extracting a subsequence we may further assume that α_n converges. But then $\alpha_{n+1} - \alpha_n \in \Gamma$ converges to 0, a contradiction with Γ being discrete. By the induction hypothesis $\Gamma/\langle a_0\rangle = \langle \tilde{a}_1, \dots, \tilde{a}_k\rangle$. We choose for each $\tilde{a}_i$ an element $a_i \in \Gamma$ representing it. Then the suitable basis in which $\Gamma = \mathbb{Z}^{k+1}$ is formed by $(a_0, a_1, \dots, a_k)$.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}a_0 \arrow[r] \arrow[d, "\cong"'] & \mathbb{Z}\{a_0, a_1, \dots, a_k\} \arrow[r] \arrow[d] & \mathbb{Z}\{\tilde{a}_1, \dots, \tilde{a}_k\} \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & \Gamma \cap \mathbb{R} \arrow[r] & \Gamma \arrow[r] & \Gamma/(\Gamma \cap \mathbb{R}) \arrow[r] & 0
\end{tikzcd}$$

COROLLARY 9.4. *The only compact connected commutative Lie group of dimension k is the torus $\mathbb{T}^k = (S^1)^k$.*

EXAMPLE 9.5. For $n \geq 3$ we have $\pi_1 \operatorname{SO}(n) \cong \mathbb{Z}/2$. Therefore $\operatorname{SO}(n)$ possesses a two-sheeted universal covering which is denoted by $\operatorname{Spin}(n) = \widetilde{\operatorname{SO}}(n)$. We will show geometrically that $\pi_1 \operatorname{SO}(3) = \mathbb{Z}/2$ in the tutorial. For higher n we have a fibration

$$\operatorname{SO}(n) \to \operatorname{SO}(n+1) \to S^n$$

whose long exact sequence of homotopy groups contains the following portion

$$0 = \pi_2(S^n) \to \pi_1(\operatorname{SO}(n)) \xrightarrow{\cong} \pi_1(\operatorname{SO}(n+1)) \to \pi_1 S^n = 0$$

10. Problems

PROBLEM 10.1. An algebra is a vector space A together with a bilinear map $\cdot : A \times A \to A$. Let A be now an associative algebra and define $[\ ,\ ] : A \times A \to A$ by $[a, b] = a \cdot b - b \cdot a$. Show that with this operation A forms a Lie algebra.

A special case of the previous is the algebra $\mathrm{End}(V)$ of endomorphisms of a vector space V together with their compositions. The induced Lie algebra is denoted by $\mathfrak{gl}(V)$. The bracket of two endomorphisms φ, ψ is

$$[\varphi, \psi] = \varphi \circ \psi - \psi \circ \varphi$$

PROBLEM 10.2. Let A be an algebra. A linear map $D : A \to A$ is called a *derivative* if for all $a, b \in A$

$$D(a \cdot b) = D(a) \cdot b + a \cdot D(b)$$

Show that derivatives form a Lie subalgebra $\mathrm{Der}(A) \subseteq \mathfrak{gl}(A)$.

PROBLEM 10.3. Let $C^\infty M = C^\infty(M, \mathbb{R})$ denote the algebra of all smooth functions on M. Then every vector field X on M determines a mapping

$$\begin{aligned} C^\infty M &\longrightarrow C^\infty(M) \\ f &\longmapsto Xf = \mathrm{d}f(X) \end{aligned}$$

Show that this mapping is a derivative (in the algebraic sense). Also show that all derivatives of $C^\infty M$ are of this form.

Let us now describe the Lie bracket of vector fields from this point of view: $[X, Y]$ is simply the vector field corresponding to the bracket of the two derivatives X and Y of $C^\infty M$. This means that $[X, Y]f = XYf - YXf$ and this formula determines a unique vector field $[X, Y]$.

It also holds that algebra homomorphisms $C^\infty N \to C^\infty M$ are in bijection with smooth maps $M \to N$. One may then rewrite the f-relatedness of vector fields X and Y as

$$\begin{tikzcd} C^\infty N \arrow[r, "f^*"] \arrow[d, "Y"'] & C^\infty M \arrow[d, "X"] \\ C^\infty N \arrow[r, "f^*"'] & C^\infty M \end{tikzcd}$$

It is then a simple matter to show that $X_i \sim_f Y_i$ implies $[X_1, X_2] \sim_f [Y_1, Y_2]$.

PROBLEM 10.4. Compute the Lie algebra of the additive Lie group $\mathbb{R}^n$.

PROBLEM 10.5. Compute the Lie algebra of the Lie group $\mathrm{GL}(n, \mathbb{R})$ from the definition.

PROBLEM 10.6. Compute the Lie algebra of the Lie group $\mathrm{GL}(n, \mathbb{R})$ from the formula $[A, B] = \frac{\partial^2}{\partial s \partial t}\Big|_{(s,t)=(0,0)} \varphi(t)\psi(s)\varphi(t)^{-1}\psi(s)^{-1}$.

PROBLEM 10.7. Compute the Lie algebra of the Lie group $S^3 = \mathrm{Sp}(1)$ of unit quaternions and show that it is isomorphic to $\mathbb{R}^3$ with the vector product $\times$.

PROBLEM 10.8. Let $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bilinear form and denote by

$$G(B) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T B A = B\} \subseteq \mathrm{GL}(n, \mathbb{R})$$

the closed subgroup of all automorphisms preserving the form B. Compute the Lie algebra of $G(B)$.

PROBLEM 10.9. Compute the Lie algebra of $\mathrm{SO}(n, \mathbb{R})$.

PROBLEM 10.10. Let A be an algebra and denote by $\mathrm{Aut}(A)$ the group of all algebra automorphisms of A. Compute its Lie algebra.

PROBLEM 10.11. Determine all Lie algebras of dimension 2 over $\mathbb{R}$.

PROBLEM 10.12. Prove that the element $\left(\begin{smallmatrix}-2 & 0\\ 0 & -1\end{smallmatrix}\right)$ of $\mathrm{GL}(2,\mathbb{R})$ lies in the component of the unit E but not in the image of exp.

PROBLEM 10.13. Let

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(3,\mathbb{R}) \;\middle|\; a,b,c \in \mathbb{R} \right\}$$

denote the Heisenberg group. Show that the bracket on $\mathrm{Lie}(G)$ is non-trivial and exp is a global diffeomorphism.

PROBLEM 10.14. Show that for $G = S^3 = \mathrm{Sp}(1)$ the map exp is not a local diffeomorphim at all points of $\mathfrak{g}$.

PROBLEM 10.15. Show that discrete subgroups of $\mathbb{R}$ are exactly those of the form $\mathbb{Z}a$ for some positive real number a. Deduce that the only Lie groups of dimension 1 are $\mathbb{R}$ and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

PROBLEM 10.16. Show that a discrete normal subgroup of a connected Lie group must lie in the centre. (Hint: $\mathrm{int}_a : H \to H$ for $a \in G$ may be connected to $\mathrm{int}_e = \mathrm{id}$. Since H is discrete these must be equal hence $\mathrm{int}_a = \mathrm{id}$ and H is central.)

PROBLEM 10.17. Let $f : M \to G$ be a smooth map from a manifold M to a Lie group G. Denote by $\delta_l f$ the $\mathfrak{g}$-valued 1-form called the *left logarithmic derivative of* f given by

$$\delta_l f(x, X) = (\lambda_{f(x)^{-1}})_* f_* X$$

(with (x, X) denoting a tangent vector $X \in T_x M$). For example

$$\delta_l \,\mathrm{id}(a, A) = (\lambda_{a^{-1}})_* A = \omega(A)$$

the Maurer-Cartan form. Compute $\delta_l \lambda_b$, $\delta_l \rho_b$, $\delta_l \mu$, $\delta_l \nu$ and $\delta_l (f \cdot g^{-1})$.

As a corollary, for a connected manifold M two maps $f, g : M \to G$ satisfy $\delta_l f = \delta_f g$ if and only if $f = c \cdot g$ for some $c \in G$. There exists also a criterion for determining whether a $\mathfrak{g}$-valued one-form is a left logarithmic derivative of a map into G. This generalizes the integral calculus of functions.

PROBLEM 10.18. Let $\tilde{G}$ be the universal covering of G. Show that $\pi_1 G \subseteq \tilde{G}$ is a discrete and normal subgroup thus lying in the centre of $\tilde{G}$.

PROBLEM 10.19. Show that the image of the adjoint representation $\mathrm{Ad} : \mathrm{Sp}(1) \to \mathrm{GL}(3,\mathbb{R})$ is $\mathrm{SO}(3,\mathbb{R})$ and that its kernel is the subgroup $\{\pm 1\}$. Thus $\mathrm{Sp}(1)$ is the 2-fold (universal) covering of $\mathrm{SO}(3,\mathbb{R})$.

PROBLEM 10.20. Let $\varphi : \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{SO}(4,\mathbb{R})$ be the map sending (a, b) to the orthogonal transformation of the quaternions $x \mapsto axb^{-1}$. Show that this map is a 2-fold (universal) covering.

PROBLEM 10.21. Compute the centre of $\mathrm{SO}(n,\mathbb{R})$ or even better its centralizer in $\mathrm{GL}(n,\mathbb{R})_+$, i.e. $C_{\mathrm{SO}(n,\mathbb{R})}\,\mathrm{GL}(n,\mathbb{R})_+$. Try to determine all connected Lie groups with Lie algebra $\mathfrak{so}(n,\mathbb{R})$.

PROBLEM 10.22. Try to determine the first few terms in the Baker-Campbell-Hausdorff formula for

$$\log(\exp X \cdot \exp Y) : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

where log is the (locally defined) inverse to exp in the case $\mathfrak{g} = \mathfrak{gl}(n)$.

A semidirect product of groups is a split short exact sequence

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] & G \arrow[r, shift left, "p"] & H \arrow[l, shift left, "i"] \arrow[r] & 1 \end{tikzcd}$$

The subgroup $K \subseteq G$ is normal being a kernel of p. The map $f : H \xrightarrow{i} G \xrightarrow{\mathrm{int}} \mathrm{Aut}(K)$ given by $f(x)(y) = xyx^{-1}$ is a group homomorphism. For $a \in G$ there are uniquely determined $k \in K$ and

$h \in H$ such that $a = k \cdot i(h)$. Namely $h = p(a)$ and $k = a \cdot i(h)^{-1}$. Therefore as sets $G \cong K \times H$ and the multiplication is given by

$$(k_1, h_1) \cdot (k_2, h_2) = k_1 \cdot i(h_1) \cdot k_2 \cdot i(h_2) = k_1 \cdot f(h_1)(k_2) \cdot i(h_1 h_2) = (k_1 \cdot f(h_1)(k_2), h_1 \cdot h_2)$$

The resulting group is denoted by $K \rtimes H = K \rtimes_f H$.

Problem 10.23. Show that $\mathrm{GA}(n, \mathbb{R})$ is a semidirect product $\mathrm{GA}(n, \mathbb{R}) \cong \mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{R})$ where the action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$ is the standard one.

Problem 10.24. Let G be a Lie group. Show that $\mu_* : TG \times TG \to TG$ endows TG with a structure of a Lie group.

Problem 10.25. Show that TG is a semidirect product $TG \cong \mathfrak{g} \rtimes G$ and identify the involved action of G on $\mathfrak{g}$.

Problem 10.26. Compute the Lie algebra of a semidirect product $K \rtimes_f H$.

Problem 10.27. Determine the Lie algebra of TG.

CHAPTER 2

Bundles

1. Bundles

The tangent bundle $p : TM \to M$ has the following property

$$(\forall x \in M)(\exists U \ni x \text{ nbhd}) : p^{-1}(U) \cong U \times \mathbb{R}^m$$

Definition 1.1. By a *bundle* (or fibre bundle) we understand a triple (E, p, M) where E and M are smooth manifolds and $p : E \to M$ is a smooth surjective[1] map such that for each $x \in M$ there exists its neighbourhood U and a diffeomorphism $\varphi : p^{-1}(U) \cong U \times F$ with F some smooth manifold and such that

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

commutes. The space E is called the *total space*, M the *base*, p the projection, $E_x = p^{-1}(x)$ the *fibre* over $x \in M$ and F the *standard fibre*.

Definition 1.2. The bundle $pr_1 : M \times F \to M$ is said to be trivial (or product). The map $\varphi : p^{-1}(U) \cong U \times F$ is referred to as a *local trivialization*.

Theorem 1.3. *Let $H \leq G$ be a closed subgroup of a Lie group G. Then the projection $G \to G/H$ is a bundle with standard fibre H.*

Proof. This is exactly the proof of Theorem 6.3. □

Examples 1.4.

- TS^2 is not globally trivial ("nelze učesati ježka").
- The Möbius band $\mathbb{R} \to L \to S^1$ is also globally nontrivial.
- The Hopf bundle: let $S^3 \subseteq \mathbb{H} = \mathbb{C}^2$ be the group of unit quaternions. The complex units S^1 form a subgroup of S^3 and the Hopf bundle is

  $$S^1 \to S^3 \to S^3/S^1 \cong \mathbb{CP}^1 \cong S^2$$

  as $S^2 = \mathbb{C} \cup \{\infty\}$. Again the bundle is not trivial: $\pi_1 S^3 = 0$ while

  $$\pi_1(S^1 \times S^2) \cong \pi_1 S^1 \times \pi_1 S^2 \cong \mathbb{Z}$$

  More generally the Hopf bundle $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ is nontrivial.

Let us consider a bundle $p : E \to M$, i.e. we have a cover $U_\alpha \subseteq M$ and local trivializations $\varphi_\alpha : p^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times F$. Denoting $U_{\alpha\beta} = U_\alpha \cap U_\beta$ we obtain

$$\begin{tikzcd} U_{\alpha\beta} \times F \arrow[dr] & p^{-1}(U_{\alpha\beta}) \arrow[l, "\varphi_\alpha", "\cong"'] \arrow[r, "\varphi_\beta", "\cong"'] \arrow[d] & U_{\alpha\beta} \times F \arrow[dl] \\ & U_{\alpha\beta} & \end{tikzcd}$$

[1] In principle surjectivity is not essential.

composing to

$$\begin{tikzcd} \varphi_{\alpha\beta} : U_{\alpha\beta} \times F \arrow[rr, "\cong"] \arrow[dr] & & U_{\alpha\beta} \times F \arrow[dl] \\ & U_{\alpha\beta} & \end{tikzcd}$$

Easily

- $\varphi_{\alpha\beta} = (\varphi_{\beta\alpha})^{-1}$,
- $\varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} = \varphi_{\alpha\gamma}$ over $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$ (the cocycle condition) and
- $\varphi_{\alpha\alpha} = \mathrm{id}$.

On the other hand given a covering U_α and a collection of maps $\varphi_{\alpha\beta}$ satisfying the above conditions there exists a bundle $p : \Phi \to M$ obtained from $S = \bigsqcup_\alpha U_\alpha \times F$ by passing to the quotient $\Phi = S/\sim$ by the relation

$$U_\alpha \times F \ni (x, a) \sim (x, \varphi_{\alpha\beta}(a)) \in U_\beta \times F$$

whenever $x \in U_{\alpha\beta}$.

Definition 1.5. A bundle $p : E \to M$ is called a *vector bundle* if each fibre E_x is given a vector space structure and local trivializations $\varphi : p^{-1}(U) \xrightarrow{\cong} U \times \mathbb{R}^k$ could be chosen in such a way that each $E_x \xrightarrow{\cong} \{x\} \times \mathbb{R}^k \cong \mathbb{R}^k$ is a linear isomorphism.

Examples 1.6.

- TM, T^*M - the tangent and cotangent bundles,
- For a submanifold $M \subseteq \mathbb{R}^n$ the normal bundle is
$$\nu(M) = \{(x, v) \mid x \in M, v \in T_xM^\perp \subseteq \mathbb{R}^n\},$$
- Let $p : E \to M$ be any bundle. The vertical tangent bundle $VE \subseteq TE$ is “the kernel of p_*”, $V_yE = T_yE_{p(y)}$,
- Consider the Grassmann manifold
$$G_k(\mathbb{R}^n) = \mathrm{O}(n)/\,\mathrm{O}(k) \times \mathrm{O}(n-k)$$
of linear subspaces of $\mathbb{R}^n$ of dimension k. Over $G_k(\mathbb{R}^n)$ we have a canonical vector bundle $\gamma_k^n \to G_k(\mathbb{R}^n)$ where
$$\gamma_k^n = \{(V, v) \in G_k(\mathbb{R}^n) \times \mathbb{R}^n \mid v \in V\}.$$
For example γ_1^2 is the Möbius band.

The transition maps $\varphi_{\alpha\beta} : U_{\alpha\beta} \times \mathbb{R}^k \to U_{\alpha\beta} \times \mathbb{R}^k$ take form

$$(x, v) \mapsto (x, \psi_{\alpha\beta}(x) \cdot v)$$

where $\psi_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(k)$ is a smooth map: the (i, j)-entry of $\psi_{\alpha\beta}(x)$ is the i-th coordinate of the second component of $\varphi_{\alpha\beta}(x, e_j)$. The cocycle condition $\psi_{\beta\gamma}\psi_{\alpha\beta} = \psi_{\alpha\gamma}$ is expressed via multplication in $\mathrm{GL}(k)$.

Remark. $\mathrm{GL}(k) \subseteq \mathrm{Diff}(\mathbb{R}^k)$. A general bundle has $\mathrm{Diff}(F)$ as a “structure group”.

Every bundle projection is a submersion: locally it is a projection. The converse is not true.

Theorem 1.7 (Ehresmann). *If $p : E \to M$ is a proper surjective submersion then it is a bundle.*

Proof. Let us identify some neighbourhood of $x \in M$ with a disc whose centre is x. By properness, $p^{-1}(D^m)$ is compact and hence every vector field is complete (when we take care of the boundary). Consider now $\frac{\partial}{\partial x_i}$ and lift it to a vector field X_i on $p^{-1}(D^m)$, i.e. X_i is such that $p_*(X_i) = \frac{\partial}{\partial x_i}$. This is possible locally by p being a submersion and globally is achieved by a partition of unity. Define

$$\begin{aligned} \varphi : D^m \times p^{-1}(0) &\longrightarrow p^{-1}(D^m) \\ (t_1, \dots, t_m, y) &\longmapsto \mathrm{Fl}_{t_1}^{X_1} \cdots \mathrm{Fl}_{t_m}^{X_m}(y) \end{aligned}$$

which is well-defined by the completeness - it lies over

$$\mathrm{Fl}_{t_1}^{\partial/\partial x_1} \cdots \mathrm{Fl}_{t_m}^{\partial/\partial x_m}(0) = (t_1, \ldots, t_m)$$

by the p-relatedness of X_i and $\frac{\partial}{\partial x_i}$. It is easy to verify that φ is a local diffeomorphism at $\{0\} \times p^{-1}(0)$, it is identity on $\{0\} \times p^{-1}(0)$ and $\frac{\partial}{\partial t_i}\varphi = X_i$ there. Since $p^{-1}(0)$ is compact, φ is a diffeomorphism onto its image on some neighbourhood $U \times p^{-1}(0)$. The surjectivity follows by integrating backwards, namely y is the image of $\Big(p(y), \mathrm{Fl}_{-p(y)_m}^{X_m} \cdots \mathrm{Fl}_{-p(y)_1}^{X_1}(y)\Big)$. $\square$

2. Basic operations with bundles

Definition 2.1. Let $p : E \to M$ and $p' : E' \to M'$ be bundles. A pair of maps $f : E' \to E$ and $\underline{f} : M' \to M$ is called a *morphism* if the diagram

$$\begin{tikzcd} E' \arrow[r, "f"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ M' \arrow[r, "\underline{f}"'] & M \end{tikzcd}$$

commutes or in other words if f preserves fibres, $f(E'_x) \subseteq E_{\underline{f}(x)}$. This determines $\underline{f}$ and is automatically smooth when f is. If moreover $M = M'$ and $\underline{f} = \mathrm{id}_M$ then f is said to be *base-preserving*.

Definition 2.2. A *product* of bundles p and p' is $p \times p' : E \times E' \to M \times M'$ with standard fibre $F \times F'$.

Definition 2.3. An *induced bundle* (or pullback) from p along a smooth map $g : M' \to M$ is the submanifold[2]

$$g^*E = \{(z, y) \in M' \times E \mid g(z) = p(y)\} \subseteq M' \times E$$

together with the projection onto the first factor. We have a diagram

$$\begin{tikzcd} g^*E \arrow[r, "pr_2"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ M' \arrow[r, "g"'] & M \end{tikzcd}$$

The universal property

$$\begin{tikzcd} E' \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & g^*E \arrow[r, "pr_2"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ & M' \arrow[r, "g"'] & M \end{tikzcd}$$

can be expressed by saying that a morphism from E' to E is the same as a base-preserving morphism from E' to the induced bundle f^*E.

If $i : N \to M$ is a submanifold inclusion then $i^*E = E|_N$ is the restriction of E to N, i.e. $i^*E \cong p^{-1}(N)$.

Definition 2.4. Let $p : E \to M$ and $p' : E' \to M$ be bundles over the same base. Their *fibre product* is

$$E \times_M E' = \Delta^*(E \times E') = (E \times E')|_\Delta$$

[2]This is so since g is transverse to any submersion p.

where $\Delta : M \to M \times M$ is the diagonal.

$$\begin{tikzcd}
E \times_M E' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
E' \arrow[r, "p'"'] & M
\end{tikzcd}$$

It is the categorical product in the category of bundles over the fixed base M.

Theorem 2.5. *If two maps $g_0, g_1 : M' \to M$ are homotopic then the induced bundles $g_0^* E$ and $g_1^* E$ are isomorphic.*

Proof. See Differential topology lecture notes. $\square$

Theorem 2.6. *Every bundle over $\mathbb{R}^n$ is trivial.*

Proof. The identity map $\mathrm{id}_{\mathbb{R}^n}$ on $\mathbb{R}^n$ is homotopic to the constant map 0. By the previous theorem

$$E \cong \mathrm{id}_{\mathbb{R}^n}^* E \cong 0^* E \cong \mathbb{R}^n \times p^{-1}(0)$$

giving a global trivialization. $\square$

Definition 2.7. A *section* of a bundle $p : E \to M$ is a smooth map $s : M \to E$ for which $p \circ s = \mathrm{id}_M$.

Examples 2.8.

- A section of TM is a vector field, a section of T^*M is a 1-form.
- A section of a trivial bundle $M \times F \to M$ is a smooth map $M \to F$.

Definition 2.9. A *local section* is a smooth map $s : U \to E$ satisfying $p \circ s = \mathrm{id}_U$ where $U \subseteq M$ is an open subset.

Example 2.10. Local sections always exist (since $F \neq \emptyset$) global sections need not. Define $\mathring{T}M = TM - \{(x, 0) \mid x \in M\}$, the space of all nonzero vectors. Easily $\mathring{T}M$ is a bundle over M and a global section of $\mathring{T}M$ is a nowhere zero vector field which does not exist for example on S^2.

Theorem 2.11. *If the standard fibre is diffeomorphic to $\mathbb{R}^k$ then global sections always exist.*

Proof. Local sections are glued together via a partition of unity (which has to be utilized in a chart). More precisely one inductively extends a section, starting with a local section in a bundle chart... FINISH!!! $\square$

Let s and s' be sections of $p : E \to M$ and $p' : E \to M$ respectively. They determine a section (s, s') of the fibre product $E \times_M E'$. A section s of p determines a section $g^* s$ of any induced bundle $g^* E$:

$$\begin{tikzcd}
M' \arrow[dr, dashed, "g^* s"'] \arrow[drr, bend left, "s \circ g"] \arrow[ddr, bend right, "\mathrm{id}_{M'}"'] & & \\
& g^* E \arrow[r, "pr_2"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
& M' \arrow[r, "g"'] & M
\end{tikzcd} \qquad g^* s : z \mapsto (z, sf(z))$$

More generally any map $t : M' \to E$ satisfying $p \circ t = g$ (a section of E along g) induces a section of the induced bundle $g^* E$. In fact this describes a bijection between sections along g and sections of the induced bundle.

Let now $p : E \to M$ and $p' : E' \to M'$ be vector bundles. A morphism $f : E' \to E$ is called *linear* if every $f|_{E'_x} : E'_x \to E_{\underline{f}(x)}$ is a linear map. Locally

$$\begin{tikzcd} U \times \mathbb{R}^k \arrow[r, "f"] \arrow[d] & V \times \mathbb{R}^l \arrow[d] \\ U \arrow[r, "\underline{f}"'] & V \end{tikzcd} \qquad f(x,v) = (\underline{f}(x), g(x)v)$$

where $g : U \to \hom(\mathbb{R}^k, \mathbb{R}^l)$ is a smooth map as $g(x)_{ij} = f_2(x, e_j)_i$.

Let $p : E \to M$ be a vector bundle, $\{U_\alpha\}$ a cover of M and $\varphi_{\alpha\beta}(x,v) = (x, \psi_{\alpha\beta}(x)v)$ the transition maps with $\psi_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(V)$ smooth into the group of linear automorphisms of the standard fibre V. Let there be given a homomorphism $f : \mathrm{GL}(V) \to \mathrm{GL}(W)$ (e.g. $W = V^{\otimes k}, S^k V, \Lambda^k V$). The compositions $f \circ \psi_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(W)$ then yield back a vector bundle with standard fibre W which we denote $f(E)$. In the construction of the dual vector bundle we obtain from $\varphi_{\alpha\beta}$ a linear map

$$U_{\alpha\beta} \times V^* \xleftarrow{\varphi^*_{\alpha\beta}} U_{\alpha\beta} \times V^*$$

going in the wrong direction. This is remedied by considering its inverse. In general we may pass from a homomorphism $f : \mathrm{GL}(V)^{\mathrm{op}} \to \mathrm{GL}(W)$ to the composition $\mathrm{GL}(V) \xrightarrow{f} \mathrm{GL}(W)^{\mathrm{op}} \xrightarrow{\nu} \mathrm{GL}(W)$ and apply the previous construction to get a vector bundle $f(E)$ with standard fibre W. Examples are $E^*, \bar{E}$. The most general case is that of a homomorphism

$$f : \mathrm{GL}(U_1)^{\mathrm{op}} \times \cdots \times \mathrm{GL}(U_k)^{\mathrm{op}} \times \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_l) \longrightarrow \mathrm{GL}(W)$$

which produces a vector bundle $f(E_1, \ldots, E_k, F_1, \ldots, F_l)$ from arbitrary vector bundles $E_1 \ldots, E_k$, $F_1, \ldots, F_l$ with standard fibres $U_1, \ldots, U_k, V_1, \ldots, V_l$.

Example 2.12. The vector bundle $\hom(E,F)$ has as fibres $\hom(E,F)_x = \hom(E_x, F_x)$ and as a special case $\hom(E, \mathbb{R}) = E^*$ where $\mathbb{R}$ here stands for the trivial bundle $M \times \mathbb{R} \to M$. This example is obtained from the general construction via the homomorphism

$$\begin{aligned} \mathrm{GL}(U)^{\mathrm{op}} \times \mathrm{GL}(V) &\longrightarrow \mathrm{GL}(\hom(U,V)) \\ (\alpha, \beta) &\longmapsto (\varphi \mapsto \beta \circ \varphi \circ \alpha) \end{aligned}$$

3. Jet bundles

Let us consider the algebra $C^\infty(\mathbb{R}^n)$ of smooth maps on $\mathbb{R}^n$. By the inductive use of the formula

$$g(x) = g(0) + \sum_{i=1}^n a_i(x) x_i$$

for a function $g : \mathbb{R}^n \to \mathbb{R}$ we derive

$$g = T_r g + R_r g$$

a decomposition of g into its Taylor polynomial $T_r g$ of order r and a remainder lying in the ideal $\mathfrak{m}_0^{r+1}$ generated by the monomials $x^I = x_1^{i_1} \cdots x_n^{i_n}$ of degree $|I| = i_1 + \cdots + i_n = r+1$. It is the $(r+1)$-st power of the ideal $\mathfrak{m}_0$ generated by the coordinate functions. The association of the Taylor polynomial or order r gives a surjective linear map

$$T_r : C^\infty(\mathbb{R}^n) \twoheadrightarrow P_r(\mathbb{R}^n)$$

onto the vector space of all polynomials of order at most r on $\mathbb{R}^n$. Clearly the kernel is the ideal $\mathfrak{m}_0^{r+1}$ and hence $P_r(\mathbb{R}^n)$ is naturally isomorphic to the quotient algebra $C^\infty(\mathbb{R}^n)/\mathfrak{m}_0^{r+1}$. The multiplication in this algebra is the truncated multiplication of polynomials. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a smooth map sending 0 to 0. Then f induces by composition an algebra homomorphism

$$f^* : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^m)$$

with the property $f^*(\mathfrak{m}_0) \subseteq \mathfrak{m}_0$ and thus $f^*(\mathfrak{m}_0^{r+1}) \subseteq \mathfrak{m}_0^{r+1}$.

$$\begin{tikzcd} C^\infty(\mathbb{R}^n) \arrow[r, "f^*"] \arrow[d, "T_r"'] & C^\infty(\mathbb{R}^m) \arrow[d, "T_r"] \\ P_r(\mathbb{R}^n) \arrow[r, dashed, "f^*"'] & P_r(\mathbb{R}^m) \end{tikzcd} \qquad \begin{tikzcd} g \arrow[r, mapsto] \arrow[d, mapsto] & g \circ f \arrow[d, mapsto] \\ T_r g \arrow[r, mapsto] & T_r(g \circ f) \end{tikzcd}$$

Therefore $T_r(g \circ f)$ only depends on $T_r g$ rather than on g. Since $P_r(\mathbb{R}^n)$ is generated as an algebra by the coordinate functions $x_1, \ldots, x_n$ we have $f^*(x_i) = T_r(x_i \circ f) = T_r(f_i)$, the Taylor polynomial of order r of the i-th component f_i. Therefore if f and f' have the same Taylor polynomial of order r then $f^* = (f')^*$ on $P_r(\mathbb{R}^n)$ and thus $T_r(g \circ f) = T_r(g \circ f')$ only depends on $T_r f$.

We have just proved that the Taylor polynomial of order r of a composition $g \circ f$ of maps g and f depends only on their respective Taylor polynomials as long as they preserve the origin. In particular we have

THEOREM 3.1. *The property of having the same Taylor polynomial of order r for maps $(\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ does not depend on the coordinates (as long as their changes preserve the origins).*

DEFINITION 3.2. Let M and N be two manifolds and $f, f' : M \to N$ two maps defined in a neighbourhood of $x \in M$. We say that f and f' *determine the same r-jet at x* (with $r \in \mathbb{N}$) if $f(x) = f'(x) = y$ and for some (any) pair of charts φ on M cenetered at x and ψ on N centered at y the maps $\psi^{-1} f \varphi$ and $\psi^{-1} f' \varphi$ have the same Taylor polynomial of order r at the origin. We write $j^r_x f$ for the class determined by the map f and

$$J^r(M, N) = \{ j^r_x f \mid x \in M, f : M \to N \text{ defined in a neighbourhood of } x \}.$$

For $X = j^r_x f$ we write $\alpha X = x$ for the *source* and $\beta X = f(x)$ for the *target* of the r-jet X. Without coordinates we can identify r-jets with source x and target y with algebra homomorphisms

$$C^\infty(N)/\mathfrak{m}_y^{r+1} \longrightarrow C^\infty(M)/\mathfrak{m}_x^{r+1}$$

There are obvious canonical projections $\pi^r_s : J^r(M, N) \to J^s(M, N)$ for $0 \leq s \leq r$. For $s = 0$ we have $J^0(M, N) \cong M \times N$ via the map (α, β). Therefore $\pi^r_0 = (\alpha, \beta)$. We denote

$$J^r_x(M, N) = \alpha^{-1}(x), \quad J^r(M, N)_y = \beta^{-1}(y), \quad J^r_x(M, N)_y = \alpha^{-1}(x) \cap \beta^{-1}(y)$$

the last being the fibre of $J^r(M, N)$ over $(x, y) \in M \times N$ via (α, β).

For $X \in J^r_x(M, N)_y$ and $Y \in J^r_y(N, Q)_z$ we define their composition $Y \circ X \in J^r_x(M, Q)_z$ either as a composition of algebra homomorphisms or via representatives $Y \circ X = j^r_x(g \circ f)$ if $X = j^r_x f$ and $Y = j^r_y g$.

DEFINITION 3.3. We say that $X \in J^r_x(M, N)_y$ is *invertible* if there exists $X^{-1} \in J^r_y(N, M)_x$ for which $X^{-1} \circ X = j^r_x \operatorname{id}_M$ and $X \circ X^{-1} = j^r_y \operatorname{id}_N$.

For $r \geq 1$ we obtain X is invertible iff its linear part $\pi^r_1 X$ is invertible. In particular for this to happen we must have $m = n$.

Let us denote $L^r_{m,n} = J^r_0(\mathbb{R}^m, \mathbb{R}^n)_0$ which we know can be identified with $\hom_{\mathrm{alg}}(P_r(\mathbb{R}^n), P_r(\mathbb{R}^m))$ or with the set of polynomials of order at most r and without constant term, $X = \sum_{1 \leq |I| \leq r} a_I x^I$. Here $a_I \in \mathbb{R}^n$ are constant. The composition of jets

$$L^r_{n,q} \times L^r_{m,n} \to L^r_{m,q}$$

is the truncated composition of polynomials (i.e. the normal composition followed by ignoring all the terms of order bigger than r). In particular it is smooth and

$$G^r_m = \operatorname{inv}(L^r_{m,m})$$

is therefore a Lie group with respect to the composition of jets, invertible jets forming an open subset (they are those where $a_1, \ldots, a_m$ are linearly independent). As a special case $G^1_m = \mathrm{GL}(m)$.

Let us consider now $X \in L^r_{m,n}$ and consider a translation by v

$$\lambda_v : x \mapsto x + v$$

The following are mutually inverse diffeomorphisms

$$\begin{aligned} \mathbb{R}^m \times L^r_{m,n} \times \mathbb{R}^n &\xrightarrow{\cong} J^r(\mathbb{R}^m, \mathbb{R}^n) \\ (u, X, v) &\longmapsto j^r_0 \lambda_v \circ X \circ j^r_u \lambda_{-u} \\ (u = \alpha Y, j^r_v \lambda_{-v} \circ Y \circ j^r_0 \lambda_u, v = \beta Y) &\longleftarrow Y \end{aligned}$$

Now we can define on $J^r(M,N)$ a smooth structure so that the projection

$$(\alpha, \beta) : J^r(M,N) \to M \times N$$

becomes a bundle. We choose charts on $U \subseteq M$ and $V \subseteq N$ giving us an identification

$$\alpha^{-1}U \cap \beta^{-1}(V) \cong U \times L^r_{m,n} \times V$$

Declaring these to be diffeomorphisms we are left to show that the effect of another choice of charts differs by a diffeomorphism preserving the projection onto $U \times V$. But this is rather easy to see using the concrete description of the involved maps.

A smooth map $f : M \to N$ induces a section $j^r f : M \to J^r(M,N)$ sending $x \mapsto j^r_x f$ of the bundle $J^r(M,N) \xrightarrow{\alpha} M$.

Example 3.4. For $r = 1$ we have $J^1(M,N) \cong \hom(TM, TN)$ or rather $\hom(p^*TM, q^*TN)$ with $p : M \times N \to M$ and $q : M \times N \to N$ the two projections. The map in one direction is provided by $j^1_x f \mapsto T_x f$ and is a diffeomorphism by an inspection in charts. As special cases $J^1_0(\mathbb{R}, M) \cong TM$ and $J^1(M, \mathbb{R})_0 \cong T^*M$.

We denote by $T^r_k M = J^r_0(\mathbb{R}^k, M) \xrightarrow{\beta} M$ the bundle which we call the bundle of k-dimensional velocities of order r. In particular $T^r_1 M$ is called the tangent bundle of order r. A smooth map $f : M \to N$ induces a morphism of bundles $T^r_k f : T^r_k M \to T^r_k N$ via the composition $j^r_0 g \mapsto j^r_0(f \circ g)$

$$\begin{tikzcd} T^r_k M \arrow[r, "T^r_k f"] \arrow[d, "\beta"'] & T^r_k N \arrow[d, "\beta"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

Dually $T^{r*}_k M = J^r(M, \mathbb{R}^k)_0$, the bundle of k-dimensional covelocities of order r. In particular $T^{r*}_1 M$ is called the cotangent bundle of order r. The bundle $T^{r*}_k M$ is a vector bundle with respect to the addition $j^r_x \varphi + j^r_x \psi = j^r_x(\varphi + \psi)$ and multiplication $\lambda \cdot j^r_x \varphi = j^r_x(\lambda \varphi)$, $\lambda \in \mathbb{R}$. On the other hand only local diffeomorphisms induce morphisms of bundles:

$$\begin{tikzcd} T^{r*}_k M \arrow[r, "T^{r*}_k f"] \arrow[d, "\alpha"'] & T^{r*}_k N \arrow[d, "\alpha"] \\ M \arrow[r, "f"'] & N \end{tikzcd} \qquad j^r_x \varphi \mapsto j^r_{f(x)}(\varphi \circ f^{-1})$$

Remark. For any smooth f we have a map on the section spaces

$$\Gamma(T^{r*}_k N) \xrightarrow{f^*} \Gamma(T^{r*}_k M)$$

Let $P^r M = \operatorname{inv} J^r_0(\mathbb{R}^m, M) \subseteq T^r_m M$ with $m = \dim M$ denote the “bundle of r-jets of maps $(\mathbb{R}^m, 0) \to (M, x)$”. The group $G^r_m = \operatorname{inv} J^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$ acts on $P^r M$ from the right via the jet composition: for a map $u : \mathbb{R}^m \to M$ and a change of coordinates $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ we have a new map $u \circ \varphi : \mathbb{R}^m \to M$

$$\begin{aligned} P^r M \times G^r_m &\longrightarrow P^r M \\ (j^r_0 u, j^r_0 \varphi) &\longmapsto j^r_0(u \circ \varphi) \end{aligned}$$

The situation is summarized in: P^rM is a bundle, the action of G^r_m preserves the fibres and is simply transitive on each of them: for j^r_0u and j^r_0v with $u(0) = v(0)$ there exists a unique $a \in G^r_m$ for which $j^r_0v = j^r_0u \cdot a$.

Example 3.5. For $r = 1$: $P^1M = \operatorname{inv}\hom(\mathbb{R}^m, TM)$ which at the fibre over $x \in M$ is the same as a basis of T_xM (namely the image of the standard basis in $\mathbb{R}^m$). We say that P^1M is the bundle of frames in TM. We then think of P^rM as a bundle of higher order frames.

Definition 3.6. Let $p : E \to M$ be a bundle. The r-th jet prolongation J^rE is the space of all jets of local sections of p. It is a manifold and bundle over M. One can either see this locally - a local section is equivalent to a map $U \to F$ and thus J^rE is locally in bijection with $J^r(U, F)$ but it is not quite obvious what the transition maps look like. A global definition is via the pullback diagram

$$\begin{tikzcd} J^rE \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & J^r(M,E) \arrow[d, "p_*"] \\ M \arrow[r, hook, "j^r\,\mathrm{id}"'] & J^r(M,M) \end{tikzcd}$$

describing it as a restriction of $J^r(M, E) \to J^r(M, M)$ along $j^r\,\mathrm{id}$. Locally

$$J^r(M,E) \cong J^r(U,V) \times_U J^r(U,F) \longrightarrow J^r(U,V)$$

which is a bundle and the restriction "forgets the first component" to get $J^r(U, F)$.

A prolongation of sections: $s : M \to E$ induces $j^rs : M \to J^rE$ but not every section of $J^rE \to M$ comes from a section of $E \to M$.

Remark. A differential equation/inequation (relation) is a subset $R \subseteq J^rE$. A solution of R is a section $s : M \to E$ for which $j^r_xs \in R$ for all $x \in M$. A formal solution is a section of $J^rE \to M$ with image in R. The jet prolongation restricts by definition to a map $\mathrm{sol} \to \mathrm{fsol}$ between the space of solutions and the space of formal solutions with fsol being much bigger. Nevertheless this map is quite often a homotopy equivalence.

4. Principal and associated bundles

Definition 4.1. Let us consider a bundle $\pi : P \to M$ and a Lie group G having a right action $r : P \times G \to P$ on P. We say that P is a *principal bundle* with a structure group G if

- the action r preserves fibres, $\pi(u \cdot a) = \pi(u)$ and
- G acts on each fibre P_x simply transitively, $u, v \in P_x \Rightarrow \exists! a \in G : v = u \cdot a$.

We write $P(M, G)$ to mean that P is a principal bundle over M with structure group G. We also say that P is a principal G-bundle.

Theorem 4.2. *Let $H \leq G$ be a closed subgroup of a Lie group G. Then the projection $G \to G/H$ is a principal H-bundle.*

Proof. This is contained in the proof of Theorem 6.3. □

Examples 4.3.

- The frame bundle $P^rM(M, G^r_m)$.
- Consider a vector bundle $E \to M$ with standard fibre $\mathbb{R}^k$. Denote by $PE \to M$ the following bundle over M

$$PE = \operatorname{inv}\hom(\mathbb{R}^k, E) \subseteq \hom(\mathbb{R}^k, E) \cong \underbrace{E \times_M \cdots \times_M E}_{k \text{ times}}$$

In the last isomorphism we identify $(u_1, \ldots, u_k)$ with a unique linear map sending e_i to u_i. Clearly this map is invertible iff $u_1, \ldots, u_k$ are linearly independent. The right action of $\mathrm{GL}(k)$ is either via composition $u \cdot a = u \circ a$ or as $(u \cdot a)_i = \sum_j u_ja_{ji}$. We obtain a principal bundle $PE(M, \mathrm{GL}(k))$ of frames in the vector bundle E.

A local section $s : U \to P$ determines a trivialization $\pi^{-1}(U) \cong U \times G$ in the following way

$$\begin{aligned} U \times G &\longrightarrow \pi^{-1}(U) \\ (x, a) &\longmapsto s(x) \cdot a \end{aligned}$$

This is easily a smooth bijection. We need to verify that it is a local diffeomorphism. This is so because the restriction to $U \times \{a\}$ is a section and hence an immersion. The restriction to $\{x\} \times G$ is an immersion by Theorem 6.5. The images of the respective derivatives are complementary. Another feature of this trivialization is that it is *equivariant.*

Alternatively we may thus characterize principal G-bundles as right G-spaces P for which there exists in a neighbourhood of every point an equivariant diffeomorphism with $\mathbb{R}^m \times G$.

Theorem 4.4. *A principal bundle is trivial if and only if it admits a global section.*

Proof. Obvious from the preceding arguments. $\square$

Definition 4.5. A manifold M^m is called parallelizable if it admits an m-tuple of linearly independent (pointwise) vector fields.

Examples 4.6.

- S^2 is not parallelizable since it does not admit even one linearly independent (i.e. nowhere zero) vector field.
- Every Lie group is parallelizable via left translations: $G \times \mathfrak{g} \to TG$ is given by $(a, A) \mapsto (\lambda_{a*})A$.

Remark. Obviously M is parallelizable if and only if P^1M is trivial.

Theorem 4.7. *The bundle P^rM is trivial if and only if M is parallelizable.*

Proof. A section of P^rM determines by composition $M \to P^rM \xrightarrow{\pi_1^r} P^1M$ a section of P^1M and hence M is parallelizable. Assume on the other hand P^1M admits a global section. The projection $P^rM \to P^1M$ is a bundle with standard fibre $\mathbb{R}^k$, the polynomials of degree at most r with zero linear part which is easily seen locally as the canonical projection $\pi_1^r : G_m^r \to G_m^1$ is a surjective homomorphism of Lie groups hence isomorphic to a projection $G_m^r \to G_m^r / \ker \pi_1^r$ which is a bundle by Theorem 4.2. We know that such bundles always admit sections. The composition $M \to P^1M \to P^rM$ is then a section of P^rM and hence it is trivial. $\square$

The local description of principal bundles via charts and transition maps simplifies as follows

$$\begin{aligned} \varphi_{\alpha\beta} : U_{\alpha\beta} \times G &\longrightarrow U_{\alpha\beta} \times G \\ (x, a) = (x, e)a &\longmapsto \varphi_{\alpha\beta}(x, e)a = (x, \psi_{\alpha\beta}(x)a) \end{aligned}$$

with $\psi_{\alpha\beta} : U_{\alpha\beta} \to G$ smooth. In other words the transition map is a left multiplication by the map $\psi_{\alpha\beta}$. Again we have $\psi_{\alpha\alpha} = e$ and $\psi_{\beta\gamma}\psi_{\alpha\beta} = \psi_{\alpha\gamma}$, the maps form a so-called G-valued cocycle. In the opposite direction from a G-valued cocycle one can construct a principal G-bundle.

We will now address the question of when two principal G-bundles P, P' are isomorphic. Let they be given by transtion maps $\psi_{\alpha\beta}$ and $\psi'_{\alpha\beta}$ respectively. Then $f : P \xrightarrow{\cong} P'$ is locally given by

$$\begin{aligned} f_\alpha : U_\alpha \times G &\longrightarrow U_\alpha \times G \\ (x, a) = (x, e)a &\longmapsto f_\alpha(x, e)a = (x, g_\alpha(x)a) \end{aligned}$$

For a different chart φ_β we have a comparison diagram

$$\begin{tikzcd}
& U_{\alpha\beta} \times G \arrow[r, "f_\alpha"] \arrow[d, "\varphi_{\alpha\beta}"'] \arrow[dl, bend right, "\varphi_\alpha"', to path={-- (\tikztotarget)}] \arrow[ddl, bend right, "\varphi_\alpha"'] & U_{\alpha\beta} \times G \arrow[d, "\varphi'_{\alpha\beta}"] \arrow[ddr, bend left, "\varphi'_\alpha"] & \\
& U_{\alpha\beta} \times G \arrow[r, "f_\beta"] \arrow[dl, "\varphi_\beta"] & U_{\alpha\beta} \times G \arrow[dr, "\varphi'_\beta"'] & \\
P \arrow[rrr, "f"] & & & P'
\end{tikzcd}$$

In the small square we see that (x,a) at top left is mapped to $(x, g_\alpha(x)\psi_{\alpha\beta}(x)a)$ at bottom right via bottom left corner and to $(x, \psi'_{\alpha\beta}(x)g_\beta(x)a)$ via top right corner. Thus we have $\psi'_{\alpha\beta} = g_\alpha \psi_{\alpha\beta} g_\beta^{-1}$.

THEOREM 4.8. *Let $\{U_\alpha\}$ be a cover of M such that both P and P' are trivialized over each U_α. Then $P \cong P'$ if and only if there exist $g_\alpha : U_\alpha \to G$ such that $\psi'_{\alpha\beta} = g_\alpha \psi_{\alpha\beta} g_\beta^{-1}$ (in this case we say that the cocycles are equivalent).*

DEFINITION 4.9. Let $p : E \to M$ be a bundle. A *subbundle* of E is a subspace $E' \subseteq E$ for which there exist local trivializations of E which also trivialize E':

$$\begin{array}{ccc} p^{-1}(U) & \cong & U \times F \\ \cup\!| & & \cup\!| \\ E' \cap p^{-1}(U) & \cong & U \times F' \end{array}$$

DEFINITION 4.10. Let $H \subseteq G$ be a Lie subgroup. A subbundle $Q \subseteq P$ of a principal bundle P is called a *reduction* of P to the subgroup H if for each $u \in Q$ we have $u \cdot a \in Q \iff a \in H$.

EXAMPLES 4.11.

- A reduction to the trivial subgroup $\{e\} \subseteq G$ is the same as a section of P, that is a trivialization of P.
- Consider a Riemannian manifold (M, g). Then $P^1M = PTM$ is a principal $\mathrm{GL}(m)$-bundle possessing a reduction to $O(m)$:

  $$PTM = \operatorname{inv}\hom(\mathbb{R}^m, TM) \supseteq \operatorname{iso}(\mathbb{R}^m, TM),$$

  the subspace of isometries. They are clearly closed under the action of $O(m)$ and more over the action is transitive so that we obtain a reduction to $O(m)$.

  In the opposite direction let $Q \subseteq \operatorname{inv}\hom(\mathbb{R}^m, TM)$ be a reduction to $O(m)$. It defines a metric on M in the following way: every $u \in Q_x$ is an isomorphism $u : \mathbb{R}^m \to T_xM$ and we declare it an isometry or in other words we transport by u the standard metric from $\mathbb{R}^m$. The result does not depend on q.

  More generally metrics on a vector bundle $p : E \to M$ are in bijection with reductions of PE to $O(k)$.
- Consider an arbitrary Lie subgroup $G \leq \mathrm{GL}(m)$. A *G-structure* on a manifold M is a reduction of P^1M to the subgroup G. Similarly for subgroups $G \leq G^r_m$ of higher order frame bundles. A reduction is then called a G-structure of r-th order.

DEFINITION 4.12. Let $P(M, G)$ and $Q(N, H)$ be two principal bundles. A bundle morphism $f : P \to Q$ is called a *morphism of principal bundles* with respect to a homomorphism $\varphi : G \to H$ of Lie groups if

$$(\forall u \in P)(\forall a \in G) : f(u \cdot a) = f(u) \cdot \varphi(a)$$

If $\varphi = \mathrm{id}$ then we speak simply of a morphism of principal bundles or a G-morphism.

EXAMPLES 4.13.

- A reduction $Q \subseteq P$ can be equivalently described as follows: the embedding $Q \to P$ is a morphism of principal bundles with respect to the embedding $H \to G$.
- Let $f : M \to N$ be a local diffeomorphism. Then

  $$J^r_0(\mathbb{R}^m, M) = T^r_m M \xrightarrow{T^r_m f} T^r_m N$$

  restricts to $f_* : P^rM \to P^rN$, a morphism of principal bundles.

Let $P(M, G)$ be a principal bundle and consider a left action $\ell : G \times F \to F$ of G on F.

DEFINITION 4.14. A bundle $p : E \to M$ with a standard fibre F is said to be an *associated bundle* to P if to each $u \in P_x$ there is given a diffeomorphism $\tilde{u} : F \to E_x$ (a so-called *frame map* determined by the frame u on E) such that the total frame map

$$\begin{aligned} \rho : P \times F &\longrightarrow E \\ (u, z) &\longmapsto \tilde{u}(z) \end{aligned}$$

is smooth and $\widetilde{u \cdot a} = \tilde{u} \circ \ell_a$. In terms of the total frame map $\rho(u \cdot a, z) = \rho(u, a \cdot z)$.

Remark. The idea is that we think of the principal bundle as consisting of coordinates choices each of which gives us an identification of the standard fibre F with the geometric fibre E_x. Hence P parametrizes these possible identifications allowing us to make constructions in coordinates in such a way that they automatically do not depend on the choice. EXPLAIN BETTER!

Remark. We will use later ρ to denote a representation. We should therefore CHANGE the above map to q.

Example 4.15. Let $p : E \to M$ be a vector bundle and $PE = \operatorname{inv}\hom(\mathbb{R}^m, E)$ the frame bundle of E, a principal $\mathrm{GL}(m)$-bundle. We will show that E is associated to PE. For that we need an action of $\mathrm{GL}(m)$ on the standard fibre of E. This being $\mathbb{R}^m$ we will use the standard action of $\mathrm{GL}(m)$. Each $u \in (PE)_x$ is by definition an invertible map $\mathbb{R}^m \to E_x$ and this is our frame map $\tilde{u}$. The equivariancy condition is then obvious since

$$\widetilde{u \cdot a} = u \circ a = \tilde{u} \circ \ell_a.$$

Also the total map $PE \times \mathbb{R}^m \to E$ is smooth since it sends $(u, v) \mapsto u(v)$.

Example 4.16. The bundle $\beta : J^r(M, N) \to N$ is associated to $P^r N$. The standard fibre is $J^r(M, \mathbb{R}^n)_0$ and the left action of $G^r_n = \operatorname{inv} J^r_0(\mathbb{R}^n, \mathbb{R}^n)_0$ is by composition. The total frame map is (as $P^r M = \operatorname{inv} J^r_0(\mathbb{R}^n, N)$)

$$\begin{aligned} \operatorname{inv} J^r_0(\mathbb{R}^n, N) \times J^r(M, \mathbb{R}^n)_0 &\longrightarrow J^r(M, N) \\ (u, X) &\longmapsto u \circ X \end{aligned}$$

Again the equivariancy is verified easily.

Example 4.17. Analogously $\alpha : J^r(M, N) \to M$ is associated to $P^r M$ via the action of G^r_m on $J^r_0(\mathbb{R}^m, N)$, $a \cdot X = X \circ a^{-1}$ and $(\alpha, \beta) : J^r(M, N) \to M \times N$ is associated to $P^r M \times P^r N$.

Theorem 4.18. *For a given principal bundle $P(M, G)$ and a G-space F there exist an associated bundle. Any two such are canonically isomorphic.*

Proof. Let us start with any associated bundle E and its total frame map

$$\rho : P \times F \to E$$

By definition ρ factors through $(P \times F)/\sim$ with $\sim$ denoting the equivalence relation $(u \cdot a, z) \sim (u, a \cdot z)$. It is a simple matter to show that the resulting map

$$\tilde{\rho} : P \times F/\sim \to E$$

is a bijection: $\rho(u, z) = \rho(u', z')$ implies that $\pi(u) = \pi(u')$ and hence $u' = u \cdot a$ so that $\rho(u', z') = \rho(u, a \cdot z')$ and hence $z = a \cdot z'$ since $\tilde{u}$ is a diffeomorphism.

We denote the quotient space $P[F] = P \times_G F$ the latter expressing a similarity to the tensor product of modules over a ring. Now we will verify that $P[F]$ bears a canonical smooth structure (as a quotient of $P \times F$) for which the projection $P[F] \to M$ is a bundle with standard fibre F. This is done locally:

$$\begin{aligned} \pi^{-1}(U_\alpha)[F] \xrightarrow{\cong} (U_\alpha \times G) \times_G F &\xrightarrow{\cong} U_\alpha \times F \\ [(x, a), z] &\longmapsto (x, az) \\ [(x, e), z] &\longleftarrow (x, z) \end{aligned}$$

the first arrow being the trivialization $\varphi_\alpha \times \mathrm{id}$. We use these to put a smooth structure on $P[F]$. We are left to exhibit the effect of changing a trivialization:

$$\begin{tikzcd} (x, z) \arrow[d, mapsto] & U_{\alpha\beta} \times F \arrow[r] \arrow[d, phantom, "\cong" rotate=-90] & U_{\alpha\beta} \times F \arrow[d, phantom, "\cong" rotate=-90] & (x, \psi_{\alpha\beta}(x) \cdot z) \\ {[(x, e), z]} \arrow[rrr, mapsto, bend right] & (U_{\alpha\beta} \times G) \times_G F \arrow[r] & (U_{\alpha\beta} \times G) \times_G F & {[(x, \psi_{\alpha\beta}(x)), z]} \arrow[u, mapsto] \end{tikzcd}$$

These are clearly smooth but we see how the associated bundle $P[F]$ is constructed from local charts using the transition maps $U_{\alpha\beta} \xrightarrow{\psi_{\alpha\beta}} G \xrightarrow{\ell} \mathrm{Diff}(F)$.

It remains to show that $P[F]$ is really associated to P. But this is provided by the quotient map $P \times F \longrightarrow P \times_G F = P[F]$. $\square$

REMARK. From now on when we speak about "the associated bundle" we mean the canonical bundle $P[F]$ constructed in the proof.

A particular case is that of a bundle associated to a principal G-bundle P via a representation $\rho : G \to \mathrm{GL}(W)$ of G on a vector space W. In this case $P[W]$ is canonically a vector bundle with standard fibre W.

Let us consider two principal bundles $P(M, G)$ and $Q(N, G)$ and a G-morphism

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & Q \arrow[d] \\ M \arrow[r, "\underline{f}"'] & N \end{tikzcd}$$

with respect to $\varphi : G \to H$. Let $E \to M$ be associated to P and $D \to N$ associated to Q with the same fibre F.

DEFINITION 4.19. We say that a bundle morphism $g : E \to D$ over the same $\underline{f}$ as above is a *morphism associated to* f if for each $u \in P$ the diagram

$$\begin{tikzcd} & F \arrow[dl, "\tilde{u}"'] \arrow[dr, "\widetilde{f(u)}"] & \\ E_x \arrow[rr, "g_x"'] & & D_x \end{tikzcd}$$

commutes.

THEOREM 4.20. *A morphism $g : P[F] \to Q[F]$ associated to f is unique,*

$$g = f[F] : [u, z] \mapsto [f(u), z]$$

$\square$

REMARK. In a similar way one can consider a morphism $f \times h : P[F] \to Q[L]$ with respect to a homomorphism $\varphi : G \to H$ of groups and a G-map $h : F \to L$ between a G-space F and an H-space L.

DEFINITION 4.21. By a *natural bundle* E over m-dimensional manifolds we understand a rule (a functor) which associates to each m-dimensional manifold M a bundle $p_M : EM \to M$ and to each local diffeomorphism $f : M \to N$ a morphism of bundles $Ef : EM \to EN$ over f in such a way that

- localization: for any open subset $U \subseteq M$ we have $EU = EM|_U = p_M^{-1}(U)$,
- functoriality: $E\,\mathrm{id}_M = \mathrm{id}_{EM}$ and $E(g \circ f) = Eg \circ Ef$.

REMARK. From the two properties it follows that Ef is also a local diffeomorphism. The association $f \mapsto Ef$ is called a *lifting* of local diffeomorphisms.

EXAMPLES 4.22.

- The tangent and the cotangent bundles.
- T_k^r, T_k^{r*} or more generally $J^r(-, N)$ and $J^r(M, -)$.
- For a left action ℓ of the group G_m^r on a manifold F we can construct a natural bundle over m-dimensional manifolds as

$$EM = P^r M[F] \to M \qquad (f : M \to N) \mapsto (Ef = P^r f[F])$$

THEOREM 4.23 (Palais-Terng). *For every natural bundle there exists $r \geq 0$, a smooth manifold F and a left action $\ell : G_m^r \times F \to F$ so that $EM = P^r M[F]$ and $Ef = P^r f[F]$.*

5. Further properties of principal and associated bundles

Let $P(M,G)$ be a principal bundle and F a left G-space. A map $\sigma : P \to F$ is called *equivariant* if $\sigma(u \cdot a) = a^{-1} \cdot \sigma(u)$.

Consider a section $s : M \to P[F] = P \times_G F$ of the associated bundle. For each $u \in P$ there is a unique $z = \sigma(u) \in F$ so that $s(x) = [u,z]$ where $x = \pi(u)$. This defines a smooth map $\sigma : P \to F$ which is equivariant by

$$[u, \sigma(u)] = s(x) = [u \cdot a, \sigma(u \cdot a)] = [u, a \cdot \sigma(u \cdot a)]$$

Another point of view is that each $u \in P_x$ gives an identification $\tilde{u} : F \to E_x$ and $\sigma(u)$ is simply $(\tilde{u})^{-1}s(x)$. This also explains why σ should be equivariant.

If on the other hand $\sigma : P \to F$ is equivariant then in the diagram

$$u \longmapsto [u, \sigma(u)]$$

$$\begin{tikzcd} P \arrow[r] \arrow[d] & P \times_G F \\ M \arrow[ur, dashed] & \end{tikzcd}$$

there exists a (unique) factorization since $M = P/G$ and u, $u \cdot a$ are carried both to the same point in $P \times_G F$. This factorization is a section of $P[F]$.

Theorem 5.1. *The above construction describes a bijection between sections of the associated bundle $P[F]$ and equivariant maps $P \to F$.*

Example 5.2. Let $P = P^1M$ and $F = \mathbb{R}^m$ with the standard action of $\mathrm{GL}(m)$. Hence $P^1M[\mathbb{R}^m] = TM$ and a section $X : M \to TM$ (i.e. a vector field) determines an equivariant map $\xi : P^1M \to \mathbb{R}^m$, the so-called frame form. It sends a basis $(u_1, \dots, u_m)$ of T_xM to the coordinates of $X(x)$ in this basis, $u \cdot \xi(u) = X(x)$.

Example 5.3. Morphisms of principal bundles $P \to Q$ are exactly equivariant maps. By the preceding they are in bijection with sections of $P[Q] \to M$.

Let $H \leq G$ be a closed subgroup. The action of G on itself via left translations passes to the quotient G/H. The associated bundle is

$$\begin{aligned} P[G/H] = P \times_G G/H &\xrightarrow{\cong} P/H \\ [u, aH] &\longmapsto (ua)H \\ [u, eH] &\longleftarrow uH \end{aligned}$$

Theorem 5.4. *There is a canonical bijection between sections of $P[G/H]$ and reductions of P to H.*

Proof. Let a section $s : M \to P[G/H]$ determine an equivariant map $\sigma : P \to G/H$. Easily σ is a submersion on every fibre and thus $Q = \sigma^{-1}(eH)$ is the desired reduction.

Let, on the other hand, $Q \subseteq P$ be a reduction to H. Then in the diagram

$$\begin{tikzcd} Q \arrow[r, hook] \arrow[d] & P \arrow[r] & P/H \\ M \arrow[urr, dashed] & & \end{tikzcd}$$

the dotted factorization exists, since $M = Q/H$, providing a section. DETAILS! $\square$

Example 5.5. Let $G \leq \mathrm{GL}(m)$ be the stabilizer of $e_1 \in \mathbb{R}^m$, the group of matrices of the form $\left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right)$. Then $\mathrm{GL}(m)/G \cong \mathbb{R}^m - \{0\}$ and thus reductions of P^1M to G are in bijection with sections of $\mathring{T}M = TM - 0$, the tangent bundle with the zero section removed. These are clearly nowhere zero vector fields.

6. Problems

PROBLEM 6.1. Determine $P[*]$ and $P[G]$.

PROBLEM 6.2. Let P be a principal G-bundle that admits a reduction Q to the subgroup $H \subseteq G$. Show that $P \cong Q \times_H G$ as principal G-bundles where the right G-action on $Q \times_H G$ is $[u, a]b = [u, ab]$.

PROBLEM 6.3. Bundles associated to P are precisely those associated to Q via an action of G.

PROBLEM 6.4. Show that $\mathrm{GL}(m)/\mathrm{O}(m) \cong \mathbb{R}^{\frac{m(m-1)}{2}}$ and apply this to the case of reductions to $\mathrm{O}(m) \subseteq \mathrm{GL}(m)$.

One possibility is to note that the mapping exp induces a diffeomorphism between the manifold of all symmetric matrices and all positively definite matrices (regardless of the fact that these are not Lie algebra/group pair).

PROBLEM 6.5. Show that $\pi^r_{r-1} : J^r(M, N) \to J^{r-1}(M, N)$ is an affine bundle.

This may be solved on the models: $L^r_{m,n} \to L^{r-1}_{m,n}$ is an affine bundle (with a fibre-preserving affine action of $G^r_m \times G^r_n$).

PROBLEM 6.6. Show that $T(G/H) \cong G \times_H \mathfrak{g}/\mathfrak{h}$ where the action of H on $\mathfrak{g}/\mathfrak{h}$ is induced by the adjoint action of H on $\mathfrak{g}$.

PROBLEM 6.7. Show that each sphere S^m is stably parallelizable, i.e. that there exists an isomorphism $TS^m \oplus \mathbb{R}^k \cong \mathbb{R}^{m+k}$ for $k \gg 0$.

PROBLEM 6.8. Show that $T\mathbb{RP}^m$ is stably isomorphic to the direct sum of m copies of the canonical line bundle over $\mathbb{RP}^m$.

PROBLEM 6.9. Show that the canonical bundle over the Stiefel manifold $S_k(\mathbb{R}^n)$ of orthonormal k-frames in $\mathbb{R}^n$ is associated to the trivial representation of $\mathrm{O}(n-k)$ on $\mathbb{R}^k$ while its orthogonal complement is associated to the standard representation of $\mathrm{O}(n-k)$ on $\mathbb{R}^{n-k}$.

PROBLEM 6.10. Show that the Stiefel manifold $S_k(\mathbb{R}^n)$ is parallelizable for $k > 2$.

The main idea is that $TS_k(\mathbb{R}^n) \cong \mathrm{O}(n) \times_{\mathrm{O}(n-k)} \mathfrak{o}(n)/\mathfrak{o}(n-k)$ and the $\mathrm{O}(n-k)$-representation $\mathfrak{o}(n)/\mathfrak{o}(n-k)$ is a direct sum of a trivial representation of dimension $k(k-1)/2$ and k copies of the standard representation $\mathbb{R}^{n-k}$. Then one observes that the sum of a trivial representation of dimension k and the standard representation induces a trivial bundle. Similarly for the Grassmann manifold $G_k(\mathbb{R}^n)$ but this time none of the two bundles is trivial.

PROBLEM 6.11. Let $E \to M$ be a vector bundle associated to a principal $\mathrm{GL}(k)$-bundle P. Define the orientation bundle (a 2-sheeted covering) $P[\mathrm{GL}(k)/\mathrm{GL}_+(k)]$ (which is isomorphic to $(\Lambda^k E - 0)/\mathbb{R}_+$). Show that if M is connected E possesses an orientation if and only if this orientation covering is trivial.

CHAPTER 3

Connections

1. Connections

Let $f : M \to N$ be a smooth map which we think of as a section (id, f) of the trivial bundle $M \times N \to M$. The derivative of f is obtained by differentiating the section and composing with the canonical projection $TM \times TN \to TN$. For a bundle which is not trivial there is no obvious way of projecting onto the tangent space of the fibre. This projection is the content of a connection on the bundle.

Definition 1.1. Let $p : E \to M$ be a bundle. A *connection* on p is a smooth linear projection $v : TE \to VE$ onto the vertical subbundle $VE = \bigcup_{x\in M} TE_x = \ker(p_* : TE \to TM)$.

We call v the *vertical projection.* An associated horizontal projection is $h = \mathrm{id} - v$. There is a short exact sequence of bundles over E

$$0 \to VE \to TE \to p^*TM \to 0$$

A vertical projection, i.e. a retraction of TE onto VE, is equivalent to a section of the projection $TE \to p^*TM$. This is our second definition of a connection.

Definition 1.2. A *connection* on $p : E \to M$ is a "lifting map" $\Gamma : E \times_M TM = p^*TM \to TE$ which is smooth, linear and satisfies $p_*(\Gamma(y, X)) = X$.

Equivalently $\Gamma(y, -)$ is a 1-jet of a section $M \to E$. The mapping $y \mapsto \Gamma(y, -)$ is then a section $E \to J^1E$.

Definition 1.3. A *connection* on $p : E \to M$ is a smooth section $\Gamma : E \to J^1E$ of the jet prolongation $J^1E \to E$.

Remark. The bundle $J^1E \to E$ is affine since $J^1(M, E) \to M \times E$ is a vector bundle, hence so is its pullback along $(p, \mathrm{id}) : E \to M \times E$ and the condition $j^1_y p \circ j^1_x s = j^1_x \mathrm{id}$ is affine.

Theorem 1.4. *Every bundle admits (globally) a connection.*

For our next formulation observe that the lifting map is completely determined by its image, a subbundle of TE.

Definition 1.5. A *connection* on $p : E \to M$ is a smooth distribution Γ on E which at each point $y \in E$ is complementary to the vertical distribution V_yE.

Definition 1.6. A vector field $\xi : E \to TE$ is called *projectable* is there exists a vector field $\underline{\xi} : M \to TM$ such that the diagram

$$\begin{tikzcd} TE \arrow[r, "p_*"] & TM \\ E \arrow[u, "\xi"] \arrow[r, "p"'] & M \arrow[u, "\underline{\xi}"'] \end{tikzcd}$$

commutes, i.e. such that ξ is p-related to $\underline{\xi}$. Loosely speaking from the top one sees only one vector over each point $x \in M$. In coordinates x^i on M and y^p on the fibre

$$\xi = \underbrace{\sum \xi^i(x) \tfrac{\partial}{\partial x^i}}_{\underline{\xi}} + \sum \xi^p(x, y) \tfrac{\partial}{\partial y^p}$$

Definition 1.7. Let $X : M \to TM$ be a vector field and $\tilde{X} : E \to TE$ given by $\tilde{X}(y) = \Gamma(y, X)$ using the lifting map of a connection. Then $\tilde{X}$ is a projectable vector field on E over X. We call this vector field the Γ-lift of X (or the horizontal lift when Γ is understood from the context).

When the section $E \to J^1E$ is given by

$$\mathrm{d}y^p = \sum F_i^p(x, y)\mathrm{d}x^i$$

the horizontal lift is $\tilde{X} = \sum X^i \frac{\partial}{\partial x^i} + \sum F_i^p(x, y)X^i \frac{\partial}{\partial y^p}$

Definition 1.8. Let $p : E \to M$ be a vector bundle. Then so is $J^1E \to M$. A connection $\Gamma : E \to J^1E$ is called *linear* if it is a linear morphism of vector bundles.

In coordinates the function $F_i^p(x, y)$ must be linear in y. We write[1]

$$F_i^p(x, y) = \sum_q \Gamma_{qi}^p(x)y^q.$$

Thus in this case $\mathrm{d}y^p = \sum_{i,q} \Gamma_{qi}^p(x)y^q\mathrm{d}x^i$. The functions Γ_{qi}^p are almost exactly the classical Christofell symbols.

We are now able to write formally the definition of the derivative of a section. Consider an arbitrary connection Γ on a bundle $p : E \to M$ and a section $s : M \to E$. We define

$$\begin{aligned} \nabla_\Gamma s(x) : T_xM &\to V_{s(x)}E \\ X &\mapsto s_*(X) - \tilde{X}(s(x)) \end{aligned}$$

The result lies in the vertical subbundle since both s_*X and $\tilde{X}(s(x))$ are lifts of X. In the first case this follows from the section property. Equivalently $\nabla_\Gamma s(x)$ is the vertical projection $v(s_*X)$ of the derivative s_*X. Using an easy adjunction

$$\nabla_\Gamma s(x) \in V_{s(x)}E \otimes T_x^*M = (VE \otimes p^*(T^*M))_{s(x)}$$

For short we write $VE \otimes T^*M$ instead of $VE \otimes p^*(T^*M)$. It is a bundle over E and by composing with p also over M.

Definition 1.9. The section $\nabla_\Gamma s : M \to VE \otimes T^*M$ is called the *covariant derivative* of s with respect to the connection Γ.

We say, that a section s is parallel, if $\nabla_\Gamma s = 0$.

In coordinates for s given by $y^p = s^p(x)$ we have $s_{*x} = \sum \frac{\partial s^p}{\partial x^i} \cdot \mathrm{d}x^i \frac{\partial}{\partial y^p}$ and further

$$\Gamma(s(x), -) = \sum F_i^p(x, s(x)) \cdot \mathrm{d}x^i \frac{\partial}{\partial y^p}$$

yielding

$$\nabla_\Gamma s(x) = \sum \left(\frac{\partial s^p}{\partial x^i}(x) - F_i^p(x, s(x))\right) \mathrm{d}x^i \frac{\partial}{\partial y^p}$$

Definition 1.10. Let $\gamma : \mathbb{R} \to M$ be a path on M defined in some neighbourhood of 0. A *section* of E *along* γ is a map $s : \mathbb{R} \to E$ for which $p(s(t)) = \gamma(t)$

$$\begin{tikzcd} & E \arrow[d, "p"] \\ \mathbb{R} \arrow[ur, "s"] \arrow[r, "\gamma"'] & M \end{tikzcd} \quad \equiv \quad \begin{tikzcd} \gamma^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ \mathbb{R} \arrow[u, bend left, "s"] \arrow[r, "\gamma"'] & M \end{tikzcd}$$

or equivalently a section of the pullback bundle.

Definition 1.11. We say that the section $s(t)$ along a path $\gamma(t)$ is *parallel* if $\dot{s}(t) \in \Gamma(s(t))$ for all t. We will see shortly that there is an induced connection on γ^*E and the condition says that the covariant derivative is 0.

[1] THE QUESTION IS WHAT IS WRONG WITH iq???

Definition 1.12. Let $f : N \to M$ be a smooth map and $E \to M$ a fibre bundle over M with a connection Γ. There is an induced connection on f^*E, given by the horizontal lifting

$$\widetilde{W}(w,y) = (W, \widetilde{f_* W}(y)) \in T(N \times_M E) \subseteq TN \times TE.$$

We note that a pair (W,Y) lies in $T(N \times_M E)$ if and only if $f_*W = p_*Y$. Thus, a vector (W,Y) is horizontal if and only if its second component Y is horizontal. In particular, a section along γ is parallel, if and only if it is a parallel section of the pullback bundle.

Let $E \to \mathbb{R}$ be a fibre bundle over $\mathbb{R}$. There is a canonical vector field X given by $t \mapsto (t,1)$. Its horizontal lift is a vector field $\tilde{X}$ on E and as such admits local integral curves. Let $y \in E_0$ lie in the fibre over 0. Then

$$\mathrm{Fl}^{\tilde{X}}(y,t) \mapsto \mathrm{Fl}^{X}(0,t) = t$$

and thus $\mathrm{Fl}^{\tilde{X}}(y)$ is a section and is clearly parallel. Thus, when $M = \mathbb{R}$, parallel sections exist locally; moreover they are unique. In particular, when γ is a path in M and $y \in E_{\gamma(0)}$ is arbitrary, there exists locally a unique parallel section along γ. We denote it $\mathrm{Pt}_\gamma(y)$.

It will be convenient to express properties of connections through parallel transport, as this is a very geometric notion. To start with, a connection on the pullback f^*E may be described via its parallel transport: a section of f^*E along a path is parallel if and only if its image in E is parallel.

Definition 1.13. A connection on a vector bundle is said to be linear (affine), when parallel sections are closed under linear combinations. In particular, the zero section must be parallel.

Let us consider now a vector bundle $p : E \to M$. We know that for a vector space W we have $TW = W \times W$. For the vertical bundle VE this means $VE \cong E \times_M E$. An isomorphism from $E \times_M E$ to VE is given by $(u,v) \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(u+tv)$. Further $VE \otimes T^*M \cong (E \times_M E) \otimes T^*M \cong E \times_M (E \otimes T^*M)$ and for a section $s : M \to E$ we write

$$\nabla_\Gamma s = (s, \nabla^\Gamma s)$$

where $\nabla^\Gamma s$ is now a section of $E \otimes T^*M \to M$.

Definition 1.14. The section $\nabla^\Gamma s$ is called the *covariant differential* of s.

In coordinates for a linear connection as above $\nabla^\Gamma s(x)$ is

$$\sum \left(\tfrac{\partial s^p}{\partial x^i} - \Gamma^p_{qi} s^q\right) \mathrm{d}x^i \tfrac{\partial}{\partial y^p}$$

For a vector field $X : M \to TM$ we might evaluate the covariant differential on X to obtain

$$\nabla^\Gamma_X s(x) = (\nabla^\Gamma s(x))(X(x)) : M \to E$$

Definition 1.15. We call this section the *covariant derivative* of the section s *with respect to* the vector field X.

$$\nabla^\Gamma_X s = \sum \left(\tfrac{\partial s^p}{\partial x^i} - \Gamma^p_{qi} s^q\right) X^i \cdot \tfrac{\partial}{\partial y^p}$$

In this way we obtain a map

$$\begin{aligned} \nabla^\Gamma : \mathfrak{X}M \times C^\infty E &\longrightarrow C^\infty E \\ (X,s) &\longmapsto \nabla^\Gamma_X s \end{aligned}$$

Theorem 1.16. *The following equalities hold*

(1) $\nabla^\Gamma_X(s_1 + s_2) = \nabla^\Gamma_X s_1 + \nabla^\Gamma_X s_2$,
(2) $\nabla^\Gamma_X(f \cdot s) = (Xf) \cdot s + f \cdot \nabla^\Gamma_X s$, *(the Leibniz rule)*
(3) $\nabla^\Gamma_{X_1+X_2} s = \nabla^\Gamma_{X_1} s + \nabla^\Gamma_{X_2} s$,
(4) $\nabla^\Gamma_{f \cdot X} s = f \cdot \nabla^\Gamma_X s$.

PROOF. We compute (2) from the coordinate expression

$$\begin{aligned}\nabla^\Gamma_X(f\cdot s)(x) &= \sum \left(\tfrac{\partial}{\partial x^i}(f\cdot s^p) - \Gamma^p_{qi} f s^q\right) X^i \cdot \tfrac{\partial}{\partial y^p}\\ &= \sum \left(\tfrac{\partial f}{\partial x^i}\cdot s^p + f\cdot \tfrac{\partial s^p}{\partial x^i} - f\cdot \Gamma^p_{qi} s^q\right) X^i \cdot \tfrac{\partial}{\partial y^p}\\ &= (Xf)\cdot s + f\cdot \nabla^\Gamma_X s\end{aligned}$$

□

THEOREM 1.17 (The Koszul principle). *Let $\nabla : \mathfrak{X}M \times C^\infty E \to C^\infty E$ be a map satisfying the conditions (1)-(4). Then there exists a unique linear connection Γ on E for which $\nabla = \nabla^\Gamma$.*

PROOF. Locally $E \cong U \times V$ where V is a vector space, $C^\infty E = C^\infty(U,V)$. Let $v \in V$ and we think of it as a constant map $U \to V$, i.e. a section $x \mapsto (x,v)$ whose derivative at $X \in T_xU$ is $(X,0)$. Thus we are forced to put

$$\tilde{X}(x,v) = (\mathrm{id}, v)_* X - (0, \nabla_X v) = (X, -\nabla_X v)$$

in order to ensure at least $\nabla_X v = \nabla^\Gamma_X v$. This formula on the other hand describes a bilinear map $E \times_M TM \to TE$, i.e. a linear connection Γ on E. It remains to show $\nabla = \nabla^\Gamma$. But a general section is locally of the form

$$s(x) = \sum a^i(x) v_i$$

and thus the formula (2) yields

$$\nabla^\Gamma_X s(x) = \sum \left((Xa^i)v_i + a^i \nabla^\Gamma_X v_i\right)$$

which reduces the general case to v. □

REMARK. Let $U \times V \xrightarrow{\cong} U \times V$ be an isomorphism of the trivial vector bundle over U. It is given by a smooth map $A : U \to \mathrm{GL}(V)$ as $(x,v) \mapsto (x, A(x)\cdot v)$. The ordinary derivative $\mathrm{d}s$ of a map $s : U \to V$ is changed to

$$\mathrm{d}(A\cdot s) = A\cdot \mathrm{d}s + \mathrm{d}A \cdot s$$

with the first part being the ordinary derivative transformed by the vector bundle morphism and the second term amounts to a map $E \times_M TM \to E$,

$$((x,v),(x,X)) \mapsto \mathrm{d}A(x,X)\cdot v$$

a linear connection. We will see now that only certain connections (so-called flat ones) arise in this way.

Let us investigate now for an arbitrary bundle $p : E \to M$ whether a given connection in the form of a distribution is integrable (i.e. involutive). For vector fields $X, Y : M \to TM$ we consider their horizontal lifts $\tilde{X}, \tilde{Y} : E \to TE$. Since $\tilde{X}$ and $\tilde{Y}$ are p-related to X and Y, also $[\tilde{X},\tilde{Y}]$ is p-related to $[X,Y]$. In other words $[\tilde{X},\tilde{Y}]$ is a lift of $[X,Y]$. If Γ is to be involutive it is necessary that $[\tilde{X},\tilde{Y}] = \widetilde{[X,Y]}$. As also the vector fields of the form $\tilde{X}$ generate Γ it is at the same time a sufficient condition. We have proved

THEOREM 1.18. *A connection Γ (considered as a distribution) is involutive if and only if $[\tilde{X},\tilde{Y}] = \widetilde{[X,Y]}$.*

DEFINITION 1.19. The mapping $C\Gamma : E \times_M \Lambda^2 TM \to VE$ given by the formula $C\Gamma(y,X,Y) = (\widetilde{[X,Y]} - [\tilde{X},\tilde{Y}])(y)$ is called the *curvature* of the connection Γ. By a dualization we think of it as a section $C\Gamma : E \to VE \otimes \Lambda^2 T^*M$.

REMARK. To make this definition correct we have to prove that the defining expression does not depend on the extension of X and Y to local vector fields. Any other choice of Y' differs by

$$Y' = Y + \sum a^i Y_i$$

where a^i are functions, that vanish at $p(y)$. Using the obvious formula $\widetilde{a^iY_i} = a^ip \cdot \widetilde{Y}_i$, we obtain

$$[\widetilde{X}, \widetilde{Y'}] = [\widetilde{X}, \widetilde{Y}] + [\widetilde{X}, \sum a^ip \cdot \widetilde{Y}] = [\widetilde{X}, \widetilde{Y}] + \sum \big(a^ip \cdot [\widetilde{X}, \widetilde{Y}_i] + \widetilde{X}(a^ip) \cdot \widetilde{Y}_i\big)$$

and similarly

$$\widetilde{[X, Y']} = \widetilde{[X, Y]} + \sum \big(a^ip \cdot \widetilde{[X, Y_i]} + (Xa^i)p \cdot \widetilde{Y}_i\big).$$

Since $a^ip(y) = 0$, and $\widetilde{X}(y)(a^ip) = (p_*\widetilde{X}(y))a^i = X(p(y))a^i$, the difference of these two equals

$$\widetilde{[X, Y']} - [\widetilde{X}, \widetilde{Y'}] = \widetilde{[X, Y]} - [\widetilde{X}, \widetilde{Y}]$$

For a linear connection on $E = TM$ we get the classical theory of connections on a manifold. The curvature is in this case a tensor of type $(1, 3)$, i.e. a section $M \longrightarrow TM \otimes (T^*M)^{\otimes 3}$ (or in fact $M \longrightarrow TM \otimes T^*M \otimes \Lambda^2T^*M$). The classical definition is $\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$. One can verify that this agrees with our (more general) definition up to the change of sign and indices of the Christoffel symbols Γ^k_{ij} as mentioned before.

Theorem 1.20. *A connection Γ is involutive if and only if the parallel transport does not locally depend on the path.*

Proof. When Γ is involutive there is an integral manifold L_y through each point $y \in E$. The composition $\varphi_y : L_y \hookrightarrow E \xrightarrow{p} M$ is a local diffeomorphism and the parallel transport of γ is simply obtained by composition $\tilde{\gamma} = \varphi_y^{-1} \circ \gamma$ the endpoint depending only on $\gamma(1)$. The converse is also true. $\square$

The integrability of Γ says that *locally in E* one can find charts of the form $U \times V$ such that the projection p becomes the projection $U \times V \to U$ and such that the distribution is $T_xU \times \{0\}$. To extend this trivialization globally we need the following notion.

Definition 1.21. A connection Γ is called *complete* if the parallel transport exists globally.

A sufficient condition is for example that the fibre is compact. Also a linear connection is always complete (a proof in the tutorial).

Theorem 1.22. *If a connection Γ is complete and involutive then there exist local trivializations $p^{-1}(U) \cong U \times F$ such that $\Gamma(x, y) = T_xU \times \{0\}$.*

Proof. The trivialization is given by the following construction. Choose a basis $X_1, \dots, X_m$ of the base M and use their lifts $\tilde{X}_1, \dots, \tilde{X}_m$ to define

$$\begin{aligned} \mathbb{R}^m \times F &\longrightarrow E \\ (t = (t^1, \dots, t^m), y) &\longmapsto \mathrm{Fl}_1^{t\tilde{X}}(y) = \mathrm{Pt}_{\mathrm{Fl}^{tX}(x)}(y, 1) \end{aligned}$$

where we denote for simplicity $tX = t^1X_1 + \cdots + t^mX_m$. The right hand side is only defined when $\mathrm{Fl}^{tX}(x)$ is defined on the interval $[0, 1]$ but such t form a neighbourhood of 0, independently of y. $\square$

We will give a geometric interpretation of the curvature $C\Gamma(y, X, Y)$. Let X and Y be extensions to commuting vector fields. Then

$$C\Gamma(y, X, Y) = -[\tilde{X}, \tilde{Y}](y) = -\tfrac{\partial}{\partial s}\big|_{s=0} \tfrac{\partial}{\partial t}\big|_{t=0} \mathrm{Fl}^{\tilde{Y}}_{-s} \mathrm{Fl}^{\tilde{X}}_{-t} \mathrm{Fl}^{\tilde{Y}}_{s} \mathrm{Fl}^{\tilde{X}}_{t}(u).$$

The flow lines are clearly parallel sections along their projections, which are the flow lines of X and Y. Thus

$$C\Gamma(y, X, Y) = -\tfrac{\partial}{\partial s}\big|_{s=0} \tfrac{\partial}{\partial t}\big|_{t=0} \mathrm{Pt}_{\mathrm{Fl}^{-sY}} \mathrm{Pt}_{\mathrm{Fl}^{-tX}} \mathrm{Pt}_{\mathrm{Fl}^{sY}} \mathrm{Pt}_{\mathrm{Fl}^{tX}}(u),$$

i.e. it is up to the sign the mixed partial derivative (the first interesting one) of the effect of transporting u parallelly along the rectangle of the flowlines.

2. Principal connections

Let us consider a principal bundle $P(M,G)$. We take $A \in \mathfrak{g}$ which we may express as $A = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \exp(tA)$. The fundamental vector field on P is

$$A^*(u) = (r(u,-))_*(A) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} (u \cdot \exp(tA)) \in V_u P$$

The reason it lies in the vertical subbundle is that $u \cdot \exp(tA)$ is a curve in $P_{\pi(u)}$. Globally we get a map

$$\begin{aligned} P \times \mathfrak{g} &\longrightarrow VP \\ (u, A) &\longmapsto A^*(u) \end{aligned}$$

and it is clearly an isomorphism of vector bundles, i.e. a trivialization of VP.

A connection on P thought of as a vertical projection $v : TP \to VP$ then yields a 1-form $\omega_\Gamma : TP \xrightarrow{v} VP \cong P \times \mathfrak{g} \to \mathfrak{g}$. The defining equation is $\omega_\Gamma(X)^* = vX$ and the vertical projection is obtained uniquely from a $\mathfrak{g}$-valued 1-form ω provided that $\omega(A^*) = A$ for all $A \in \mathfrak{g}$ (expressing that the map v is really a projection onto the vertical subbundle, $vA^* = A^*$).

THEOREM 2.1. *The following conditions are equivalent for a connection Γ on a principal bundle, where we abbreviate $Xa = (r_a)_*(X)$ for a vector $X \in TP$ (this in fact defines an action of G on TP)*

(1) $v(Xa) = (vX)a$,
(2) $h(Xa) = (hX)a$,
(3) $\tilde{X}(ua) = \tilde{X}(u)a$,
(4) *the horizontal distribution is equivariant,* $\Gamma(ua) = \Gamma(u)a$,
(5) $\omega_\Gamma(Xa) = \mathrm{Ad}(a^{-1})\omega_\Gamma X$,
(6) *the section* $\Gamma : P \to J^1P$ *satisfies* $\Gamma(u) = j^1_x s \implies \Gamma(ua) = j^1_x(sa)$.

A connection satisfying these conditions is called principal.

PROOF. (1) and (2) are equivalent since $v + h = \mathrm{id}$ and id is equivariant. (2) is also equivalent to (3) since they both say that the action of G preserves horizontal vectors (it preserves lifts by definition). For point (4) note that the condition (1) is automatically satisfied on vertical vectors and on horizontal ones (those in the kernel) it is plainly (4).

The most interesting is (5), we compute

$$\begin{aligned} v(Xa) &= (\omega(Xa))^* \\ (vX)a &= (\omega X)^* a = \big(\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} u \cdot \exp(t \cdot \omega X)\big) \cdot a = \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} u \cdot \exp(t \cdot \omega X)a \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} (ua) \cdot (a^{-1} \exp(t \cdot \omega X)a) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} (ua) \exp(\mathrm{Ad}(a^{-1})(t \cdot \omega X)) \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} (ua) \exp(t \cdot \mathrm{Ad}(a^{-1})\omega X) = (\mathrm{Ad}(a^{-1})\omega X)^* \end{aligned}$$

Thus $v(Xa) = (vX)a$ iff $\omega(Xa) = \mathrm{Ad}(a^{-1})\omega X$.

For (6) observe that the lift $\tilde{X}(u)$ can be expressed as $\tilde{X}(u) = s_* X$. Therefore $\tilde{X}(u)a = (s_* X)a = (sa)_* X$ and this equals $\tilde{X}(ua)$ iff sa represents $\Gamma(ua)$. □

REMARK. In the lecture I have used quite a lot parallel sections in the explanations. It might be worth to start already here. A connection is principal if and only if the action preserves parallel sections along paths.

COROLLARY 2.2. *For every $\mathfrak{g}$-valued 1-form ω on P satisfying*

(1) $\omega(Xa) = \mathrm{Ad}(a^{-1})\omega(X)$
(2) $\omega A^* = A$

there exists a unique principal connection Γ on P whose connection form is ω.

PROOF. $\Gamma = \ker \omega$. □

Let us consider a left G-space F and the associated bundle $E = P[F] = P \times_G F$. Let Γ be a principal connection on P.

Definition 2.3. An *associated connection* Γ_F on $P[F]$ is defined via horizontal lifting as follows:

$$\widehat{X}[u,z] = q_*(\widetilde{X}(u), 0(z))$$

We have to verify that the definition does not depend on the choice of the representatives. This follows from

$$q_*(\widetilde{X}(ua), 0(z)) = q_*(\widetilde{X}a, 0(z)) = q_*(\widetilde{X}, a0(z)) = q_*(\widetilde{X}, 0(az)).$$

Another useful description of the associated connection uses parallel sections. Let s be a parallel section along a curve γ. Then $[s,z]$ is again a parallel section where z is a constant map at $z \in F$.

Lemma 2.4. *When W is a vector space and the action of G on W is linear, then the associated connection Γ_F is linear.*

Proof. This follows from the description via parallel sections: when $s : \mathbb{R} \to G$ is parallel along a path γ, and $w_1, w_2 \in W$, then the linear combination

$$a_1[s, w_1] + a_2[s, w_2] = [s, a_1 w_1 + a_2 w_2]$$

of the two parallel sections $[s, w_1]$ and $[s, w_2]$ is again parallel. □

We will now bring the equivalence of vector bundles and principal $\mathrm{GL}(k)$-bundles further. We now know that a principal connection induces a linear connection on the vector bundle. To get back consider a vector bundle $E \to M$ and a linear connection Γ on it. The total space of the frame bundle PE is naturally an open subset

$$PE \subseteq E \times_M \cdots \times_M E$$

in the k-fold fibre product of E with itself. Let $u = (u_1, \ldots, u_k) \in PE$ be a frame in E_x and let us consider a path γ in M through x and a frame $u = (u_1, \ldots, u_k) \in PE$ in E_x. Let $s_i(t)$ be a parallel transport of u_i along γ. Then $s(t) = (s_1(t), \ldots, s_k(t))$ is a path in PE covering the path γ and we declare it to be parallel. Thus we define the horizontal lift of $X = \dot{\gamma}(0)$ to u by

$$\tilde{X}(u) = \tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} s(t) = (\tilde{X}(u_1), \ldots, \tilde{X}(u_k)) \in T(E \times_M \cdots \times_M E) \subseteq TE \times \cdots \times TE.$$

Since linear combinations of parallel sections are parallel it is easy to see that this connection is principal. We have proved:

Theorem 2.5. *There is a natural bijection between linear connections on E and principal connections on PE.* □

3. The covariant differential on associated bundles

A section s of the associated bundle $P[F] \to M$ can be described via an equivariant map $\sigma : P \to F$ using the diagram

$$\begin{tikzcd}
 & F \\
 & P \times F \arrow[u] \arrow[d] \\
P \arrow[uur, bend left, "\sigma"] \arrow[ur, "(\mathrm{id},\sigma)"] \arrow[r, "{[\mathrm{id},\sigma]}"] \arrow[d, "\pi"'] & P \times_G F \\
M \arrow[ur, "s"'] &
\end{tikzcd}$$

Now for $X \in T_x M$ we have $s_* X = [\mathrm{id}, \sigma]_* \tilde{X} = q_*(\tilde{X}, \sigma_* \tilde{X})$ and so

$$\nabla_\Gamma s(X) = v(s_* X) = q_*(\tilde{X}, \sigma_* \tilde{X}) - q_*(\tilde{X}, 0) = q_*(0, \sigma_* \tilde{X})$$

since $[\tilde{X}, 0]$ is the horizontal lift of X. The moral is that the covariant differential is no more than "a derivative in the direction of horizontal vectors".

We will make this even more explicit in the case of vector bundles. In a vector space, let us denote by tr the transport of a vector to 0, i.e. the map $TW = W \times W \xrightarrow{\mathrm{pr}_2} W$. We have used this map fibrewise to define the covariant differential. The relevant formula, that we need is

$$\begin{aligned}\operatorname{tr} q_*(0(u), W(w)) &= \operatorname{tr} \left.\tfrac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} q(u, w + tW)\\ &= \operatorname{tr} \left.\tfrac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} ([u, w] + t[u, W])\\ &= [u, W] = q(u, \operatorname{tr} W(w)).\end{aligned}$$

This enables us to write

$$\nabla^\Gamma_X s(x) = \operatorname{tr} \nabla_\Gamma s(X(x)) = \operatorname{tr} q_*(0(u), \sigma_* \tilde{X}(u)) = [u, \operatorname{tr} \sigma_* \tilde{X}(u)] = [u, \mathrm{d}\sigma(\tilde{X})].$$

In particular, the equivariant map corresponding to this covariant differential $\nabla^\Gamma_X s$ is $\mathrm{d}\sigma(\tilde{X})$.

LATER, we will need the following.

Lemma 3.1. *$\sigma_* A^*(u) = -\ell_A(\sigma(u))$, where ℓ_A is the fundamental vector field corresponding to $A \in \mathfrak{g}$ on the G-space F. In particular the derivative along vertical vectors does not depend on σ_{*u} but only on $\sigma(u)$.*

Note. As $\sigma_* A^*$ is not equivariant it does not induce a section of VE.

Proof. This is an easy computation

$$\sigma_* A^*(u) = \left.\tfrac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \sigma(u \cdot \exp(tA)) = \left.\tfrac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(-tA) \cdot \sigma(u) = -\ell_A(\sigma(u))$$

□

We will now generalize this form of the covariant differential to forms of higher degree. We start a bit more generally with a smooth manifold P and a vector space W.

Definition 3.2. A *W-valued k-form* on P is a smooth antisymmetric multilinear map

$$\varphi : TP \times_P \cdots \times_P TP \longrightarrow W \qquad \text{or} \qquad \varphi : \Lambda^k TP \longrightarrow W$$

We write $\varphi \in \Omega^k(P; W)$.

Let $\varphi = \sum \varphi^j e_j$ where $\varphi^j \in \Omega^k(M)$ and (e_j) a basis of W. We define

$$\mathrm{d}\varphi = \sum (\mathrm{d}\varphi^j) e_j$$

which is a W-valued $(k+1)$-form that does not depend on the choice of the basis since a change of basis is linear as is the differential.

Let $\rho : G \to \mathrm{GL}(W)$ be a representation and $P(M, G)$ a principal bundle.

Definition 3.3. We say that $\varphi \in \Omega^k(P; W)$ *is of type ρ* if

$$\varphi(A_1 a, \ldots, A_k a) = \rho(a^{-1})\varphi(A_1, \ldots, A_k)$$

If this is the case we write $\varphi \in \Omega(P; \rho)$. Observe that the left hand side is simply

$$\varphi(r_{a*} A_1, \ldots, r_{a*} A_k) = r_a^* \varphi(A_1, \ldots, A_k)$$

Therefore the condition may be rewritten simply as $r_a^* \varphi = \rho(a^{-1})\varphi$.

Example 3.4. The form ω_Γ of a principal connection Γ is of type Ad

Definition 3.5. We say that φ is *horizontal* if $\varphi(A_1, \ldots, A_k) = 0$ whenever one of A_i is vertical. In this way φ can be thought of as a map

$$\Lambda^k HP = \Lambda^k(TP/VP) \longrightarrow W$$

Theorem 3.6. *Horizontal k-forms of type ρ are in bijection with $P[W]$-valued k-forms on M, i.e. vector bundle morphisms*

$$\begin{tikzcd} \Lambda^k TM \arrow[rr, "\underline{\varphi}"] \arrow[dr] & & P[W] \arrow[dl] \\ & M & \end{tikzcd}$$

Proof. A horizontal k-form $\varphi : \Lambda^k TP \to W$ of type ρ induces, as we observed, a G-map $\Lambda^k HP \to W$ or equivalently a G-map $\tilde{\varphi} : P \times_M \Lambda^k TM \to W$. We have seen how to identify any equivariant map $P \to W$ with a section of $P[W]$ and in the present situation we just carry $\Lambda^k TM$ over[2] to obtain $\underline{\varphi} : \Lambda^k TM \to P[W]$:

$$\underline{\varphi}(X_1, \dots, X_k) = [u, \tilde{\varphi}(u, X_1, \dots, X_k)] = [u, \varphi(\tilde{X}_1(u), \dots, \tilde{X}_k(u))]$$

wehre $u \in P$ is any point lying over the same point as all the vectors X_i. This formula says that for vector fields $X_1, \dots, X_k$ the section $\underline{\varphi}(X_1, \dots, X_k)$ of $P[W]$ corresponds to the equivariant map $\varphi(\tilde{X}_1, \dots, \tilde{X}_k) : P \to W$ which may be interpreted as: having a correspondence for sections/equivariant maps and vector fields/equivariant horizontal vector fields gives a correspondence for forms. □

Remark. When the representation ρ is trivial, $\forall a \in G : \rho(a) = \mathrm{id}$, then $\varphi(\tilde{X}_1(u), \dots, \tilde{X}_k(u))$ does not depend on the choice of u over x and defines a map $\Lambda^k TM \to W$. This corresponds to $\Lambda^k TM \to P[W] \cong M \times W \xrightarrow{pr} W$.

Let $\varphi \in \Omega^k(P, \rho)$ then $\mathrm{d}\varphi \in \Omega^{k+1}(P, \rho)$ is of the same type since

$$r_a^* \mathrm{d}\varphi = \mathrm{d} r_a^* \varphi = \mathrm{d}(\rho(a^{-1}) \circ \varphi) = \rho(a^{-1}) \circ \mathrm{d}\varphi$$

by linearity of the map $\rho(a^{-1})$. The horizontality on the other hand needs not be preserved by d.

Definition 3.7. An *exterior covariant differential* of a W-valued k-form on P is a $(k+1)$-form

$$D\varphi(X_0, \dots, X_k) = \mathrm{d}\varphi(hX_0, \dots, hX_k)$$

Clearly $D\varphi$ is horizontal. If φ is moreover of type ρ then so is $D\varphi$ since both the horizontal projection h and $\mathrm{d}\varphi$ are equivariant. Therefore we get a diagram

$$\begin{tikzcd}
\Omega^k(P,\rho) \arrow[r, "\mathrm{d}"] & \Omega^{k+1}(P,\rho) \arrow[d, "h^*"] \\
\Omega^k_{\mathrm{hor}}(P,\rho) \arrow[u, hook] \arrow[r, "D"] \arrow[d, leftrightarrow, "\cong"'] & \Omega^{k+1}_{\mathrm{hor}}(P,\rho) \arrow[d, leftrightarrow, "\cong"] \\
\Omega^k(M, P[W]) \arrow[r, dashed, "\nabla"] & \Omega^{k+1}(M, P[W])
\end{tikzcd}$$

with $h^*\psi(X_0, \dots, X_k) = \psi(hX_0, \dots, hX_k)$.

Definition 3.8. Let $\varphi \in \Omega^k(P, U)$ be a U-valued k-form on P and $\psi \in \Omega^\ell(P, V)$ a V-valued ℓ-form. We define their wedge product $\varphi \wedge \psi \in \Omega^{k+\ell}(P, U \otimes V)$ by the formula

$$(\varphi \wedge \psi)(X_1, \dots, X_{k+\ell}) = \sum \varphi(X_{\sigma(1)} \ \dots, X_{\sigma(k)}) \otimes \psi(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)})$$

where the sum runs over the (k, ℓ)-shuffles, i.e. those permutations of $\{1, \dots, k+\ell\}$ for which

$$\sigma(1) < \dots < \sigma(k), \ \sigma(k+1) < \dots < \sigma(k+\ell).$$

When there is a linear map $f : U \otimes V \to W$, we further define

$$f(\varphi, \psi) = f \circ (\varphi \wedge \psi) \in \Omega^{k+\ell}(P, W).$$

This wedge product of vector-valued forms enjoys properties similar to the real-valued one. We present the version with the linear map f applied. The Leibniz rule reads

$$\mathrm{d}f(\varphi, \psi) = f(\mathrm{d}\varphi, \psi) + (-1)^{|\varphi|} f(\varphi, \mathrm{d}\psi).$$

(The coordinates of $\varphi \wedge \psi$ are the wedge-products of the coordinates of φ and ψ and thus the usual Leibniz rule applies. The version with f follows by the linearity of f.)

[2]Another possibility is to view $P' = P \times_M \Lambda^k TM$ as a pullback bundle of P along the projection $\Lambda^k TM \to TM$. Then $\tilde{\varphi}$ is a G-map $P' \to W$ and these are in bijection with sections of $P'[W]$ which are exactly the maps as in the statement of the theorem.

Next, we have the following graded commutativity

$$\varphi \wedge \psi = (-1)^{|\varphi|\cdot|\psi|} \operatorname{ex} \psi \wedge \varphi$$

where $\operatorname{ex} : U \otimes V \to V \otimes U$ is the canonical isomorphism $u \otimes v \mapsto v \otimes u$. It follows, that for a symmetric $f : W \otimes W \to W$, the product $f(\varphi, \psi)$ will be graded symmetric, and similarly for the skew-symmetric one. This applies, in particular, to the Lie bracket $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, so that

$$[\varphi, \psi] = (-1)^{|\varphi|\cdot|\psi|+1}[\psi, \varphi].$$

Similarly, the graded Jacobi identity reads

$$[\varphi, [\psi, \chi]] = [[\varphi, \psi], \chi] + (-1)^{|\varphi|\cdot|\psi|}[\psi, [\varphi, \chi]].$$

(The adjoint action $[\varphi, -]$ is a graded derivation.)

All these properties pass to covariant differential, so that in particular we have the following Leibniz rule

$$Df(\varphi, \psi) = f(D\varphi, \psi) + (-1)^{|\varphi|} f(\varphi, D\psi).$$

We will now explain, how this translates to the covariant differential ∇ on $P[W]$-valued forms on M. Let us have forms $\underline{\varphi} \in \Omega^k(M, P[U])$ and $\underline{\psi} \in \Omega^\ell(M, P[V])$. These correspond to horizontal equivariant forms $\varphi \in \Omega^k(P, U)$ and $\psi \in \Omega^\ell(P, V)$ and their wedge-product then lies in

$$\varphi \wedge \psi \in \Omega^{k+\ell}_{\mathrm{hor}}(P, U \otimes V) \cong \Omega^{k+\ell}(M, P[U \otimes V]).$$

The associated bundle $P[U \otimes V] \cong P[U] \otimes P[V]$ via the isomorphism $[u, a \otimes b] \sim [u, a] \otimes [u, b]$. Thus we obtain a wedge product $\underline{\varphi} \wedge \underline{\psi} \in \Omega^{k+\ell}(M, P[U] \otimes P[V])$, which can be defined directly as

$$\underline{\varphi} \wedge \underline{\psi}(X_1, \ldots, X_{k+\ell}) = \sum \underline{\varphi}(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) \otimes \underline{\psi}(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+\ell)}).$$

The Leibniz rule then becomes $\nabla f(\underline{\varphi}, \underline{\psi}) = f(\nabla\underline{\varphi}, \underline{\psi}) + (-1)^{|\underline{\varphi}|} f(\underline{\varphi}, \nabla\underline{\psi})$.

We may now describe this form of the covariant differential explicitly. Locally, we may write

$$\underline{\varphi} = \sum s_i \mathrm{d}g_{i,1} \wedge \cdots \wedge \mathrm{d}g_{i,k}$$

for some sections s_i and functions $g_{i,1}, \ldots, g_{i,k}$. Then

$$\nabla\underline{\varphi} = \sum \nabla s_i \wedge \mathrm{d}g_{i,1} \wedge \cdots \wedge \mathrm{d}g_{i,k}.$$

(The first wedge product is induced by the multiplication $P[W] \otimes P[\mathbb{R}] \to P[W]$, where $P[\mathbb{R}] = M \times \mathbb{R}$ is the trivial bundle with the trivial connection on it.)

Another formula is obtained from the above in the usual manner. We obtain

$$\begin{aligned}\nabla\underline{\varphi}(X_0, \ldots, X_k) &= \sum_i (-1)^i \nabla_{X_i} \underline{\varphi}(X_0, \ldots, \hat{X}_i, \ldots, X_k) \\ &\quad + \sum_{i<j} (-1)^{i+j} \underline{\varphi}([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k)\end{aligned}$$

It is instructive to compute the second covariant differential of a section s, i.e. a 0-form. We have

$$(\nabla\nabla s)(X, Y) = \nabla_X(\nabla s)(Y) - \nabla_Y(\nabla s)(X) - (\nabla s)[X, Y] = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s.$$

In general, this is not zero. However, it corresponds to a 2-form $D^2\sigma \in \Omega^2_{\mathrm{hor}}(P; \rho)$, which we will able to compute; it is purely algebraic $\Omega \cdot \sigma$. First, we will relate the curvature form Ω to the connection form ω.

Consider a principal connection Γ on P and its curvature

$$C\Gamma : P \times_M \Lambda^2 TM \longrightarrow VP$$

defining a $\mathfrak{g}$-valued 2-form Ω on P by the formula

$$\Omega_u(Y, Z) := \omega C\Gamma(u, \pi_* Y, \pi_* Z)$$

Theorem 3.9. *$\Omega = D\omega$, i.e. the curvature Ω is a covariant derivative of the form of the connection.*

PROOF. We first express Ω using the defining equation for ω:

$$\Omega_u(Y,Z)^* = C\Gamma(u,\pi_*Y,\pi_*Z) = -v[\widetilde{\pi_*Y},\widetilde{\pi_*Z}] = -v[hY,hZ]$$

Therefore $\Omega_u(Y,Z) = -\omega[hY,hZ]$. Now we compute

$$D\omega(Y,Z) = \mathrm{d}\omega(hY,hZ) = (hY)\underbrace{\omega(hZ)}_{0} - (hZ)\underbrace{\omega(hY)}_{0} - \omega[hY,hZ] = \Omega(Y,Z)$$

□

Next, we will show how the covariant differential is related to the ordinary differential. First, we need two lemmas.

LEMMA 3.10. *The flow of the fundamental vector field A^* is $\mathrm{Fl}_t^{A^*}(u) = u\exp tA$, i.e. $\mathrm{Fl}_t^{A^*}$ is the right translation by $\exp tA$.*

PROOF. This is a very simple computation

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_0} u\exp tA = \tfrac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} u\exp t_0A\exp sA = A^*(u\exp t_0A).$$

□

LEMMA 3.11. *Let X be a vector field on M and $\tilde{X}$ its horizontal lift; let $A \in \mathfrak{h}$. Then $[A^*,\tilde{X}] = 0$.*

PROOF. This is an application of the Lie derivative version of the Lie bracket,

$$[A^*,\tilde{X}](u) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} (\mathrm{Fl}_{-t}^{A^*})_*\tilde{X}(\mathrm{Fl}_t^{A^*} u) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \tilde{X}(u\exp tA)\exp(-tA) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \tilde{X}(u) = 0,$$

where we use the equivariance of the horizontal lifts to conclude that the path is constant at $\tilde{X}(u)$ and thus its derivative is zero. □

Lastly, it should be clear, that the Lie bracket of two vertical vector fields is again vertical (they are tangent to the fibres and thus i-related to vector fields on the fibres, where i denotes the inclusion; consequently their bracket is i-related to some vector field on the fibre and thus is vertical). There is in fact a formula

LEMMA 3.12. *It holds $[A^*,B^*] = [A,B]^*$.*

PROOF. Using the action map $r : P\times G \to P$, we have that $(0,\lambda_A)$ is r-related to A^* and thus $(0,[\lambda_A,\lambda_B]) = (0,\lambda_{[A,B]})$ is r-related to $[A^*,B^*]$. Since it is also r-related to $[A,B]^*$, the two must be equal. (There is also a computation similar to the previous proof.) □

We may now prove the following theorem.

THEOREM 3.13. *For a horizontal equivariant k-form $\varphi \in \Omega^k_{\mathrm{hor}}(P;\rho)$, the following holds*

$$D\varphi = \mathrm{d}\varphi + \omega\cdot\varphi,$$

where $\omega \in \Omega^1(P;\mathfrak{g})$ is the connection form and the dot denotes the infinitesimal action $\mathfrak{g}\otimes W \to W$.

PROOF. For simplicity, we will restrict to the case $k = 1$. Thus, we may write

$$\mathrm{d}\varphi(\tilde{X}+A^*,\tilde{Y}+B^*) = \mathrm{d}\varphi(\tilde{X},\tilde{Y}) + \mathrm{d}\varphi(A^*,\tilde{Y}) + \mathrm{d}\varphi(\tilde{X},B^*) + \mathrm{d}\varphi(A^*,B^*).$$

The first term equals $D\varphi(\tilde{X}+A^*,\tilde{Y}+B^*)$, we will proceed with the remaining terms,

$$\mathrm{d}\varphi(A^*,\tilde{Y}) = A^*\varphi\tilde{Y} - \tilde{Y}\varphi A^* - \varphi[A^*,\tilde{Y}] = A^*\varphi\tilde{Y},$$

since both A^* is vertical and $[A^*,\tilde{Y}] = 0$. We have

$$A^*\varphi\tilde{Y}(u) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \varphi\tilde{Y}(u\exp tA) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \varphi(\tilde{Y}(u)\exp tA) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \exp(-tA)\varphi\tilde{Y}(u)$$

by the equivariancy of $\tilde{Y}$ and of φ. Thus, this expression equals the infinitesimal action of $-A = -\omega(A^*)$ on $\varphi\tilde{Y}(u)$. In the end, we have

$$\mathrm{d}\varphi(A^*,\tilde{Y}) = -\omega(A^*)\varphi(\tilde{Y}) = -\omega(\tilde{X}+A^*)\varphi(\tilde{Y}+B^*).$$

Similarly $\mathrm{d}\varphi(\tilde{X}, B^*) = \omega(\tilde{Y} + B^*)\varphi(\tilde{X} + A^*)$. Finally we observe that

$$\mathrm{d}\varphi(A^*, B^*) = A^*\varphi B^* - B^*\varphi A^* - \varphi[A^*, B^*] = 0$$

since all A^*, B^* and $[A^*, B^*] = [A, B]^*$ are vertical. Putting together, we have

$$\mathrm{d}\varphi(\tilde{X} + A^*, \tilde{Y} + B^*) = D\varphi(\tilde{X} + A^*, \tilde{Y} + B^*) - \omega(\tilde{X} + A^*)\varphi(\tilde{Y} + B^*) + \omega(\tilde{Y} + B^*)\varphi(\tilde{X} + A^*),$$

i.e. $\mathrm{d}\varphi = D\varphi - \omega \cdot \varphi$. □

Corollary 3.14. *For a horizontal equivariant k-form $\varphi \in \Omega^k_{\mathrm{hor}}(P; \rho)$, it holds $D^2\varphi = \Omega \cdot \varphi$, where $\Omega \in \Omega^2_{\mathrm{hor}}(P; \mathrm{Ad})$ is the curvature form.*

Proof. We have $D^2\varphi = h^*\mathrm{d}(\mathrm{d}\varphi + \omega\cdot\varphi) = h^*(\mathrm{d}\omega\cdot\varphi - \omega\cdot\mathrm{d}\varphi)$. As ω is vertical, the composition with the horizontal projection is zero. Thus

$$D^2\varphi = h^*\mathrm{d}\omega \cdot h^*\varphi = D\omega \cdot \varphi = \Omega \cdot \varphi.$$

□

Thus, we see that the section $\nabla^2 s(X, Y) = \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s$ corresponds to $D^2\sigma(\tilde{X}, \tilde{Y}) = \Omega(\tilde{X}, \tilde{Y}) \cdot \sigma$. We will now show, that this purely algebraic expression corresponds to the curvature at s. Thus, we compute

$$\begin{aligned}
[u, \Omega(\tilde{X}(u), \tilde{Y}(u)) \cdot \sigma(u)] &= \mathrm{tr}\ \tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} [u, \exp(t\Omega(\tilde{X}(u), \tilde{Y}(u))) \cdot \sigma(u)] \\
&= \mathrm{tr}\ \tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} [u \exp(t\Omega(\tilde{X}(u), \tilde{Y}(u))), \sigma(u)] \\
&= \mathrm{tr}\, q_*(\Omega(\tilde{X}(u), \tilde{Y}(u))^*(u), 0) \\
&= \mathrm{tr}\, q_*(\widetilde{[X, Y]}(u) - [\tilde{X}, \tilde{Y}](u), 0) \\
&= \mathrm{tr}(\widehat{[X, Y]} - [\widehat{X}, \widehat{Y}])(q(u, \sigma(u))) \\
&= \mathrm{tr}\, C\Gamma(q(u, \sigma(u)), X, Y) = \mathrm{tr}\, C\Gamma(s(x), X, Y),
\end{aligned}$$

i.e. it is the transport of the curvature at $s(x)$ to the zero section.

The curvature form Ω is of type Ad. This means, that when G is commutative, the form Ω is of the trivial type and we may think of it as a $\mathfrak{g}$-valued form on M. Let us see what happens, when we integrate this form over a 2-dimensional $D^2 \to M$. Each rectangle in D^2 contributes the value consisting of the difference, that results in transporting parallelly some $u \in P$ along this rectangle, or more precisely the corresponding tangent vector. Thus, when adding the actual group elements, we must obtain as the integral, the effect of going along the boundary of D^2 and computing the group element responsible for the difference. In particular, when we integrate in this way over a closed surface, we must get 0. A different situation happens in the universal cover of G, i.e. the Lie algebra $\mathfrak{g}$. Then, this integral can be non-zero and measures the number of times, the parallel transport wraps around (take e.g. $G = S^1$). This kind of ideas lead to Chern-Weil forms to follow. They start with a map of Lie algebras $\mathfrak{g} \to \mathbb{R}$ and measure this effect. When, e.g. this map is $\mathrm{tr} : \mathfrak{gl}(n, \mathbb{C}) \to \mathbb{C}$, the corresponding "homomorphism" of groups is $\log\det : \mathrm{GL}(n, \mathbb{C}) \to \mathbb{C}$. Since only $\det : \mathrm{GL}(n, \mathbb{C}) \to \mathbb{C}^\times$ is well-defined, we may measure the number of times det wraps around 0 by computing the integral $\int \mathrm{tr}\,\Omega$, or still better we may normalize to $\frac{1}{2\pi i}\int \mathrm{tr}\,\Omega$ to get an integer-value and thus an interesting integer-valued form $c_2 = \frac{1}{2\pi i}\,\mathrm{tr}\,\Omega$, called the first Chern form.

4. The structure equation

Let U, V, W be vector spaces and $f : U \otimes V \to W$ a linear map. Let $\varphi : TM \to U$ and $\psi : TM \to V$ be 1-forms. Consider the antisymmetrization of

$$TM \otimes TM \xrightarrow{\varphi\otimes\psi} U \otimes V,$$

a 2-form $\varphi\wedge\psi : \Lambda^2 TM \to U \otimes V$. By composing with f we obtain a 2-form $f(\varphi, \psi) : \Lambda^2 TM \to W$, explicitly

$$f(\varphi, \psi)(X, Y) = f(\varphi(X) \otimes \psi(Y)) - f(\varphi(Y) \otimes \psi(X))$$

Applying this construction to the Lie algebra bracket $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ and $\omega \in \Omega^1(M, \mathfrak{g})$

$$[\omega, \omega](X, Y) = [\omega X, \omega Y] - [\omega Y, \omega X] = 2[\omega X, \omega Y]$$

On a principal bundle with a principal connection Γ we have the form of the connection $\omega \in \Omega^1(P, \mathrm{Ad})$.

Theorem 4.1 (The structure equation). $\mathrm{d}\omega + \frac{1}{2}[\omega, \omega] = \Omega$.

Corollary 4.2 (The second Bianchi identity). $\mathrm{d}\Omega = [\Omega, \omega]$. *In particular* $D\Omega = 0$.

Proof. Applying a linear map f to $\mathrm{d}(\varphi \wedge \psi) = \mathrm{d}\varphi \wedge \psi - \varphi \wedge \mathrm{d}\psi$ one obtains

$$\mathrm{d}(f(\varphi, \psi)) = f(\mathrm{d}\varphi, \psi) - f(\varphi, \mathrm{d}\psi)$$

Thus using the structure equation

$$\mathrm{d}\Omega = \mathrm{d}(\mathrm{d}\omega + \tfrac{1}{2}[\omega, \omega]) = \tfrac{1}{2}[\mathrm{d}\omega, \omega] - \tfrac{1}{2}[\omega, \mathrm{d}\omega] = [\mathrm{d}\omega, \omega]$$

since $\omega \wedge \mathrm{d}\omega = tw \circ \mathrm{d}\omega \wedge \omega$ and $[\ ,\ ]$ is anticommutative, $[\ ,\ ] \circ tw = -[\ ,\ ]$. Using the structure equation again

$$[\mathrm{d}\omega, \omega] = [\Omega - \tfrac{1}{2}[\omega, \omega], \omega] = [\Omega, \omega]$$

since $[[\omega, \omega], \omega] = 0$ by the Jacobi identity:

$$\begin{aligned}
[[\omega, \omega], \omega](X, Y, Z) &= [[\omega, \omega](X, Y), \omega Z] - [[\omega, \omega](X, Z), \omega Y] + [[\omega, \omega](Y, Z), \omega X] \\
&= 2([[\omega X, \omega Y], \omega Z] - [[\omega X, \omega Z], \omega Y] + [[\omega Y, \omega Z], \omega X]) = 0.
\end{aligned}$$

The last part follows from $D\Omega = h^*\mathrm{d}\Omega = [h^*\Omega, h^*\omega]$ and $h^*\omega = 0$. □

The proof of the structure equation. First we deal with a Lie group G thought of as a principal G-bundle over a point. The vertical projection in this case is the identity and thus there exists a unique connection. Since

$$A^*(a) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\, a \cdot \exp(tA) = (\lambda_a)_* A$$

the unique connection form is $\omega_G(a, X) = (\lambda_{a^{-1}})_* X$. This is the canonical $\mathfrak{g}$-valued 1-form on G called the *Maurer-Cartan form*. The structure equation reduces in this case to

Theorem 4.3 (Maurer-Cartan equation). $\mathrm{d}\omega_G + \frac{1}{2}[\omega_G, \omega_G] = 0$.

Note. The curvature must be zero since $TM = 0$.

Proof. Observe that any $X \in T_aG$ extends to a left-invariant vector field $\lambda_{\omega_G X}$ and denote for short $\omega_G X = A$ and $\omega_G Y = B$. Thus

$$\begin{aligned}
\mathrm{d}\omega_G(X, Y) &= \mathrm{d}\omega_G(\lambda_A, \lambda_B) = \lambda_A(\omega_G \lambda_B) - \lambda_B(\omega_G \lambda_A) - \omega_G[\lambda_A, \lambda_B] \\
&= \lambda_A(\mathrm{const}) - \lambda_B(\mathrm{const}) - \omega_G \lambda_{[A,B]} = -[A, B] = -\tfrac{1}{2}[\omega_G, \omega_G](X, Y).
\end{aligned}$$

□

Let us now proceed to the general case with P a principal G-bundle over M and a principal connection on P. For $A \in \mathfrak{g}$ the fundamental vector field $A^* : P \to VP$ is given by

$$A^*(u) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\, u \cdot \exp(tA)$$

The derivative at a general t_0 is

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_0}\, u \cdot \exp(tA) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_0}\, u \cdot \exp(t_0 A) \cdot \exp((t - t_0)A) = A^*(u \cdot \exp(t_0 A))$$

In particular $\mathrm{Fl}_t^{A^*} = u \cdot \exp(tA)$ or in other words $\mathrm{Fl}_t^{A^*} = r_{\exp(tA)}$.

Lemma 4.4. *For arbitrary horizontal vector field* Y *on* P*,* $[A^*, Y]$ *is also horizontal.*

Proof. We determine the Lie bracket by

$$[A^*, Y](u) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\, (\mathrm{Fl}_{-t}^{A^*})_* Y(\mathrm{Fl}_t^{A^*}(u)) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\, (r_{\exp(tA)})_* Y(u \cdot \exp(tA))$$

Here $Y(u \cdot \exp(tA))$ is horizontal by assumption and the action preserves horizontality. Therefore the curve lies in H_uP and so does its derivative. □

The proof of the structure equation splits into three cases by bilinearity

- both X and Y vertical: then $\Omega(X,Y)=0$ as Ω is horizontal. The restriction $\omega|_{TP_x}$ of the connection form to the fibre is the Maurer-Cartan form ω_G and the Maurer-Cartan equation finishes this case.
- $X=A^*$ vertical and Y horizontal: still $\Omega(X,Y)=0$ by horizontality. The left hand side is

$$\mathrm{d}\omega(A^*,Y)=A^*(\underbrace{\omega Y}_{0})-Y(\underbrace{\omega A^*}_{\text{const.}})-\omega\ \underbrace{[A^*,Y]}_{\text{horizontal}}=0$$
$$\tfrac{1}{2}[\omega,\omega](A^*,Y)=[\omega A^*,\underbrace{\omega Y}_{0}]=0$$

- both X and Y horizontal: then $\frac{1}{2}[\omega,\omega](X,Y)=0$ and the structure equation says

$$\mathrm{d}\omega(X,Y)\overset{?}{=}D\omega(X,Y)=\mathrm{d}\omega(hX,hY)$$

which is satisfied by the horizontality of X and Y.

□

LEMMA 4.5. *A differential k-form φ on P projects to a k-form ψ on M (i.e. $\varphi=\pi^*\psi$) if and only if the following two conditions are satisfied*

(1) *φ is horizontal and*
(2) *$\varphi(X_1a,\dots,X_ka)=\varphi(X_1,\dots,X_k)$, i.e. φ is of the type given by the trivial representation $e:G\to\mathrm{GL}(1)$, $a\mapsto e=\mathrm{id}$.*

PROOF. This is a special case of the bijection $\Omega^k_{\mathrm{hor}}(P,\rho)\cong\Omega^k(M,P[W])$ for $W=\mathbb{R}$ with the trivial action so that $P[W]=M\times\mathbb{R}$. □

Now we will construct k-forms on M from the curvature Ω. We denote by $\mathcal{I}^k(G)$ the set of all symmetric multilinear maps

$$f:\mathfrak{g}\times\cdots\times\mathfrak{g}\longrightarrow\mathbb{R}$$

satisfying $f(\mathrm{Ad}(a)A_1,\dots,\mathrm{Ad}(a)A_k)=f(A_1,\dots,A_k)$. In other words f is equivariant with respect to the trivial action of G on $\mathbb{R}$. The curvature form $\Omega:\Lambda^2TP\to\mathfrak{g}$ then induces

$$\Lambda^2TP\otimes\cdots\otimes\Lambda^2TP\xrightarrow{\Omega\otimes\cdots\otimes\Omega}\mathfrak{g}\otimes\cdots\otimes\mathfrak{g}\xrightarrow{f}\mathbb{R}$$

Antisymmetrizing we obtain $f(\Omega):\Lambda^{2k}TP\to\mathbb{R}$.

THEOREM 4.6. *The $(2k)$-form $f(\Omega)$ on P projects to a $(2k)$-form $\underline{f}(\Omega)$ on M.*

PROOF. Easily $f(\Omega)$ is horizontal since Ω is and

$$f(\Omega)(X_1,\dots,X_{2k})=\sum f(\Omega(X_{\sigma(1)},X_{\sigma(2)}),\dots,\Omega(X_{\sigma(2k-1)},X_{\sigma(2k)}))$$

the sum being taken over $(2,\dots,2)$-shuffles. The equivariancy follows from Ω being of type Ad and f being equivariant. □

DEFINITION 4.7. The form $\underline{f}(\Omega)$ is called the *Chern-Weil form.*

LEMMA 4.8. *When an r-form φ on P projects to an r-form $\underline{\varphi}$ on M then $\mathrm{d}\underline{\varphi}=\underline{D\varphi}$ for any connection on P.*

PROOF. First we express

$$\underline{D\varphi}(X_0,\dots,X_r)=D\varphi(\tilde{X}_0,\dots,\tilde{X}_r)=\mathrm{d}\varphi(\tilde{X}_0,\dots,\tilde{X}_r)$$

To compute $\mathrm{d}\underline{\varphi}$ we use $\underline{\varphi}(X_0,\dots,X_r)=\varphi(\tilde{X}_0,\dots,\tilde{X}_r)$ and differentiate

$$\begin{aligned}\mathrm{d}\underline{\varphi}(X_0,\dots,X_r) &= \sum(-1)^i X_i\cdot\underline{\varphi}(X_0,\dots,\widehat{X_i},\dots,X_r)\\ &\qquad+\sum(-1)^{i+j}\underline{\varphi}([X_i,X_j],X_0,\dots,\widehat{X_i},\dots,\widehat{X_j}\dots,X_r)\\ &=\sum(-1)^i\tilde{X}_i\cdot\varphi(\tilde{X}_0,\dots,\widehat{\tilde{X}_i},\dots,\tilde{X}_r)\\ &\qquad+\sum(-1)^{i+j}\varphi(\widetilde{[X_i,X_j]},\tilde{X}_0,\dots,\widehat{\tilde{X}_i},\dots,\widehat{\tilde{X}_j}\dots,\tilde{X}_r)\end{aligned}$$

This is exactly $\mathrm{d}\varphi(\tilde{X}_0,\dots,\tilde{X}_r)$ when $\widetilde{[X_i,X_j]}$ is replaced by $[\tilde{X}_i,\tilde{X}_j]$. But since the difference is a vertical vector and φ is horizontal this makes no difference. □

THEOREM 4.9. *All Chern-Weil forms $\underline{f}(\Omega)$ are closed.*

PROOF. By the previous lemma $\mathrm{d}\underline{f}(\Omega)=\mathrm{d}\underline{f(\Omega)}=\underline{Df(\Omega)}$ and

$$Df(\Omega)=D(f\circ(\Omega\wedge\dots\wedge\Omega))=\sum f\circ(\Omega\wedge\dots\wedge\Omega\wedge\underbrace{D\Omega}_{0}\wedge\Omega\wedge\dots\wedge\Omega)=0$$

with $D\Omega=0$ by the Bianchi identity. □

LEMMA 4.10. *Let φ be a horizontal 1-form of type ρ. Then*

$$D\varphi(X,Y)=\mathrm{d}\varphi(X,Y)+\omega(X)\cdot\varphi(Y)-\omega(Y)\cdot\varphi(X)$$

where the dot stands for the infinitesimal action $A\cdot w=\rho_*(A)(w)$ with $\rho_*:\mathfrak{g}\to\mathfrak{gl}(W)$ the derivative of ρ. We may write simply

$$D\varphi=\mathrm{d}\varphi+\omega\cdot\varphi$$

PROOF. Again we split the proof into three cases.

- both X and Y horizontal. Then $D\varphi(X,Y)=\mathrm{d}\varphi(X,Y)$ and $\omega(X)=0=\omega(Y)$.
- both $X=A^*$, $Y=B^*$ vertical. Then $D\varphi(X,Y)=0$ and
$$\mathrm{d}\varphi(A^*,B^*)=A^*\underbrace{\varphi(B^*)}_{0}-B^*\underbrace{\varphi(A^*)}_{0}-\varphi[A^*,B^*]=-\varphi[A,B]^*=0$$
As $\varphi(A^*)=0=\varphi(B^*)$ the equality holds trivially.
- $X=A^*$ vertical and $Y=\tilde{Z}$ horizontal. Still $D\varphi(X,Y)=0$ and
$$\mathrm{d}\varphi(A^*,\tilde{Z})=A^*\varphi(\tilde{Z})-\tilde{Z}\underbrace{\varphi(A^*)}_{0}-\varphi[A^*,\tilde{Z}]$$
In the last term
$$[A^*,\tilde{Z}]=\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(r_{\exp(-tA)})_*\tilde{Z}(u\cdot\exp(tA))=\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\tilde{Z}(u)=0$$
so that
$$\begin{aligned}\mathrm{d}\varphi(A^*,\tilde{Z})=A^*\varphi(\tilde{Z})&=\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\varphi(\tilde{Z}(u\cdot\exp(tA))=\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\varphi(\tilde{Z}(u)\cdot\exp(tA))\\ &=\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\rho(\exp(-tA))\varphi(\tilde{Z}(u))=-\rho_*(A)\cdot\varphi(\tilde{Z}(u))\end{aligned}$$
Since $A=\omega(X)$ this equals $-\omega(X)\cdot\varphi(Y)$. As $\omega(Y)\cdot\varphi(X)=0$ the equality holds.

□

We are now aiming at the independence of the cohomology class of the Chern-Weil form under the choice of the principal connection. Therefore let Γ_0 and Γ_1 be two principal connections with associated forms ω_0 and ω_1. Put $\alpha=\omega_1-\omega_0\in\Omega^1_{\mathrm{hor}}(P,\mathrm{Ad})$, horizontal by $\omega_1(A^*)=A=\omega_0(A^*)$. Then a covariant derivative with respect to some principal connection ω is

$$D\alpha=\mathrm{d}\alpha+[\omega,\alpha]$$

since α is of type $\rho=\mathrm{Ad}$ and $\rho_*=\mathrm{ad}=[\ ,\ ]$. We consider a 1-parameter family of connections

$$\omega_t=\omega_0+t\alpha=(1-t)\omega_0+t\omega_1$$

(note that principal connections form an affine space in $\Omega^1(P,\mathfrak{g})$ as both conditions - being of type Ad and the reproduction of vertical vector fields - are affine). We denote by Ω_t the curvature associated to ω_t and D_t the covariant differential.

LEMMA 4.11. *$\frac{\mathrm{d}}{\mathrm{d}t}\Omega_t = D_t\alpha$.*

PROOF. To explain the formula $\Omega_t(u) \in \hom(\Lambda^2 T_u P, \mathfrak{g})$ and the derivative is taken in this vector space. Differentiate the structure equation

$$\Omega_t = \mathrm{d}\omega_t + \tfrac{1}{2}[\omega_t, \omega_t]$$

to obtain

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\Omega_t &= \tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}(\omega_t) + \tfrac{1}{2}[\tfrac{\mathrm{d}}{\mathrm{d}t}\omega_t, \omega_t] + \tfrac{1}{2}[\omega_t, \tfrac{\mathrm{d}}{\mathrm{d}t}\omega_t] \\
&= \mathrm{d}(\tfrac{\mathrm{d}}{\mathrm{d}t}\omega_t) + \tfrac{1}{2}[\alpha, \omega_t] + \tfrac{1}{2}[\omega_t, \alpha] = \mathrm{d}\alpha + [\omega_t, \alpha] = D_t\alpha
\end{aligned}$$

□

DEFINITION 4.12. Define a horizontal $(2k-1)$-form on P

$$f(\alpha, \underbrace{\Omega_t, \dots, \Omega_t}_{k-1}) = f \circ (\alpha \wedge \Omega_t \wedge \dots \wedge \Omega_t) : \Lambda^{2k-1}TP \longrightarrow \mathbb{R}$$

It projects onto a $(2k-1)$-form $\underline{f}(\alpha, \Omega_t, \dots, \Omega_t) \in \Omega^{2k-1}(M)$. Let

$$\Phi = k \cdot \int_0^1 \underline{f}(\alpha, \Omega_t, \dots, \Omega_t) \in \Omega^{2k-1}(M)$$

THEOREM 4.13. *It holds $\mathrm{d}\Phi = \underline{f}(\Omega_1) - \underline{f}(\Omega_0)$ so that the forms $\underline{f}(\Omega_0)$ and $\underline{f}(\Omega_1)$ determine the same class in the de Rham cohomology.*

PROOF. Since $\underline{f}(\Omega_1) - \underline{f}(\Omega_0) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\underline{f}(\Omega_t)\ \mathrm{d}t$ we compute

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\underline{f(\Omega_t, \dots, \Omega_t)} &= \sum \underline{f(\Omega_t, \dots, \tfrac{\mathrm{d}}{\mathrm{d}t}\Omega_t, \dots, \Omega_t)} = k \cdot \underline{f(\tfrac{\mathrm{d}}{\mathrm{d}t}\Omega_t, \Omega_t, \dots, \Omega_t)} \\
&= k \cdot \underline{f(D_t\alpha, \Omega_t, \dots, \Omega_t)} = k \cdot \underline{D_t f(\alpha, \Omega_t, \dots, \Omega_t)} \\
&= k \cdot \mathrm{d}(\underline{f(\alpha, \Omega_t, \dots, \Omega_t)})
\end{aligned}$$

Thus

$$\underline{f}(\Omega_1) - \underline{f}(\Omega_0) = \int_0^1 k \cdot \mathrm{d}(\underline{f}(\alpha, \Omega_t, \dots, \Omega_t))\ \mathrm{d}t = \mathrm{d}\left(\int_0^1 k \cdot \underline{f}(\alpha, \Omega_t, \dots, \Omega_t)\ \mathrm{d}t\right) = \mathrm{d}\Phi$$

□

THEOREM 4.14 (reformulation). *For each $f \in \mathcal{I}^k(G)$ the de Rham class of the Chern-Weil form $\underline{f}(\Omega)$ does not depend on the connection Γ.* □

EXAMPLE 4.15. Consider the example $G = \mathrm{GL}(k)$, i.e. the example of vector bundles of dimension k. Here $\mathfrak{g} = \mathfrak{gl}(k)$ and $\mathrm{Ad}(a)(A) = aAa^{-1}$. The trace of a matrix is a map $\mathrm{tr} : \mathfrak{gl}(k) \to \mathbb{R}$ satisfying $\mathrm{tr}(aAa^{-1}) = \mathrm{tr}(A)$. Therefore $\mathrm{tr} \in \mathcal{I}^1(\mathrm{GL}(k))$ and yields a class $[\underline{\mathrm{tr}}(\Omega)] \in H^2(M)$. In the tutorial we will show that $\mathcal{I}^1(\mathrm{GL}(k)) = \langle \mathrm{tr} \rangle$. There exist higher traces

$$\begin{aligned}
\mathrm{tr}_j : \mathfrak{gl}(k) \otimes \dots \otimes \mathfrak{gl}(k) &\longrightarrow \mathbb{R} \\
X_1 \otimes \dots \otimes X_j &\longmapsto \mathrm{tr}(X_1 \cdots X_j)
\end{aligned}$$

which exhibits a cyclic symmetry. Fully symmetrizing we get $\mathrm{sym}(\mathrm{tr}_j) \in \mathcal{I}^j(\mathrm{GL}(k))$. "A bit of representation theory" implies that all Chern-Weil forms are generated by tr_j via the wedge product and linear combinations.

We will show now that for j odd these classes are zero. This will follow from the fact that every principal $\mathrm{GL}(k)$-bundle P admits a reduction Q to $\mathrm{O}(k)$. This means that for the connection induced from a principal connection on Q the curvature Ω takes values in $\mathfrak{o}(k)$ (this will be shown at the tutorial), the algebra of anti-symmetric matrices. Since an odd power of an anti-symmetric matrix is again anti-symmetric the trace $\underline{\mathrm{tr}}_{2i-1}(\Omega)$ must be zero.

For complex vector bundles there are non-zero classes in all even dimensions up to the dimension of the vector bundle.

5. The canonical form on P^1M (solder form)

Let $\pi : P^1M \to M$ be the bundle of frames on M, $P^1M = PTM \ni u = (u_1, \dots, u_m)$ a basis of T_xM, $x = \pi(u)$. It is a principal $\mathrm{GL}(m)$-bundle whose fibre can be described as $\operatorname{inv}\hom(\mathbb{R}^m, T_xM)$. The action of $\mathrm{GL}(m)$ on P^1M is then given as precomposition. Alternatively $(u_1, \dots, u_m)a = (\sum u_i a_{i1}, \dots, \sum u_i a_{im})$.

Definition 5.1. The *canonical form* on P^1M is the $\mathbb{R}^m$-valued 1-form θ defined by

$$\pi_* X = \theta^1(X)u_1 + \cdots + \theta^m(X)u_m$$

In other words, the components of $\theta(X)$ are the coordinates of π_*X in the basis u. Using the frame map $\rho : P^1M \times \mathbb{R}^m \to TM$, $(u, \alpha) \mapsto \sum \alpha^i u_i$ the definition becomes

$$\rho(u, \theta(X)) = \pi_*(X)$$

Theorem 5.2. *$\theta \in \Omega^1_{\mathrm{hor}}(P, \mathrm{id})$ where $\mathrm{id} : \mathrm{GL}(m) \to \mathrm{GL}(m)$ is the standard representation of $\mathrm{GL}(m)$ on the vector space $\mathbb{R}^m$.*

Proof. The horizontality is obvious since $\pi_*X = 0$ for a vertical vector X. Since the action preserves fibres we have

$$(ua) \cdot \theta(Xa) = \pi_*(Xa) = \pi_*X = u \cdot \theta(X)$$

implying $a\theta(Xa) = \theta(X)$ as both are coordinates of π_*X in the basis u. Finally $\theta(Xa) = a^{-1}\theta(X)$. □

Lemma 5.3. *Under the identification*

$$\mathfrak{X}M = C^\infty TM = C^\infty P^1M[\mathbb{R}^m] \cong \mathrm{map}_{\mathrm{GL}(m)}(P^1M, \mathbb{R}^m)$$

a vector field X corresponds to $\theta(\tilde{X})$.

Proof. The section of $P^1M[\mathbb{R}^m]$ corresponding to $u \mapsto \theta(\tilde{X}(u))$ sends x to

$$[u, \theta(\tilde{X}(u))] \sim u \cdot \theta(\tilde{X}(u)) = X(x)$$

where $u \in P^1M_x$ is arbitrary and $\sim$ denotes the identification $P^1M[\mathbb{R}^m] \cong TM$. □

Definition 5.4. Let Γ be a principal connection on P^1M. The covariant differential $D\theta$ is called the *torsion form* of the connection Γ.

Another definition of θ is as the equivariant horizontal 1-form corresponding to $\mathrm{id} \in \Omega^1(M; TM)$. To compute, what it does, we start with

$$X = \mathrm{id}(X) = [u, \theta(\tilde{X}(u))] = \sum u^i \theta_i \tilde{X}(u),$$

i.e. the components θ_i are the coordinates of $X = \pi_*\tilde{X}$ in the basis u.

Theorem 5.5. *$D\theta = 0$ if and only if the connection Γ has no torsion.*

Proof. Since the identity corresponds to θ, $\nabla \mathrm{id}$ corresponds to $D\theta$ and we have

$$(\nabla \mathrm{id})(X, Y) = \nabla_X \mathrm{id}(Y) - \nabla_Y \mathrm{id}\, X - \mathrm{id}[X, Y] = \nabla_X Y - \nabla_Y X - [X, Y].$$

Another proof: We will show that the section corresponding to $D\theta(\tilde{X}, \tilde{Y})$ is

$$\nabla_X Y - \nabla_Y X - [X, Y]$$

for any vector fields X and Y on M. But

$$D\theta(\tilde{X}, \tilde{Y}) = \tilde{X}(\theta\tilde{Y}) - \tilde{Y}(\theta\tilde{X}) - \theta[\tilde{X}, \tilde{Y}]$$

where $\underline{\theta\tilde{Y}} = Y$ by the last lemma and hence $\underline{\tilde{X}(\theta\tilde{Y})} = \nabla_X Y$. By horizontality of θ the last term can be simplified $\underline{\theta[\tilde{X}, \tilde{Y}]} = \underline{\theta\widetilde{[X, Y]}} = [X, Y]$. □

For the next theorem denote by dot the following pairing

$$- \cdot - : \mathfrak{gl}(m) \otimes \mathbb{R}^m \to \mathbb{R}^m$$

THEOREM 5.6 (The first structure equation). *It holds $D\theta = \mathrm{d}\theta + \omega \cdot \theta$ where $\omega \in \Omega^1(P^1M, \mathfrak{gl}(m))$ is the form of the connection and $\theta \in \Omega^1(P, \mathbb{R}^m)$ is the canonical form.*

PROOF. We have shown this more generally, $D\varphi = \mathrm{d}\varphi + \rho_* \omega \cdot \varphi$. □

REMARK. By differentiating covariantly once more we obtain

$$D^2\theta = h^*\mathrm{d}(\mathrm{d}\theta + \omega \cdot \theta) = h^*(\mathrm{d}\omega \cdot \theta - \omega \cdot \mathrm{d}\theta) = h^*\mathrm{d}\omega \cdot h^*\theta = D\omega \cdot \theta = \Omega \cdot \theta$$

We have not used any specific property of θ and thus for $\varphi \in \Omega^k_{\text{hor}}(P, \rho)$ it holds generally that $D^2\varphi = \rho_* \Omega \cdot \varphi$. In particular the covariant differential does not in general square to zero.

6. The second tangent space TTM

What is $TTM = T(TM)$? Locally one has

$$TT\mathbb{R}^m = T(\mathbb{R}^m \times \mathbb{R}^m) = \underbrace{\mathbb{R}^m \times \mathbb{R}^m}_{\text{base}} \times \underbrace{\mathbb{R}^m \times \mathbb{R}^m}_{\text{fibre}}$$

Let us write the coordinates as (x, X, Y, α). For $\sigma : \mathbb{R}^2 \to \mathbb{R}^m$ with coordinates s and t on $\mathbb{R}^2$ we have

$$\begin{aligned} \tfrac{\partial}{\partial t}\big|_{t=0} \sigma : \mathbb{R} &\longrightarrow T\mathbb{R}^m \\ s &\longmapsto (\sigma(s,0), \tfrac{\partial}{\partial t}\sigma(s,0)) \end{aligned}$$

Differentiating again we obtain $\frac{\partial}{\partial s}\big|_{s=0} \frac{\partial}{\partial t}\big|_{t=0} \sigma \in TT\mathbb{R}^m$ with coordinates

$$(\sigma(0), \tfrac{\partial}{\partial t}\sigma(0), \tfrac{\partial}{\partial s}\sigma(0), \tfrac{\partial^2}{\partial s \partial t}\sigma(0))$$

We have a well defined map $TT\mathbb{R}^m \to TT\mathbb{R}^m$ by

$$\tfrac{\partial}{\partial s}\big|_{s=0} \tfrac{\partial}{\partial t}\big|_{t=0} \sigma \longmapsto \tfrac{\partial}{\partial t}\big|_{t=0} \tfrac{\partial}{\partial s}\big|_{s=0} \sigma$$

which plainly swaps the middle coordinates but in this form it clearly does not depend on coordinates and thus induces a map

$$\kappa : TTM \to TTM$$

on the second tangent bundle of any manifold M.

Let $\tilde{\ } : TM \times_M TM \to TTM$ be a lifting map of a linear connection on TM. Locally $((x,X),(x,Y)) \mapsto (x, X, Y, (\Gamma^k_{ij} X^i Y^j))$. We can use κ to introduce a new lifting map $\hat{\ }$ via the diagram

$$\begin{array}{ccc} (x, Y, X, (\Gamma^k_{ij} Y^i X^j)) & & (x, X, Y, (\Gamma^k_{ji} X^i Y^j)) \end{array}$$

$$\begin{tikzcd} TTM \arrow[r, "\kappa"] & TTM \\ TM \times_M TM \arrow[u, "\tilde{\ }"] & TM \times_M TM \arrow[l, "ex"'] \arrow[u, dashed, "\hat{\ }"'] \end{tikzcd}$$

$$\begin{array}{ccc} ((x,Y),(x,X)) & & ((x,X),(x,Y)) \end{array}$$

with ex denoting exchanging the two factors.

THEOREM 6.1. *$\hat{\ } = \kappa \circ \tilde{\ } \circ ex$ prescribes a linear connection on TM, the so-called* conjugate connection $\bar{\Gamma}$. *In coordinates $\bar{\Gamma}^k_{ij} = \Gamma^k_{ji}$.*

Let us denote the "translation" map by

$$\begin{aligned} \mathrm{tr} : VTM \cong TM \times_M TM &\xrightarrow{pr_2} TM \\ (x, X, 0, Z) &\longmapsto (x, Z) \end{aligned}$$

LEMMA 6.2. *For any vector fields* $X, Y \in \mathfrak{X}M$

$$\operatorname{tr}(TY \circ X - \kappa \circ TX \circ Y) = [X, Y]$$

PROOF. In coordinates $X : x \mapsto (x, X)$, $Y : x \mapsto (x, Y)$ so that

$$TY \circ X : x \longmapsto (x, Y, X, \sum X^j \tfrac{\partial Y^i}{\partial x^j} \tfrac{\partial}{\partial x^i})$$
$$TX \circ Y : x \longmapsto (x, X, Y, \sum Y^j \tfrac{\partial X^i}{\partial x^j} \tfrac{\partial}{\partial x^i})$$
$$TY \circ X - \kappa \circ TX \circ Y : x \longmapsto (x, Y, 0, [X, Y]) \xrightarrow{\operatorname{tr}} (x, [X, Y])$$

where e.g. the last coordinate of $TY \circ X$ is the derivative of Y along X. □

THEOREM 6.3. *The following holds*

(1) $\bar{\nabla}_X Y = \nabla_Y X + [X, Y]$,
(2) *the section corresponding to* $D\theta(\tilde{X}, \tilde{Y})$ *is* $\nabla_X Y - \bar{\nabla}_X Y$.

PROOF. Once (1) is proved, (2) follows from $\underline{D\theta(\tilde{X}, \tilde{Y})} = \nabla_X Y - \nabla_Y X - [X, Y]$. To prove (1) we observe that by definition

$$\nabla_Y X = \operatorname{tr}(v(TX \circ Y)) = \operatorname{tr}(TX \circ Y - \tilde{Y}(X)) = \operatorname{tr}(\kappa \circ TX \circ Y - \kappa(\tilde{Y}(X)))$$

using $\operatorname{tr} \circ \kappa = \operatorname{tr}$ and analogously

$$\bar{\nabla}_X Y = \operatorname{tr}(TY \circ X - \hat{Y}(X)) = \operatorname{tr}(TY \circ X - \kappa(\tilde{Y}(X)))$$

Subtracting the two formulas reduces the theorem to the previous lemma. □

7. Morphisms of connections

Let $p_i : E_i \to M$, $i = 1, 2$ be two bundles and also $q_i : D_i \to N$. Further let $f_i : E_i \to D_i$ be bundle morphisms over the same base map $\underline{f} : M \to N$. We obtain

$$f_1 \times f_2 = f_1 \times_{\underline{f}} f_2 : E_1 \times_M E_2 \to D_1 \times_N D_2$$

the so-called *fibre product* of f_1 and f_2.

Let now $\tilde{}: E \times_M TM \to TE$ be a lifting map for a connection Γ on $E \to M$ and $\hat{}: D \times_N TN \to TD$ a lifting map for a connection Δ on $D \to N$, $f : D \to E$ a bundle morphism over $\underline{f} : N \to M$.

DEFINITION 7.1. Connections Δ and Γ are calles *f-related* if the diagram

$$\begin{tikzcd} TD \arrow[r, "f_*"] & TE \\ D \times_N TN \arrow[u, "\hat{}"] \arrow[r, "f \times \underline{f}_*"] & E \times_M TM \arrow[u, "\tilde{}"'] \end{tikzcd}$$

commutes. In other words f is required to preserve horizontal vectors, i.e. $f_* \hat{X}(y) = \widetilde{\underline{f}_* X}(f(y))$ or $f_* \Delta(y) \subseteq \Gamma(f(y))$.

We also say that f is a *morphism of connections* Δ and Γ.

DEFINITION 7.2. An *induced connection* $g^*\Gamma$ on the pullback bundle g^*E

$$\begin{tikzcd} g^*E \arrow[r, "\bar{g}"] \arrow[d] & E \arrow[d] \\ N \arrow[r, "g"'] & M \end{tikzcd}$$

is determined by the requirement that the horizontal distribution $g^*\Gamma$ is the preimage of the horizontal distribution Γ, i.e. $g^*\Gamma(x, y) = (\bar{g}_{*(x,y)})^{-1}\Gamma(y)$.

THEOREM 7.3. *The distribution* $g^*\Gamma$ *gives a connection on* g^*E.

PROOF. By the diagram

$$\begin{tikzcd}
(X,Y) \arrow[r, mapsto] \arrow[d, mapsto] & Y \arrow[d, mapsto] \\
X \arrow[r, mapsto] & g_*X = p_*Y
\end{tikzcd}$$

for $(X,Y) \in g^*\Gamma(x,y)$ necessarily $Y = \widetilde{g_*X}(y)$ so that for each $X \in T_xN$ there is a unique $Y \in T_yE$ with $(X,Y) \in g^*\Gamma(x,y)$. $\square$

Another characterization of $g^*\Gamma$ is via the jets of sections: a section $s : M \to E$ representing the horizontal subspace of Γ, i.e. $\Gamma(sg(x)) = j^1_{g(x)}s$, induces a section $g^*s : N \to g^*E$

$$\begin{tikzcd}
N \arrow[r, dashed, "g^*s"'] \arrow[rr, bend left, "s\circ g"] \arrow[dr, "\mathrm{id}"'] & g^*E \arrow[r] \arrow[d] & E \arrow[d] \\
& N \arrow[r, "g"'] & M
\end{tikzcd}$$

and $g^*\Gamma(x, sg(x)) = j^1_x(g^*s)$. In this way one obtains the horizontal spaces at all points.

THEOREM 7.4. *The connections $g^*\Gamma$ and Γ are $\bar{g}$-related.*

PROOF. Follows immediately from the definition. $\square$

THEOREM 7.5. *In the diagram*

$$\begin{tikzcd}
D \arrow[r, "\tilde{f}"'] \arrow[rr, bend left, "f"] \arrow[d] & f^*E \arrow[r] \arrow[d] & E \arrow[d] \\
N \arrow[r, equal] & N \arrow[r, "\underline{f}"'] & M
\end{tikzcd}$$

the connections Δ and Γ are f-related if and only if Δ and $\underline{f}^*\Gamma$ are $\tilde{f}$-related.

PROOF. Easy. $\square$

THEOREM 7.6. *If Δ and Γ are f-related then the following diagram commutes*

$$\begin{tikzcd}
VD \arrow[r, "f_*"] & VE \\
D \times_N \Lambda^2 TN \arrow[u, "C\Delta"] \arrow[r, "f\times\Lambda^2\underline{f}_*"'] & E \times_M \Lambda^2 TM \arrow[u, "C\Gamma"']
\end{tikzcd}$$

We say that $C\Delta$ and $C\Gamma$ are f-related.

PROOF. Let $y \in D$, $X, Y \in T_{q(y)}N$ where $q : D \to N$ is the bundle projection. Suppose first that it is possible to extend X and Y to vector fields X and Y that are $\underline{f}$-related to vector fields X' and Y' on M. Then

$$C\Gamma(f(y), X', Y') = -v[\tilde{X}', \tilde{Y}'](f(y))$$

But $\hat{X}$ and $\hat{Y}$ are f-related to $\tilde{X}'$ and $\tilde{Y}'$ so that $[\hat{X}, \hat{Y}]$ is also f-related to $[\tilde{X}', \tilde{Y}']$ and also $\widehat{[X,Y]}$ is f-related to $\widetilde{[X', Y']}$). Subtracting we obtain that $C\Delta(-, X, Y)$ is f-related to $C\Gamma(-, X', Y')$ which is exactly the commutativity of the diagram from the theorem.

In general the extensions X' and Y' might not exist. They do exist for f an immersion and a submersion. But it is possible to decompose f into a composition of such, namely

$$\begin{tikzcd} D \arrow[r, "\tilde{f}"] \arrow[d] & f^*E \arrow[r] \arrow[d] & pr_2^*E \arrow[r] \arrow[d] & E \arrow[d] \\ N \arrow[r, equal] & N \arrow[r, "(\mathrm{id},\underline{f})"'] & N \times M \arrow[r, "pr_2"'] & M \end{tikzcd}$$

The graph $(\mathrm{id}, \underline{f})$ of $\underline{f}$ is obviously an immersion while the projection pr_2 is a submersion. The extensions are easy to construct. □

8. Problems

Problem 8.1. Show that a linear connection is complete.

Problem 8.2. Show that for a vector bundle $E \to M$ the vector space $C^\infty E$ of all smooth sections of E is naturally a module over $C^\infty M$. Further show that if E and F are two vector bundles over M then there is a bijection between linear morphisms $E \to F$ and $C^\infty M$-linear homomorphisms $C^\infty E \to C^\infty F$.

Problem 8.3. Show that if $\varphi : C^\infty E_1 \times C^\infty E_2 \longrightarrow C^\infty F$ is bilinear over $C^\infty M$ then the value of $\varphi(s_1, s_2)$ at x depends only on $s_1(x)$ and $s_2(x)$ and this dependence describes a bilinear morphism $E_1 \times_M E_2 \longrightarrow F$ of bundles over M.

One may reduce to the previous problem by showing that

$$C^\infty E_1 \otimes_{C^\infty M} C^\infty E_2 \cong C^\infty(E_1 \otimes_M E_2)$$

Problem 8.4. Apply the previous problem to the curvature

$$C\Gamma : C^\infty(p^*TM) \times C^\infty(p^*TM) \longrightarrow C^\infty VE$$

Problem 8.5. Show that an exterior derivative of a 1-form $\varphi \in \Omega^1(M)$ satisfies

$$\mathrm{d}\varphi(X, Y) = X\varphi(Y) - Y\varphi(X) - \varphi[X, Y]$$

for any two vector fields $X, Y \in \mathfrak{X}M$. Generalize to higher degrees.

Problem 8.6. Show that $\mathcal{I}^1(\mathrm{GL}(k)) = \langle \mathrm{tr} \rangle$.

Decomposition into symmetric and antisymmetric matrices yields an easy reduction to linear forms on symmetric matrices which are $\mathrm{O}(k)$-invariant. Since each is equivalent to a diagonal one this gives the result.

Problem 8.7. Let $Q \subseteq P$ be a reduction of a principal G-bundle P to $H \subseteq G$. Prove the following two characterizations of principal connections on P induced from principal connections on Q:

- Γ is a principal connection tangent to Q, i.e. $\Gamma|_Q \subseteq TQ$,
- ω is a principal connection whose restriction $\omega|_Q$ to Q takes values in $\mathfrak{h}$.

Problem 8.8. Show that the canonical form $\theta : TP^1M \to \mathbb{R}^m$ corresponds to $\mathrm{id} : TM \to TM$.

Problem 8.9. Show that under the identification of sections $s \in C^\infty(P[W])$ and equivariant maps $\sigma : P \to W$ we get that $\nabla_X s$ corresponds to $D\sigma(\tilde{X})$. Maybe not a good problem...

Problem 8.10. Let E, F be two vector bundles associated to $P(M, G)$ and let $s \in C^\infty E$ and $t \in C^\infty F$ be two sections. Then $\nabla_X(s \otimes t) = \nabla_X s \otimes t + s \otimes \nabla_X t$.

Problem 8.11. Let $P \to M$ be a principal $\mathrm{GL}(k)$-bundle and $Q \subseteq P$ a reduction to $\mathrm{O}(k)$ which is equivalent to a scalar product $g \in C^\infty(E \otimes E)^*$. Let Γ be a principal connection of P. Show that the following conditions are equivalent.

- Γ reduces to Q,
- $\nabla g = 0$ (g is then called covariantly constant),

- the parallel transport on E preserves the scalar product.

PROBLEM 8.12. Let $P(M,G)$ be a principal bundle with a principal connection ω and let $\rho : G \to \mathrm{GL}(W)$ be a representation, $E = P \times_G W$ the associated vector bundle. Describe the curvature $C\Gamma_W$ of the associated bundle in terms of Ω.

PROBLEM 8.13. Describe for $X, Y \in \mathfrak{X}M$ and $s \in C^\infty E$ the expression

$$\nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$$

in terms of the equivariant map $\sigma : P \to W$ corresponding to s.

PROBLEM 8.14. Let $\iota : Q \hookrightarrow P$ be an inclusion of a reduction Q of P to a subgroup $H \subseteq G$. Let Γ_Q be a reduction of a principal connection Γ_P on P. Show that Γ_Q and Γ_P are ι-related and that Ω_Q is a restriction of Ω_P to Q. Deduce that $\Omega_P|_Q$ takes values in $\mathfrak{h}$.

As a consequence for a principal $\mathrm{GL}(k)$-connection ω the curvature is trace-free, $\operatorname{tr}\Omega = 0$, when the connection reduces to $\mathrm{O}(k)$. Namely any element of P may be expressed as $u \cdot a$ with $u \in Q$ and $a \in \mathrm{GL}(k)$. Then

$$\operatorname{tr}\Omega(\tilde{X}(u\cdot a), \tilde{Y}(u \cdot a)) = \operatorname{tr}(a^{-1}\Omega(\tilde{X}(u), \tilde{Y}(u))a) = \operatorname{tr}\Omega(\tilde{X}(u), \tilde{Y}(u)) = 0$$

since $\Omega(\tilde{X}(u), \tilde{Y}(u))$ is an antisymmetric matrix and as such has zero trace.

CHAPTER 4

Riemannian geometry

1. Interpretation of Riemannian geometry

Let us start with a motivation which I presented in the tutorials. Let $P \to M$ be a principal $\mathrm{GL}(k)$-bundle, $Q \subseteq P$ a reduction to $\mathrm{O}(k)$ and Γ a principal connection on P. The question arises how to recognize whether Γ is associated to a principal connection on Q where we think of P as $Q \times_{\mathrm{O}(k)} \mathrm{GL}(k)$ to make sense of this. In such a situation we say that *the connection* Γ *reduces to* Q. In this special case a reduction to $\mathrm{O}(k)$ is the same as a choice of a scalar product

$$g : E \otimes E \to \mathbb{R}$$

on the associated bundle $E = P \times_{\mathrm{GL}(k)} \mathbb{R}^k$.

Theorem 1.1. *The following conditions are equivalent*

(1) Γ *reduces to* Q*,*
(2) $\nabla g = 0$ *(we say that* g *is covariant constant),*
(3) *the parallel transport on* E *preserves the scalar product.*

Lemma 1.2. *Let* E_1 *and* E_2 *be two vector bundles associated to* $P(M, G)$*,* $s_1 \in C^\infty E_1$*,* $s_2 \in C^\infty E_2$ *two sections. Then*

$$\nabla_X(s_1 \otimes s_2) = \nabla_X s_1 \otimes s_2 + s_1 \otimes \nabla_X s_2$$

Proof. Let s_i be associated to an equivariant map $\sigma_i : P \to W_i$, i.e. $s_i(\pi(u)) = [u, \sigma_i(u)]$. We have an isomorphism

$$\begin{aligned} (P \times_G W_1) \otimes (P \times_G W_2) &\xrightarrow{\cong} P \times_G (W_1 \otimes W_2) \\ [u, v_1] \otimes [u, v_2] &\longmapsto [u, v_1 \otimes v_2] \end{aligned}$$

under which the section $s_1 \otimes s_2$ becomes $\sigma_1 \otimes \sigma_2 : P \to W_1 \otimes W_2$ since

$$(s_1 \otimes s_2)(\pi(u)) = [u, \sigma_1(u)] \otimes [u, \sigma_2(u)] = [u, (\sigma_1 \otimes \sigma_2)(u)]$$

In our correspondence $\nabla_X s_i$ becomes $\mathrm{d}\sigma_i(\tilde{X})$ and thus $\nabla_X(s_1 \otimes s_2)$ corresponds to

$$\mathrm{d}(\sigma_1 \otimes \sigma_2)(\tilde{X}) = \mathrm{d}\sigma_1(\tilde{X}) \otimes \sigma_2 + \sigma_1 \otimes \mathrm{d}\sigma_2(\tilde{X})$$

because the coordinates of $\sigma_1 \otimes \sigma_2$ are products of coordinates of σ_1 and of σ_2. The right hand side then corresponds to the formula from the statement. □

Proof of the theorem. We start with "(1) ⇒ (2)". The scalar product section g of $(E \otimes E)^* \cong Q \times_{\mathrm{O}(k)} (\mathbb{R}^k \otimes \mathbb{R}^k)^*$ corresponds to an equivariant map $\gamma : Q \to (\mathbb{R}^k \otimes \mathbb{R}^k)^*$ which we now identify. Let $u \in Q_x$ be an orthonormal basis of E_x thought of as a map $u : \mathbb{R}^k \xrightarrow{\cong} E_x$. Then

$$g(\pi(u)) = [u, \gamma(u)] : E_x \otimes E_x \xrightarrow{u^{-1} \otimes u^{-1}} \mathbb{R}^k \otimes \mathbb{R}^k \xrightarrow{\gamma(u)} \mathbb{R}$$

Thus $\gamma(u)$ is the scalar product g expressed in the *orthonormal* basis u, i.e. $\gamma(u)$ is the standard scalar product (independently of u) and γ is constant. Thus $\mathrm{d}\gamma = 0$ and hence $\nabla g = 0$.

We now reformulate (2) slightly. Let $\mathrm{ev} : (\mathbb{R}^k \otimes \mathbb{R}^k)^* \otimes \mathbb{R}^k \otimes \mathbb{R}^k \longrightarrow \mathbb{R}$ be the evaluation map $h \otimes v \otimes w \mapsto (h(v \otimes w)$. Then ev induces a (linear) morphism of induced vector bundles

$$P \times_{\mathrm{GL}(k)} ((\mathbb{R}^k \otimes \mathbb{R}^k)^* \otimes \mathbb{R}^k \otimes \mathbb{R}^k) \longrightarrow P \times_{\mathrm{GL}(k)} \mathbb{R} = M \times \mathbb{R}$$

For sections g, s_1 and s_2 we therefore obtain

$$\begin{aligned}\nabla_X(g(s_1\otimes s_2)) &= \nabla_X \operatorname{ev}(g\otimes s_1\otimes s_2) = \operatorname{ev}\nabla_X(g\otimes s_1\otimes s_2)\\ &= \operatorname{ev}(\nabla_X g\otimes s_1\otimes s_2 + g\otimes\nabla_X s_1\otimes s_2 + g\otimes s_1\otimes\nabla_X s_2)\\ &= \nabla_X g(s_1\otimes s_2) + g(\nabla_X s_1\otimes s_2) + g(s_1\otimes\nabla_X s_2)\end{aligned}$$

In other words

$$\nabla_X\langle s_1, s_2\rangle = \nabla_X g(s_1\otimes s_2) + \langle\nabla_X s_1, s_2\rangle + \langle s_1, \nabla_X s_2\rangle$$

Now we are ready to prove "(2) $\Rightarrow$ (3)". For vectors $v_1, v_2 \in E_x$ and a path $\gamma : \mathbb{R} \to M$ through x let us denote the parallel transport of v_i along γ by $\tilde\gamma_i$. The definition gives $\nabla_{\dot\gamma}\tilde\gamma_i = 0$ and thus

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\langle\tilde\gamma_1,\tilde\gamma_2\rangle = \nabla_{\dot\gamma}\langle\tilde\gamma_1,\tilde\gamma_2\rangle = \langle\nabla_{\dot\gamma}\tilde\gamma_1,\tilde\gamma_2\rangle + \langle\tilde\gamma_1,\nabla_{\dot\gamma}\tilde\gamma_2\rangle = 0$$

The scalar product $\langle\tilde\gamma_1,\tilde\gamma_2\rangle$ is therefore constant which is exactly what (3) asserts.

Finally we prove "(3) $\Rightarrow$ (1)". Let $x \in M$ and represent $X \in T_xM$ as the velocity $\dot\gamma(0)$ of a path $\gamma : \mathbb{R} \to M$. Choose an orthonormal basis $u = (u_1, \ldots, u_k)$ of E_x. By (3) the parallel transports $\tilde\gamma_i$ of u_i along γ form an orthonormal basis at the points $\gamma(t)$ on the path. The derivatives $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\tilde\gamma_i$ are the horizontal lifts $\tilde X(u_i)$ and they constitute a horizontal lift

$$\tilde X(u) = (\tilde X(u_1), \ldots, \tilde X(u_k)) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(\tilde\gamma_1, \ldots, \tilde\gamma_k)$$

of X at u. Since the path takes place in Q, $\tilde X(u) \in T_uQ$ and therefore the connection reduces to Q. □

A second exercise was to identify the curvature of the associated connection. The idea is that this should be determined by the curvature of the principal connection which is equivalent to the curvature form Ω. Thus one should be able to express the curvature $C\Gamma_W$ translated from VE to to the zero section E using Ω. More precisely $\operatorname{tr} C\Gamma$ is a bundle morphism

$$\operatorname{tr} C\Gamma_W : E\times_M \Lambda^2TM \longrightarrow W$$

and as such is induced by an equivariant map

$$P\times_M\Lambda^2TM \longrightarrow \operatorname{map}(W, W)$$

In fact we will see that the curvature is linear and $\operatorname{map}(W, W)$ can be replaced by $\hom(W, W) = \mathfrak{gl}(W)$.

Theorem 1.3. *Let $P(M, G)$ be a principal bundle equipped with a principal connection ω, $\rho : G \to \mathrm{GL}(W)$ a linear representation, $E = P\times_G W$ the associated vector bundle. Then the curvature $\operatorname{tr} C\Gamma_W(-, X, Y) : E \to E$ of the associated connection is induced by the map $P \to \mathfrak{gl}(W)$, $u \mapsto \rho_*\Omega(\tilde X, \tilde Y)(u)$.*

Proof. As we do not want to confuse the Lie bracket with points in $E = P\times_G W$ (i.e. classes $[u, v]$ of pairs $(u, v) \in P\times W$) we will use the quotient map

$$q : P\times W \longrightarrow P\times_G W = E$$

to denote the latter. The horizontal lifts $\hat X$ on E are given by

$$\hat X(q(u, v)) = q_*(\tilde X(u), 0_v)$$

using the horizontal lift $\tilde X$ for P. We need to compute their Lie bracket. For this we observe that we have $(\tilde X, 0) \sim_q \hat X$ and $(\tilde Y, 0) \sim_q \hat Y$. Therefore

$$[(\tilde X, 0), (\tilde Y, 0)] \sim_q [\hat X, \hat Y]$$

Since q is a submersion this determines $[\hat X, \hat Y]$:

$$[\hat X, \hat Y](q(u, v)) = q_{*(u,v)}[(\tilde X, 0), (\tilde Y, 0)] = q_{*(u,v)}([\tilde X, \tilde Y], 0)$$

Subtracting from $\widehat{[X, Y]}(q(u, v)) = q_{*(u,v)}(\widetilde{[X, Y]}, 0)$ we get

$$C\Gamma_W(q(u, v), X, Y) = q_*(C\Gamma(u, X, Y), 0_v) = q_*(\Omega(\tilde X, \tilde Y)^*(u), 0_v)$$

The last step is to use $A^*(u) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} u \cdot \exp(tA)$ to simplify to

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} q(u \cdot \exp(t \cdot \Omega(\tilde{X}, \tilde{Y})(u)), v) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} q(u, \exp(t \cdot \Omega(\tilde{X}, \tilde{Y})(u)) \cdot v)$$

When composed with the translation this can be written as

$$q(u, \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \exp(t \cdot \Omega(\tilde{X}, \tilde{Y})(u)) \cdot v)) = q(u, \rho_*(\Omega(\tilde{X}, \tilde{Y})(u)) \cdot v)$$

□

The last exercise was to express the section $\nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$ via an equivariant map

THEOREM 1.4. *The following formula holds.*

$$\nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s = \operatorname{tr} C\Gamma(s, X, Y)$$

PROOF. If s corresponds to σ then $\nabla_X s$ corresponds to $\tilde{X}\sigma$ and the whole formula to

$$(\tilde{X}\tilde{Y} - \tilde{Y}\tilde{X} - \widetilde{[X,Y]})\sigma = ([\tilde{X}, \tilde{Y}] - \widetilde{[X,Y]})\sigma = -C\Gamma_W(-, X, Y)\sigma$$

Now we express the result using Ω to get

$$-C\Gamma_W(u, X, Y)\sigma = -\big(\Omega(\tilde{X}, \tilde{Y})(u)\big)^* \sigma(u) = \ell_{\Omega(\tilde{X},\tilde{Y})(u)}(\sigma(u)) = \rho_*(\Omega(\tilde{X}, \tilde{Y})(u)) \cdot \sigma(u)$$

By the previous theorem this corresponds to the section

$$\operatorname{tr} C\Gamma_W(q(u, \sigma(u)), X, Y) = \operatorname{tr} C\Gamma_W(s, X, Y)$$

□

2. The curvatures of a Riemannian space

For a smooth manifold M a Riemannian structure is a section $g : M \to S^2_+ T^*M$ of the bundle of symmetric positive definite bilinear forms. We say that (M, g) is a Riemannian manifold (a manifold M equipped with a Riemannian metric g).

DEFINITION 2.1. A *Levi-Civita connection* ∇ on TM is characterized by its three properties

(1) it is linear,
(2) torsion-free, i.e. $D\theta = \mathrm{d}\theta + \omega \cdot \theta = 0$,
(3) $\nabla g = 0$, i.e. ∇ comes from a connection on the subbundle $Q^1M \subseteq P^1M$ of orthonormal frames.

Let us consider the curvature of the Levi-Civita connection

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

a section R of $TM \otimes \Lambda^2 T^*M \otimes T^*M$ with factors corresponding to the value of R, the X and Y entries, and the Z entry. We have shown that the equivariant map inducing R is

$$\begin{aligned} Q^1M &\longrightarrow \mathbb{R}^m \\ u &\longmapsto \Omega(\tilde{X}, \tilde{Y})(u) \cdot \theta\tilde{Z}(u) \end{aligned}$$

where $\theta\tilde{Z}$ is the map corresponding to Z. The map corresponding to g is the constant map with value the standard scalar product.

THEOREM 2.2. *For $X, Y, Z, U \in \mathfrak{X}M$ the following holds*

$$g(R(X, Y)Z, U) = -g(R(X, Y)U, Z)$$

PROOF. By what we have said the left hand side corresponds to

$$\langle \Omega(\tilde{X}, \tilde{Y}) \cdot \theta\tilde{Z}, \theta\tilde{U} \rangle$$

As Ω takes values in the Lie algebra $\mathfrak{o}(m)$ of all skew-symmetric matrices (anti-self-adjoint maps) the result follows. □

The tensor field R of type $(0,4)$ sending

$$(X,Y,Z,U) \longmapsto R(X,Y,Z,U) = -g(R(X,Y)Z,U)$$

is called the *covariant form* of the curvature tensor field R of type $(1,3)$. In coordinates

$$R = \sum R_{ijkl} \cdot \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k \otimes \mathrm{d}x^l$$

and we have so far proved

$$R_{ijkl} = -R_{jikl} \qquad R_{ijkl} = -R_{ijlk}$$

THEOREM 2.3 (The first Bianchi identity). $R_{ijkl} + R_{jkil} + R_{kijl} = 0.$

PROOF. Our previous (more general) first Bianchi identity claimed $D^2\theta = \Omega \cdot \theta$. Since in our case $D\theta = 0$ we have

$$0 = (\Omega \cdot \theta)(\tilde{X}, \tilde{Y}, \tilde{Z}) = \Omega(\tilde{X}, \tilde{Y}) \cdot \theta\tilde{Z} + \Omega(\tilde{Y}, \tilde{Z}) \cdot \theta\tilde{X} + \Omega(\tilde{Z}, \tilde{X}) \cdot \theta\tilde{Y}$$

Multiplying by $\theta\tilde{U}$ and converting to the section form yields the result. □

THEOREM 2.4. *As an algebraic consequence of the previous identities*

$$R_{ijkl} = R_{klij}$$

PROOF. Consider two instances of the first Bianchi identity

$$\begin{aligned} R_{ijkl} + R_{jkil} + R_{kijl} &= 0 \\ R_{ijlk} + R_{jlik} + R_{lijk} &= 0 \end{aligned}$$

Subtracting we obtain

$$2R_{ijkl} + R_{jkil} + R_{kijl} - R_{jlik} - R_{lijk} = 0$$

Changing the indices according to $\left(\begin{smallmatrix} i & j & k & l \\ k & l & i & j \end{smallmatrix}\right)$ one gets

$$2R_{klij} + R_{likj} + R_{iklj} - R_{ljki} - R_{jkli} = 0$$

and finally subtracting the last two equalities one obtains

$$2R_{ijkl} - 2R_{klij} = 0$$

□

DEFINITION 2.5. For a linear connection on M we define its *Ricci tensor field* of type $(0,2)$ by the formula

$$R_{ij} = \sum_k R^k_{ijk}$$

i.e. $R(X,Y)$ is the trace of $R(-,X)Y : TM \to TM$.

THEOREM 2.6. *The Ricci tensor field of the Levi-Civita connection is symmetric*

PROOF. Let u_i be an orthonormal basis of T_xM. Then $\operatorname{tr}(R(-,X)Y)$ equals

$$\sum_i g(R(u_i,X)Y,u_i) = \sum g(R(Y,u_i)u_i,X) = \sum g(R(u_i,Y)X,u_i)$$

which is $\operatorname{tr}(R(-,Y)X)$. □

DEFINITION 2.7. The function $s = \sum_{i,j} \tilde{g}^{ij} R_{ij}$ is called the *scalar curvature* of the Riemannian space.

$$s : M \xrightarrow{\text{Ricci}} T^*M \otimes T^*M \xrightarrow{\cong} TM \otimes TM \xrightarrow{g} \mathbb{R}$$

Using an orthonormal frame u_i we can write

$$s = \sum_{i,j} g(R(u_i,u_j)u_j,u_i) = \sum_{i,j} R(u_i,u_j,u_i,u_j)$$

Observe that the covariant form of the curvature is a section

$$M \to \Lambda^2 T^* M \otimes \Lambda^2 T^* M$$

For a pair of vectors u, v consider $R(u, v, u, v)$. Changing the basis to $(u', v') = (u, v) \cdot A$ we obtain $u' \wedge v' = \det A \cdot u \wedge v$ and thus

$$R(u', v', u', v') = (\det A)^2 \cdot R(u, v, u, v)$$

THEOREM 2.8. *Let $p \subseteq T_x M$ be a two-dimensional linear subspace, $u, v \in p$ a basis. Then the number*

$$K(p) = \frac{R(u, v, u, v)}{g(u, u)g(v, v) - g(u, v)^2}$$

does not depend on the choice of the basis u, v of p.

DEFINITION 2.9. The number $K(p)$ is called the *sectional curvature* of (M, g) in the direction of the two-dimensional subspace p.

PROOF. We know that

$$g(u, u)g(v, v) - g(u, v)^2 = \begin{vmatrix} g(u, u) & g(u, v) \\ g(u, v) & g(v, v) \end{vmatrix}$$

equals the square of the volume of the parallelpiped determined by u, v. In particular by passage to $(u', v') = (u, v) \cdot A$ this expression gets multiplied by $(\det A)^2$. □

REMARK. If $u, v \in p$ is an orthonormal basis then $K(p) = R(u, v, u, v)$ since the denominator is 1.

INFORMATION 2.10. (without proof) Using geodesics to transport the disc $D(r, p)$ centred at $0 \in p$ and of radius r to M, we obtain a two-dimensional submanifold $V(r, p) = \exp(D(r, p)) \subseteq M$. It holds

$$K(p) = 12 \lim_{r \to 0} \frac{\pi r^2 - \operatorname{vol} V(r, p)}{\pi r^4}$$

The normalization is such that for the unit sphere $K(p) = 1$.

DEFINITION 2.11. We say that a Riemannian space (M, g) has *constant curvature* if its sectional curvature is the same at all points and in all directions.

THEOREM 2.12 (Schur). *Let (M, g) be a connected Riemannian space of dimension at least 3. If $K(p)$ depends only on the point x then M has a constant curvature.*

PROOF. Later. □

REMARK (The Cartan's viewpoint). The curvature

$$\Omega \in \Omega^2_{\mathrm{hor}}(P, \mathfrak{gl}(m)) \cong \Omega^2(M, \mathrm{End}(TM))$$

corresponds to an equivariant map

$$P \to \hom(\Lambda^2 \mathbb{R}^m, \mathfrak{gl}(m))$$

with its image in fact lying in the $\mathrm{GL}(m)$-submodule $\hom_B(\Lambda^2 \mathbb{R}^m, \mathfrak{o}(m))$ of tensors satisfying the Bianchi identities. This submodule happens to decompose into three irreducible components and thus the curvature decomposes correspondingly. The components are respectively the scalar curvature, the traceless Ricci and Weyl curvature.

THEOREM 2.13. *Let (M, g) be a Riemannian manifold, $Q^1 M$ the principal* $\mathrm{O}(m)$*-bundle of orthonormal frames. Then on $Q^1 M$ there exists a unique torsion-free principal connection. It is called the* Levi-Civita connection.

PROOF. We are searching for $\omega : T(Q^1M) \to \mathfrak{o}(m)$, it is already determined on the vertical subbundle $V(Q^1M)$. To determine the horizontal distribution we need to solve

$$0 = D\theta = \mathrm{d}\theta + \omega \cdot \theta$$

expressing that ω is torsion-free. As $D\theta$ is horizontal independently of ω this condition is automatically satisfied for vertical vectors. We use θ to make an identification of some complementary subspace with $\mathbb{R}^m$, i.e. $H_u(Q^1M) \cong \mathbb{R}^m$. In this way the above equation becomes

$$\Lambda^2\mathbb{R}^m \xleftarrow{\cong} \Lambda^2 H_u(Q^1M) \xrightarrow{\mathrm{d}\theta+\omega\cdot\theta} \mathbb{R}^m$$

The mapping $\omega \mapsto \mathrm{d}\theta + \omega \cdot \theta$ is affine with the associated linear map $\omega \mapsto \omega \cdot \theta$. We will now show that it is bijective. Then there exists a unique ω for which $\omega \cdot \theta = -\mathrm{d}\theta$ verifying the theorem. At each $u \in Q^1M$ the map $\omega_u \mapsto \omega_u \cdot \theta_u$ becomes under our identification the map

$$\begin{aligned} \hom(\mathbb{R}^m, \mathfrak{o}(m)) &\longrightarrow \hom(\Lambda^2\mathbb{R}^m, \mathbb{R}^m) \\ \alpha &\longmapsto (\beta : v \wedge w \mapsto \alpha(v)w - \alpha(w)v) \end{aligned}$$

Since the vector spaces have the same dimension it is sufficient to prove injectivity. Denoting $\alpha(e_i)e_j = \sum a_{ij}^k e_k$ we have the antisymmetry relation $a_{ij}^k = -a_{ik}^j$ and the coordinates of the image β are simply

$$b_{ij}^k = \beta(e_i \wedge e_j)^k = (\alpha(e_i)e_j)^k - (\alpha(e_j)e_i)^k = a_{ij}^k - a_{ji}^k$$

Clearly the kernel consists precisely of those a_{ij}^k symmetric in the lower indices. Thus $a_{ij}^k = a_{ji}^k = -a_{jk}^i$ and repeating this cyclic permutation three times

$$a_{ij}^k = -a_{jk}^i = +a_{ki}^j = -a_{ij}^k$$

In the end $a_{ij}^k = 0$ and the map is injective.

The equivariancy of ω follows by uniqueness from

$$0 = r_a^*(\mathrm{d}\theta + \omega \cdot \theta) = \mathrm{d}(r_a^*\theta) + r_a^*\omega \cdot r_a^*\theta = a^{-1}\mathrm{d}\theta + r_a^*\omega \cdot a^{-1}\theta$$

(where we use equivariancy of θ) and

$$0 = a^{-1}(\mathrm{d}\theta + \omega \cdot \theta) = a^{-1}\mathrm{d}\theta + \mathrm{Ad}(a^{-1})\omega \cdot a^{-1}\theta$$

□

3. Normal coordinates

Let ∇ be a linear connection on M. A path $\gamma : I \to M$, where $I \subseteq \mathbb{R}$ is an interval, is called *geodesic* if the tangent vector field $\dot\gamma$ along γ is parallel. We will now express this condition in coordinates using the classical Christoffel symbols

$$\nabla_{\partial x^i}\partial x^j = \sum \Gamma_{ij}^k \partial x^k$$

If $\gamma(t)$ is given in coordinates by functions $x^i(t)$ then $\dot\gamma$ is given by $\frac{\mathrm{d}x^i}{\mathrm{d}t}(t)$ and the differential equation describing geodesic paths becomes

$$\tfrac{\mathrm{d}^2x^k}{\mathrm{d}t^2} + \sum \Gamma_{ij}^k \tfrac{\mathrm{d}x^i}{\mathrm{d}t}\tfrac{\mathrm{d}x^j}{\mathrm{d}t} = 0$$

We see immediately from the shape of the equation that the geodesic paths are preserved by affine reparametrizations $t = a\tau + b$, $a \neq 0$. For $X \in T_xM$ there exists a unique geodesic path γ with $\dot\gamma(0) = X$ which we denote by γ_X.

THEOREM 3.1. *The rule* $\exp_x(X) = \gamma_X(1)$ *defines on some neighbourhood* U_x *of* $0 \in T_xM$ *a smooth map* $\exp_x : U_x \to M$ *which is a local diffeomorphism at 0.*

PROOF. Let us denote the unit ball in T_xM by B_x and observe that by compactness the geodesic paths γ_X are defined on $[-\varepsilon, \varepsilon]$ for $X \in B_x$. For $X \in \varepsilon B_x$ they are defined on $[-1, 1]$ by affine reparametrization. Since

$$(\exp_x)_{*0}(X) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \exp_x(tX) = \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \gamma_X(t) = X$$

the derivative at 0 is $(\exp_x)_{*0} = \mathrm{id}$. □

DEFINITION 3.2. This map is called the *exponential map* of the connection ∇.

REMARK.

- $\exp_x$ needs not be defined globally. For $\mathbb{R}^m$ with the classical connection (the Levi-Civita connection of the standard metric) $\exp_0 = \mathrm{id}_{\mathbb{R}^m}$ and thus also for any open subset $U \subseteq \mathbb{R}^m$. In this case $\exp_0$ is not defined globally. For compact manifolds $\exp_x : TM \to M$ is always defined globally.
- $\exp_x$ needs not be a global diffeomorphism, e.g. for S^m with the standard connection the whole sphere of radius π centred at $0 \in T_xS^m$ is mapped to the opposite point $-x$.

DEFINITION 3.3. The local coordinate chart determined by $\exp_x$ for a linear torsion-free connection ∇ on M is called the *normal coordinate chart.*

THEOREM 3.4. *In the normal coordinate chart at x it holds $\Gamma^k_{ij}(x) = 0$.*

PROOF. The geodesic going through $x = 0$ with speed X has a coordinate expression a^it. Then the differential equation for the geodesic becomes

$$0 = \tfrac{\mathrm{d}^2x^k}{\mathrm{d}t^2} + \sum \Gamma^k_{ij}(x) \tfrac{\mathrm{d}x^i}{\mathrm{d}t} \tfrac{\mathrm{d}x^j}{\mathrm{d}t} = \sum \Gamma^k_{ij}(ta^i)a^ia^j$$

For $t = 0$ we get $\sum \Gamma^k_{ij}(0)a^ia^j = 0$ for arbitrary a^i. Since Γ^k_{ij} is symmetric in the lower indices $\Gamma^k_{ij}(0)$ prescribes a symmetric bilinear form with vanishing associated quadratic form. Hence it must be zero itself. □

Let us compute the coordinate expression of the curvature tensor

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

We first compute

$$\nabla_{\partial x^i}\nabla_{\partial x^j}\partial x^k = \nabla_{\partial x^i} \sum \Gamma^h_{jk}\partial x^h = \sum \tfrac{\partial \Gamma^l_{jk}}{\partial x^i}\partial x^l + \sum \Gamma^h_{jk}\Gamma^l_{ih}\partial x^l$$

which in the normal coordinates is simply $\sum \frac{\partial \Gamma^l_{jk}}{\partial x^i}\partial x^l$. Thus

$$R^l_{ijk} = \tfrac{\partial \Gamma^l_{jk}}{\partial x^i} - \tfrac{\partial \Gamma^l_{ik}}{\partial x^j}$$

THEOREM 3.5 (First Bianchi identity). *For any torsion-free connection it holds $R^l_{ijk} + R^l_{jki} + R^l_{kij} = 0$ or equivalently $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$.*

PROOF. We compute in the normal coordinates where

$$\begin{aligned} R^l_{ijk} &= \tfrac{\partial \Gamma^l_{jk}}{\partial x^i} - \tfrac{\partial \Gamma^l_{ik}}{\partial x^j} \\ R^l_{jki} &= \tfrac{\partial \Gamma^l_{ki}}{\partial x^j} - \tfrac{\partial \Gamma^l_{ji}}{\partial x^k} \\ R^l_{kij} &= \tfrac{\partial \Gamma^l_{ij}}{\partial x^k} - \tfrac{\partial \Gamma^l_{kj}}{\partial x^i} \end{aligned}$$

Adding up all terms cancel by symmetry. □

We will now explain what is meant by $\nabla_X R$. Here R is a tensor of type $(1, 3)$, i.e. a section of $P^1M \times_{\mathrm{GL}(M)} \hom(\otimes^3\mathbb{R}^m, \mathbb{R}^m)$. Using the evaluation map

$$\mathrm{ev} : \hom(\otimes^3\mathbb{R}^m, \mathbb{R}^m) \otimes \mathbb{R}^m \otimes \mathbb{R}^m \otimes \mathbb{R}^m \longrightarrow \mathbb{R}^m$$

the corresponding *linear* map on the associated bundles is

$$\mathrm{ev} : \hom(\otimes^3TM, TM) \otimes TM \otimes TM \otimes TM \longrightarrow TM$$

The covariant derivative commutes with linear maps thus

$$\nabla_X(R(Y, Z)U) = (\nabla_X R)(Y, Z, U) + R(\nabla_X Y, Z)U + R(Y, \nabla_X Z)U + R(Y, Z)\nabla_X U$$

THEOREM 3.6 (Second Bianchi identity). *For every torsion-free linear connection it holds $(\nabla_X R)(Y, Z, U) + (\nabla_Y R)(Z, X, U) + (\nabla_Z R)(X, Y, U) = 0$.*

PROOF. In the normal coordinates we write the components of ∇R as $R^l_{ijk;h}$. To compute them we plug

$$X = \partial x^h \qquad Y = \partial x^i \qquad Z = \partial x^j \qquad U = \partial x^k$$

into our formula for $(\nabla_X R)(Y, Z, U)$. At the origin

$$\nabla_X Y(0) = \sum \Gamma^l_{hi}(0)\partial x^l = 0$$

and similarly for the other covariant derivatives. Thus

$$R^l_{ijk;h}(0) = (\nabla_X R)(Y, Z, U)(0) = \nabla_X(R(Y, Z)U)(0) = \frac{\partial}{\partial x^h}\left(\frac{\partial\Gamma^l_{jk}}{\partial x^i} - \frac{\partial\Gamma^l_{ik}}{\partial x^j}\right)(0)$$

Summing with the cyclic permutations we obtain the result. □

REMARK. To relate this version of the second Bianchi identity to the more general one $D\Omega = 0$ we will show that the tensor[1]

$$(\nabla_X R)(Y, Z, U) + R(T(X, Y), Z)U + \text{cyclic permutations in } X, Y, Z$$

corresponds to an equivariant map $D\Omega(\tilde{X}, \tilde{Y}, \tilde{Z}) \cdot \theta\tilde{U}$. We have

$$R(Y, Z)U \sim \Omega(\tilde{Y}, \tilde{Z}) \cdot \theta\tilde{U}$$

which when differentiated along $\tilde{X}$ yields

$$\begin{aligned}
\nabla_X(R(Y, Z)U) &\sim \tilde{X}\Omega(\tilde{Y}, \tilde{Z}) \cdot \theta\tilde{U} + \Omega(\tilde{Y}, \tilde{Z}) \cdot \tilde{X}\theta\tilde{U}\\
R(\nabla_X Y, Z)U &\sim \Omega(\widetilde{\nabla_X Y}, \tilde{Z}) \cdot \theta\tilde{U}\\
R(Y, \nabla_X Z)U &\sim \Omega(\tilde{Y}, \widetilde{\nabla_X Z}) \cdot \theta\tilde{U}\\
R(Y, Z)\nabla_X U &\sim \Omega(\tilde{Y}, \tilde{Z}) \cdot \tilde{X}\theta\tilde{U}
\end{aligned}$$

Subtracting one obtains

$$(\nabla_X R)(Y, Z, U) \sim \tilde{X}\Omega(\tilde{Y}, \tilde{Z}) \cdot \theta\tilde{U} - \Omega(\widetilde{\nabla_X Y}, \tilde{Z}) \cdot \theta\tilde{U} - \Omega(\tilde{Y}, \widetilde{\nabla_X Z}) \cdot \theta\tilde{U}$$

For torsion one has $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ so that

$$R(T(X, Y), Z)U \sim \Omega(\widetilde{\nabla_X Y}, \tilde{Z}) \cdot \theta\tilde{U} + \Omega(\widetilde{\nabla_Y X}, \tilde{Z}) \cdot \theta\tilde{U} - \Omega([\tilde{X}, \tilde{Y}]), \tilde{Z}) \cdot \theta\tilde{U}$$

Adding up all the cyclic permutations (with respect to X, Y, Z only) of the last two correspondences one gets

$$\sum (\nabla_X R)(Y, Z, U) + R(T(X, Y), Z)U \sim \sum \left(\tilde{X}\Omega(\tilde{Y}, \tilde{Z}) - \Omega([\tilde{X}, \tilde{Y}]), \tilde{Z})\right) \cdot \theta\tilde{U}$$

On the right it is easy to recognize the formula for $D\Omega(\tilde{X}, \tilde{Y}, \tilde{Z}) \cdot \theta\tilde{U}$.

REMARK. We have derived a general form of the second Bianchi identity

$$(\nabla_X R)(Y, Z, U) + R(T(X, Y), Z)U + \text{cyclic permutations in } X, Y, Z = 0$$

Analogously one can prove for a general connection that

$$R(X, Y)Z + \text{cyclic} = T(T(X, Y), Z) + (\nabla_X T)(Y, Z) + \text{cyclic}$$

by showing that the right hand side corresponds to $D^2\theta = \Omega \cdot \theta$ which we know that corresponds to the left hand side.

LEMMA 3.7. *Let V be a finite dimensional vector space and $R_0, R_1 : V^{\otimes 4} \to \mathbb{R}$ two linear maps satisfying*

(a) $R_i(x \otimes y \otimes z \otimes u) = -R_i(y \otimes x \otimes z \otimes u)$,
(b) $R_i(x \otimes y \otimes z \otimes u) = -R_i(x \otimes y \otimes u \otimes z)$,
(c) $R_i(x \otimes y \otimes z \otimes u) + R_i(y \otimes z \otimes x \otimes u) + R_i(z \otimes x \otimes y \otimes u) = 0$.

If $R_0(x \otimes y \otimes x \otimes y) = R_1(x \otimes y \otimes x \otimes y)$ then $R_0 = R_1$.

[1]Thinking of ∇R as a section of $T^*M \otimes \Lambda^2 T^*M \otimes T^*M \otimes TM$ the summation over cyclic permutations corresponds to the antisymmetrization in the first three variables.

Proof. We set $R = R_1 - R_0$. Multiplying out $0 = R(x \otimes (y+u) \otimes z \otimes (y+u))$ we obtain

$$0 = R(x \otimes y \otimes x \otimes y) + R(x \otimes u \otimes x \otimes u) + R(x \otimes y \otimes x \otimes u) + R(x \otimes u \otimes x \otimes y)$$

where the first two terms are zero hence so must be the sum of the last two. According to Theorem 2.4 this sum equals $2R(x \otimes y \otimes x \otimes u)$. Thus

$$R((x+tz) \otimes y \otimes (x+tz) \otimes u) = 0$$

and taking derivative we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} R((x+tz) \otimes y \otimes (x+tz) \otimes u) = R(x \otimes y \otimes z \otimes u) + R(z \otimes y \otimes x \otimes u) = 0$$

Thus R is fully anti-symmetric in the first three indices and thus

$$R(x \otimes y \otimes z \otimes u) = R(y \otimes z \otimes x \otimes u) = R(z \otimes x \otimes y \otimes u)$$

We can now rewrite (c) as $3R(x \otimes y \otimes z \otimes u) = 0$ and $R = 0$, i.e. $R_0 = R_1$. □

Theorem 3.8 (Schur). *Let (M, g) be a connected Riemannian space of dimension at least 3 such that $K(p)$ depends only on the point x for which $p \subseteq T_xM$. Then M has constant curvature.*

Proof. The tensor field $R_1(X, Y, Z, U) = g(X, Z)g(Y, U) - g(Y, Z)g(X, U)$ satisfies (a), (b) and (c). At each point $x \in M$ it holds

$$K(x) = \frac{R(X, Y, X, Y)}{R_1(X, Y, X, Y)}$$

where we denote by $K(x)$ the common value of $K(p)$ for all $p \subseteq T_xM$. By the previous lemma $R = K(x)R_1$ since they agree on tensors of type $X \otimes Y \otimes X \otimes Y$. We want to show that $K(x)$ is a constant function.

To determine in what sense is R_1 constant we get back to the curvature tensor of type $(1, 3)$.

$$R_1(X, Y)Z = g(Y, Z)X - g(X, Z)Y \sim \langle \theta\tilde{Y}, \theta\tilde{Z}\rangle\theta\tilde{X} - \langle \theta\tilde{X}, \theta\tilde{Z}\rangle\theta\tilde{Y}$$

This is summarized in the diagram

$$\begin{tikzcd} P^1M \arrow[r, "\theta\tilde{X}\otimes\theta\tilde{Y}\otimes\theta\tilde{Z}"] & \mathbb{R}^m \otimes \mathbb{R}^m \otimes \mathbb{R}^m \arrow[d] & x \otimes y \otimes z \arrow[d, mapsto] \\ & \mathbb{R}^m & \langle y, z\rangle x - \langle x, z\rangle y \end{tikzcd}$$

showing that the map form of R_1 is the constant map

$$P^1M \to \hom(\otimes^3\mathbb{R}^m, \mathbb{R}^m)$$

sending everything to the above $x \otimes y \otimes z \mapsto \langle y, z\rangle x - \langle x, z\rangle y$. In particular $\nabla_X R_1 = 0$ and thus

$$\nabla_X R = \nabla_X(K \cdot R_1) = XK \cdot R_1 + K\nabla_X R_1 = XK \cdot R_1$$

Now we use the second Bianchi identity

$$(\nabla_X R)(Y, Z, U) + (\nabla_Y R)(Z, X, U) + (\nabla_Z R)(X, Y, U) = 0$$

which in our case takes form

$$\begin{aligned} &XK \cdot (g(Z, U)Y - g(Y, U)Z) + YK \cdot (g(X, U)Z - g(Z, U)X) \\ &+ZK \cdot (g(Y, U)X - g(X, U)Y) = 0 \end{aligned}$$

Take an orthonormal system X, Y, Z and set $U = Z$; then substitute to obtain

$$XK \cdot Y - YK \cdot X = 0$$

Since X and Y are linearly independent $XK = 0 = YK$. As they were also arbitrary the derivative of K is zero and K is locally constant. By connectedness it is globally constant. □

Theorem 3.9. *For an arbitrary Riemannian space with constant curvature K*

$$R(X, Y)Z = K(g(Y, Z)X - g(X, Z)Y)$$

Proof. See the proof of the last theorem. □

4. The second fundamental form of a hypersurface

THEOREM 4.1. *Let (N, g) be a Riemannian manifold, $M \subseteq N$ its submanifold, ∇^N and ∇^M the Levi-Civita connections on N and M. Then for all $X \in T_xM$ and $Y \in \mathfrak{X}M$ and any extension $\bar{Y} \in \mathfrak{X}N$ of Y the following holds*

$$\nabla^M_X Y \equiv \nabla^N_X \bar{Y} \mod \nu_x$$

where ν_x is the orthogonal complement of T_xM in T_xN. In other words $\nabla^M_X Y$ is the orthogonal projection of $\nabla^N_X \bar{Y}$ onto T_xM.

PROOF. Let us define ∇ by the formula from the statement, i.e. $\nabla_X Y$ is the orthogonal projection of $\nabla^N_X \bar{Y}$ onto T_xM. We will show that ∇ is metric and torsion-free[2]. This will imply $\nabla^M = \nabla$ by uniqueness.

$$\nabla_X(g(Y, Z)) = X(g(Y, Z)) = X(g(\bar{Y}, \bar{Z}))$$

and the other terms of $(\nabla_X g)(Y, Z)$ are

$$-g(\nabla_X Y, Z) = -g(\nabla^N_X \bar{Y}, \bar{Z})$$

since the difference of $\nabla_X Y$ and $\nabla^N X \bar{Y}$ is orthogonal to $\bar{Z}$. Similarly

$$-g(Y, \nabla_X Z) = -g(Y, \nabla^N_X \bar{Z})$$

and adding these three equalities we obtain

$$(\nabla_X g)(Y, Z) = (\nabla^N_X g)(\bar{Y}, \bar{Z}) = 0$$

so that ∇ is metric.

Projecting the equality $\nabla^N_{\bar{X}} \bar{Y} - \nabla^N_{\bar{Y}} \bar{X} = [\bar{X}, \bar{Y}]$ onto T_xM we obtain

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

since $[\bar{X}, \bar{Y}]$ is already tangent to M: in effect X is ι-related to $\bar{X}$ (for $\iota : M \hookrightarrow N$ the inclusion), analogously for Y and thus the same holds for their bracket. □

The normal projection of $\nabla^N_X \bar{Y}$ only depends on the value of X and Y at x. Before we start with the verification let us denote the normal projection by π and $B(X, Y) = \pi \nabla^N_X \bar{Y}$. Then we compute

$$B(X, fY) = \pi \nabla^N_X (\bar{f}\bar{Y}) = \pi(\underbrace{X\bar{f} \cdot \bar{Y}}_{\text{tangent}} + \bar{f} \cdot \nabla^N_X \bar{Y}) = f(x) \cdot \pi(\nabla^N_X \bar{Y}) = f(x) B(X, Y)$$

Thus B is tensorial. Moreover it is symmetric as

$$B(X, Y) - B(Y, X) = \pi([\bar{X}, \bar{Y}]) = \pi[X, Y] = 0$$

so that $B : S^2TM \to \nu M$ with the target being the normal bundle of M.

DEFINITION 4.2. The tensor B is called the *second fundamental form* of the submanifold $M \subseteq N$.

Let us consider a special case $M \subseteq \mathbb{E}_{m+1}$, a hypersurface in a Euclidean space with its standard metric and orientation. An orientation of a manifold M is a continuous choice of an orientation of T_xM for all $x \in M$. The normal bundle ν is one-dimensional and the orientation of M determines an orientation of ν by declaring $u \in \nu_x$ positive iff $(u, e_1, \ldots, e_m)$ is positive in $\mathbb{E}_{m+1}$ with $(e_1, \ldots, e_m)$ positive in T_xM. The unique unit positive vector n_x thus provides a global trivialization of ν by

$$\begin{aligned} M \times \mathbb{R} &\longrightarrow \nu \\ (x, t) &\longmapsto t \cdot n_x \end{aligned}$$

Let us denote by D the covariant derivative on $\mathbb{E}_{m+1}$ and ∇ the covariant derivative on M. Then

$$B(X, X)_x = \langle D_X X, n_x \rangle \cdot n_x$$

[2]Verifying that it is a linear connection is easy.

where $X \in T_xM$. We represent X by a path $\gamma : \mathbb{R} \to M$ with $\dot\gamma(0) = X$ and extend X to a vector field in such a way that $X(\gamma(t)) = \dot\gamma(t)$. Then

$$D_X X = \tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} X(\gamma(t)) = \tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \dot\gamma(t) = \ddot\gamma(0)$$

and consequently $B(X,X)_x = \langle \ddot\gamma(0), n_x \rangle \cdot n_x$.

THEOREM 4.3. *The normal acceleration $\langle \ddot\gamma(0), n_x \rangle \cdot n_x$ depends only on $\dot\gamma(0) = X$ and equals $B(X,X)$.* □

DEFINITION 4.4. In the case of an oriented hypersurface M in $\mathbb{E}_{m+1}$ by the *second fundamental form* we understand the map

$$h : S^2TM \to \mathbb{R} \qquad h(X,Y) = \langle B(X,Y), n_x \rangle$$

or in other words $B(X,Y) = h(X,Y)n_x$.

In a local coordinate chart $f(u_1, \dots, u_m) : \mathbb{R}^m \to M$ the basis of T_xM is formed by $f_i = \frac{\partial f}{\partial u_i}$. We denote $f_{ij} = \frac{\partial^2 f}{\partial u_i \partial u_j}$. A path $\gamma(t)$ on M ha a coordinate expression $u_i = u_i(t)$, we obtain $\dot\gamma(t) \in T_{\gamma(t)}$ the velocity, $\ddot\gamma = \frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2}$ the acceleration and $\langle \frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2}, n_x \rangle$ the normal acceleration. Since $\gamma(t) = f(u(t))$ we may write

$$\tfrac{\mathrm{d}\gamma}{\mathrm{d}t} = \sum f_i(u(t)) \tfrac{\mathrm{d}u_i}{\mathrm{d}t}$$

and thus

$$\ddot\gamma(t) = \sum (f_{ij}\Big(u(t)\Big) \cdot \tfrac{\mathrm{d}u_i}{\mathrm{d}t}\tfrac{\mathrm{d}u_j}{\mathrm{d}t} + \underbrace{f_i(u(t))\tfrac{\mathrm{d}^2u_i}{\mathrm{d}t^2}}_{\text{tangent to } M}\Big)$$

so that

$$\langle \ddot\gamma, n_x \rangle = \sum \underbrace{\langle f_{ij}(u(t)), n_x \rangle}_{h_{ij}} \cdot \tfrac{\mathrm{d}u_i}{\mathrm{d}t}\tfrac{\mathrm{d}u_j}{\mathrm{d}t}$$

REMARK. The first fundamental form is $g_{ij} = \langle f_i, f_j \rangle$ or more geometrically the scalar product g on M.

Let X, Y be vector fields on $M \subseteq \mathbb{E}_{m+1}$ and $\bar X, \bar Y$ their extensions to vector fields on $\mathbb{E}_{m+1}$. Then $D_{\bar X}\bar Y = \nabla_X Y + h(X,Y) \cdot n$ where n is the (choice of a) unit normal vector field on M. The curvature of $\mathbb{E}_{m+1}$ is zero while the curvature for the hypersurface M is

$$g(R(X,Y)Z,U) = g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, U).$$

THEOREM 4.5 (Gauss formula). *For a hypersurface $M \subseteq \mathbb{E}_{m+1}$ it holds*

$$g(R(X,Y)Z,U) = h(Y,Z)h(X,U) - h(X,Z)h(Y,U)$$

PROOF. By the metricity of the connection

$$\begin{aligned} g(\nabla_X\nabla_Y Z, U) &= Xg(\nabla_Y Z, U) - g(\nabla_Y Z, \nabla_X U) \\ &= \bar X \langle D_{\bar Y}\bar Z, \bar U \rangle - \langle D_{\bar Y}\bar Z, D_{\bar X}\bar U \rangle + h(Y,Z)h(X,U) \end{aligned}$$

and similarly

$$g(\nabla_{[X,Y]}Z, U) = \langle D_{\overline{[X,Y]}}\bar Z, \bar U \rangle = \langle D_{[\bar X,\bar Y]}\bar Z, \bar U \rangle.$$

Therefore $g(R(X,Y)Z,U)$ equals

$$\langle R(\bar X,\bar Y)\bar Z, \bar U \rangle + h(Y,Z)h(X,U) - h(X,Z)h(Y,U)$$

with the first term zero since the curvature of $\mathbb{E}_{m+1}$ vanishes. □

The sectional curvature in the direction of a plane $p \subseteq T_xM$ spanned by v_1 and v_2 is defined by

$$K(p) = \frac{-g(R(v_1,v_2)v_1,v_2)}{g(v_1,v_1)g(v_2,v_2) - g(v_1,v_2)^2}$$

Consider now a surface $M \subseteq \mathbb{E}_3$ with a local parametrization $f(u_1, u_2) : \mathbb{R}^2 \to M$ and compute the sectional curvature $K(x) = K(T_xM)$ by substituting $\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}$ into the Gauss formula

$$K(x) = \frac{h(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_1})h(\frac{\partial}{\partial u_2}, \frac{\partial}{\partial u_2}) - h(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2})^2}{g(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_1})g(\frac{\partial}{\partial u_2}, \frac{\partial}{\partial u_2}) - g(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2})^2} = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2} = \frac{\det h}{\det g}$$

This is the classical Gauss curvature from the differential geometry of curves and surfaces.

COROLLARY 4.6 (Theorema Egregium). *The Gauss curvature belongs to the inner geometry of a surface, i.e. it does not depend on the isometric embedding $M \hookrightarrow \mathbb{E}_3$.*

REMARK. The Gauss curvature is a product of the curvatures in the principal directions - the eigenvectors of h.

5. The geodesic curves of a Riemannian space

DEFINITION 5.1. Let $f : N \to M$ be a smooth map. A *vector field along* f is a smooth map F for which the diagram

$$\begin{tikzcd}
 & TM \arrow[d] \\
N \arrow[ur, "F"] \arrow[r, "f"'] & M
\end{tikzcd}$$

commutes. In other words it is a section of the pullback bundle $f^*TM \to N$.

For a linear connection ∇ on M the induced connection on f^*TM will be also denoted by ∇. Then $\nabla F : TN \to f^*TM$ and for a vector field $X \in \mathfrak{X}N$, $\nabla_X F : N \to f^*TM$, i.e. $\nabla_X F : N \to TM$ is again a vector field along f.

$$\begin{tikzcd}
TN \arrow[r, "F_*"] \arrow[drr, "\nabla F"] & TTM \arrow[r, "v"] & VTM \arrow[d] \arrow[r, phantom, "\cong"] & TM \times_M TM \arrow[dl, "pr_2"] \\
N \arrow[u, "X"] \arrow[rr, "\nabla_X F"'] & & TM &
\end{tikzcd}$$

DEFINITION 5.2. Let $\gamma : \mathbb{R} \to M$ be a path and $v : \mathbb{R} \to TM$ a vector field along γ. It is said to be *transported parallelly along* γ if $\nabla v = 0$ or equivalently

$$\nabla_{\dot\gamma} v := \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}} v = 0$$

DEFINITION 5.3. A path $\gamma(t)$ is *geodesic* if $\dot\gamma(t)$ is parallel along γ.

REMARK (on Cartan's point of view). Consider the principal bundle $P^1M \to M$ of frames on M. There are two forms on P^1M, the connection form ω and the canonical form θ. Combined together they provide

$$(\theta, \omega) : TP^1M \to \mathfrak{ga}(m) = \mathbb{R}^m \rtimes \mathfrak{gl}(m)$$

a trivialization of the tangent bundle TP^1M. Here $\mathfrak{ga}(m)$ is the Lie algebra of matrices of the form

$$\begin{pmatrix} 0 & 0 \\ v & A \end{pmatrix},$$

i.e. the Lie algebra of the Lie group $\mathrm{GA}(m)$ of all affine isomorphisms of $\mathbb{R}^m$. This is an example of a Cartan connection of type $(\mathfrak{ga}(m), \mathfrak{gl}(m))$. Taking $v \in \mathbb{R}^m$ and thinking of it as the matrix $\left(\begin{smallmatrix} 0 & 0 \\ v & 0 \end{smallmatrix}\right)$ in $\mathfrak{ga}(m)$ we obtain a vector field $(\omega, \theta)^{-1}(v)$ on P^1M, horizontal by definition. Let $\tilde\gamma(t)$ be its integral curve and $\gamma(t) = \pi \circ \tilde\gamma(t)$. Then $\frac{\mathrm{d}}{\mathrm{d}t}\tilde\gamma$ is a horizontal vector field along γ and hence is transported parallelly.

We will now show that γ is a geodesic. By definition $\theta(\dot{\tilde\gamma}) = v$, the coordinates of the projection $\pi_*\dot{\tilde\gamma}(t) = \dot\gamma(t)$ in the basis $\tilde\gamma(t) = (u_i(t))$. Since $u_i(t)$ are parallel along $\gamma(t)$ so is their constant linear combination $\dot\gamma(t)$ and thus $\nabla_{\dot\gamma}\dot\gamma = 0$.

Now we draw some consequences of the geodesicity of γ. Firstly

$$\tfrac{\mathrm{d}}{\mathrm{d}t}|\dot\gamma(t)|^2 = \tfrac{\mathrm{d}}{\mathrm{d}t}g(\dot\gamma(t),\dot\gamma(t)) = 2g(\nabla_{\dot\gamma}\dot\gamma(t),\dot\gamma(t)) = 0$$

implying that $|\dot\gamma(t)|$ is constant. By a reparametrization we may assume that $|\dot\gamma(t)| = 1$. In this case we say that γ *is parametrized by the arc length* and use s for the parameter instead of t.

Let now C be a curve, i.e. a 1-dimensional submanifold. Locally we parametrize C by the arc length as $\gamma : \mathbb{R} \to C$. The *geodesic curvature* of C is defined as

$$K_g(C) = |\nabla_{\dot\gamma}\dot\gamma|.$$

DEFINITION 5.4. A curve C is called a *geodesic* if its parametrization by the arc length is a geodesic curve, i.e. $K_g(C) = 0$.

REMARK (the Frenet's formulas). For a planar curve we define $e_1 = \dot\gamma(s)$ the tangent unit vector field along C and e_2 (a choice of) the unit normal vector field. Then $\nabla_{\dot\gamma}\dot\gamma = \pm K_g \cdot e_2$ since

$$2g(\nabla_{\dot\gamma}\dot\gamma,\dot\gamma) = \tfrac{\mathrm{d}}{\mathrm{d}s}g(\dot\gamma,\dot\gamma) = \tfrac{\mathrm{d}}{\mathrm{d}s}1 = 0$$

and thus $\nabla_{\dot\gamma}\dot\gamma$ is a vector field perpendicular to $\dot\gamma$ and of length K_g.

For a connected Riemannian manifold (M, g) we define

$$d(x, y) = \inf\{\ell(\gamma) \mid \gamma : [0, 1] \to M, \gamma(0) = x, \gamma(1) = y\}$$

where $\ell(\gamma) = \int_a^b |\dot\gamma(t)|\ \mathrm{d}t$ is the length of a (piecewise) smooth curve γ. Easily $d(x, y) \geq 0$ and $d(x, z) \leq d(x, y) + d(y, z)$ (when considering smooth curves only one needs to use smoothing of the concatenation).

Let us choose $x \in M$ and using the scalar product g_x on T_xM we denote by $N(x, r)$ the open ball centred at 0_x of radius r. For small r the exponential map $\exp_x$ is defined on $N(x, r)$ and is a diffeomorphism onto $U(x, r) \subseteq M$.

THEOREM 5.5. *For any $r > 0$ for which $\exp_x : N(x, r) \to U(x, r)$ is a diffeomorphism the following holds*

(a) *Every point $y \in U(x, r)$ may be joined with x by a unique geodesic inside $U(x, r)$.*
(b) *The length of the geodesic from (a) is exactly $d(x, y)$.*
(c) *$U(x, r)$ is the set of all $y \in M$ for which $d(x, y) < r$.*

REMARK. It follows that $d(x, y) = 0$ iff $x = y$ and d is a metric on M, $U(x, r)$ being the ball in this metric.

PROOF. Firstly (a) follows from the fact that geodesics emanating from x are exactly the images under $\exp_x$ of the rays from 0_x. For (b) we will need the following lemma in which we denote by g^0 the Riemannian metric on T_xM given by the scalar product g_x at each $v \in T_xM$.

LEMMA 5.6 (Gauss lemma). *Let $v \in T_xM$ lie in the domain of $\exp_x$. Then for arbitrary $w \in T_xM$*

$$g^0((v, v), (v, w)) = g(\exp_{x*}(v, v), \exp_{x*}(v, w))$$

i.e. $\exp_{x*}$ preserves the scalar product whenever one of the vectors is radial.

We will prove the lemma later. Let us denote by $\mathrm{pr} : TT_xM \to TT_xM$ the radial projection,

$$\mathrm{pr}(v, w) = \left(v, \frac{\langle v, w\rangle}{\langle v, v\rangle} \cdot v\right).$$

Let $\gamma : [0, 1] \to N(x, r)$ be a path and $\delta = \exp_x \cdot\gamma$ its image in M. The length is

$$\ell(\delta) = \int_0^1 |\dot\delta|\ \mathrm{d}t$$

Decomposing $\dot\gamma(t)$ into the radial part and the complement the orthogonality is preserved by $\exp_{x*}$ by Gauss lemma. In particular

$$\begin{aligned}|\dot\delta(t)|^2 = |\exp_{x*}\dot\gamma(t)|^2 &= |\exp_{x*}\mathrm{pr}\,\dot\gamma(t)|^2 + |\exp_{x*}(\dot\gamma(t) - \mathrm{pr}\,\dot\gamma(t))|^2 \\ &\geq |\exp_{x*}\mathrm{pr}\,\dot\gamma(t)|^2 = |\mathrm{pr}\,\dot\gamma(t)|^2\end{aligned}$$

with equality only for $\dot\gamma(t)$ radial. Therefore

$$\ell(\delta) \geq \int_0^1 |\operatorname{pr}\dot\gamma(t)|\ \mathrm{d}t \geq \left|\int_0^1 |\operatorname{pr}\dot\gamma(t)|_{\mathrm{or}}\ \mathrm{d}t\right|$$

where we write $|\operatorname{pr}\dot\gamma(t)|_{\mathrm{or}}$ for the *oriented* length (the sign being that of w/v)

$$|(v,w)|_{\mathrm{or}} = |\operatorname{pr}(v,w)|_{\mathrm{or}} = \mathrm{d}n(v,w)$$

where $n : N(x,r) - \{0_x\} \to \mathbb{R}_+$ is the norm $|\cdot|$. Thus

$$\ell(\delta) \geq \left|\int_0^1 \mathrm{d}n(\dot\gamma(t))\ \mathrm{d}t\right| = |n(\gamma(1)) - n(\gamma(0))| = |\gamma(1)|$$

The equality occurs iff γ is radial and positively oriented hence a reparametrization of a linear path in $N(x,r)$. The path δ is then a reparametrization of a geodesic taking care of paths staying inside $U(x,r)$. But if δ left $U(x,r)$ then its beginning would be a path from x to a point z of the same geodesic distance from x as that of y. The length of this part of δ would then be at least this geodesic distance proving (b). The very same argument proves (c). □

Definition 5.7. A space with a linear connection, i.e. a manifold M together with a linear connection on TM, is called *complete* if every geodesic path $\gamma : I \to M$ extends to the whole $\mathbb{R}$.

Remark. Equivalently the vector fields $(\theta,\omega)^{-1}(v)$ are complete.

Theorem 5.8. *If (M,g) is complete as a metric space then it is complete with respect to the Levi-Civita connection.*

Proof. Let $\gamma : (a,b) \to M$ be a geodesic path parametrized by the arc length and let b_n be a sequence in (a,b) converging to b. By the previous theorem $d(\gamma(b_n),\gamma(b_m)) \leq |b_n - b_m|$ and thus $\gamma(b_n)$ is Cauchy. Let $x \in M$ be its limit point. In a neighbourhood of x every geodesic parametrized by the arc length is defined on an interval of a uniform radius by compactness. Thus γ can be prolonged. □

We will later prove the reverse implication.

Let M be an oriented 2-dimensional Riemannian manifold. The sectional curvature is a function $K : M \to \mathbb{R}$, $K(x) = K(T_xM)$. Further there is a volume 2-form $\mathrm{vol}_g = e_1^* \wedge e_2^*$ where e_1^*, e_2^* is an oriented orthonormal basis of T^*M.

Definition 5.9. The 2-form $\kappa = K \cdot \mathrm{vol}_g$ is called the *curvature 2-form* on M.

Consider on M a one-parameter family of curves $\gamma : I \times J \to U \subseteq M$ for which

- γ is a diffeomorphism $I \times J \xrightarrow{\cong} U$,
- for each $s \in J$ the curve $\gamma(-,s)$ is parametrized by the arc length, $|\frac{\partial}{\partial t}\gamma(-,s)| = 1$.

Let us denote $\dot\gamma(t,s) = \frac{\partial}{\partial t}\gamma(t,s)$, a vector field on U. Then $g(\nabla_{\dot\gamma}\dot\gamma,\dot\gamma) = 0$. We denote by ν the unit vector field orthogonal to $\dot\gamma$, namely that for which $(\dot\gamma,\nu)$ is a positive basis. On U define a 1-form $\omega = g(\nabla\dot\gamma,\nu)$, i.e. $\omega(X) = g(\nabla_X\dot\gamma,\nu)$.

Lemma 5.10. $\mathrm{d}\omega = -\kappa$.

Proof. It is enough to verify on the basis, $\mathrm{d}\omega(\dot\gamma,\nu) = -\kappa(\dot\gamma,\nu)$. To determine the right hand side $\mathrm{vol}_g(\dot\gamma,\nu) = 1$ and

$$K = R(\dot\gamma,\nu,\dot\gamma,\nu) = -g(R(\dot\gamma,\nu)\dot\gamma,\nu).$$

Putting together $-\kappa(\dot\gamma,\nu) = g(R(\dot\gamma,\nu)\dot\gamma,\nu)$ while $\mathrm{d}\omega(\dot\gamma,\nu)$ is

$$\begin{aligned}
&\dot\gamma\omega(\nu) - \nu\omega(\dot\gamma) - \omega[\dot\gamma,\nu] \\
&\quad= \nabla_{\dot\gamma} g(\nabla_\nu\dot\gamma,\nu) - \nabla_\nu g(\nabla_{\dot\gamma}\dot\gamma,\nu) - g(\nabla_{[\dot\gamma,\nu]}\dot\gamma,\nu) \\
&\quad= g(\nabla_{\dot\gamma}\nabla_\nu\dot\gamma,\nu) - g(\nabla_\nu\nabla_{\dot\gamma}\dot\gamma,\nu) - g(\nabla_{[\dot\gamma,\nu]}\dot\gamma,\nu) + g(\nabla_\nu\dot\gamma,\nabla_{\dot\gamma}\nu) - g(\nabla_{\dot\gamma}\dot\gamma,\nabla_\nu\nu) \\
&\quad= g(R(\dot\gamma,\nu)\dot\gamma,\nu) + g(\nabla_\nu\dot\gamma,\nabla_{\dot\gamma}\nu) - g(\nabla_{\dot\gamma}\dot\gamma,\nabla_\nu\nu)
\end{aligned}$$

and the last two terms are zero by the following argument. Since $g(\dot\gamma,\dot\gamma)=1$ the derivative $\nabla_X\dot\gamma$ is orthogonal to $\dot\gamma$ and thus $\nabla_X\dot\gamma \parallel \nu$. Similarly $\nabla_Y\nu \parallel \dot\gamma$ and so

$$g(\nabla_X\dot\gamma, \nabla_Y\nu)=0$$

for arbitrary vectors X, Y. □

Let us denote by $B(r)$ the open disc in $\mathbb{R}^2$ of radius r and by $S(r)$ the circle of radius r both centred at the origin.

Definition 5.11. We say that a curve C is *simple closed* if there exists a diffeomorphism $\varphi : B(1+\varepsilon)\to U \subseteq M$ onto a neighbourhood U of C such that $\varphi(S(1)) = C$. The set $\varphi(B(1))$ is called the *interior* of the curve C.

Notation. For a curve C we have the curve integral $\int_C f\ \mathrm{d}s$ and for a 2-dimensional region D we have $\iint_D f\ \mathrm{d}\sigma$ both defined by multiplying a function f by the respective volume form associated to the induced metric.

The oriented geodesic curvature is $K_g = g(\nabla_{\dot\gamma}\dot\gamma, \nu)$. This depends on the choice on ν which we make in such a way that $(\dot\gamma, \nu)$ is positively oriented.

Theorem 5.12 (Gauss-Bonet). *Let C be a simple closed curve with the oriented geodesic curvature K_g and let D be its interior. Then*

$$\int_C K_g\ \mathrm{d}s = 2\pi - \iint_D K\ \mathrm{d}\sigma$$

Proof. Let us choose $\varphi : B(1+\delta) \xrightarrow{\cong} U$ with $\varphi(S(1)) = C$ and $\varphi(B(1)) = D$. We may assume[3] that in a small neighbourhood of the origin $\varphi = \exp_{\varphi(0)}$. Around the origin we consider a small circle C_ε and on the annulus D_ε we construct the 1-form ω corresponding to the (local) parametrization of C by the arc length

$$D_\varepsilon = S^1\times[\varepsilon,1]\to U$$

By the Stokes theorem

$$\int_C \omega - \int_{C_\varepsilon}\omega = \int_{D_\varepsilon}\mathrm{d}\omega = -\int_{D_\varepsilon}\kappa = -\iint_{D_\varepsilon} K\ \mathrm{d}\sigma$$

and also

$$\int_C \omega = \int_{S^1}\omega(\dot\gamma)\ \mathrm{d}s = \int_{S^1} g(\nabla_{\dot\gamma}\dot\gamma,\nu)\ \mathrm{d}s = \int_C K_g\ \mathrm{d}s$$

Clearly $\lim_{\varepsilon\to0}\iint_{D_\varepsilon} K\ \mathrm{d}\sigma = \iint_D K\ \mathrm{d}\sigma$ and thus it remains to show that

$$\lim_{\varepsilon\to0}\int_{C_\varepsilon} K_g(C_\varepsilon)\ \mathrm{d}s = 2\pi$$

The rough idea is that in the Euclidean plane $K_g(C_\varepsilon) = 1/\varepsilon$ and thus

$$\int_{C_\varepsilon} K_g(C_\varepsilon)\ \mathrm{d}s = \int_0^{2\pi\cdot\varepsilon} 1/\varepsilon\ \mathrm{d}t = 2\pi.$$

As $\varepsilon\to0$ the geometry approaches the Euclidean geometry and thus the limit formula holds. Now for a more precise proof.

First we need a lemma about describing the geodesic curvature when the parametrization is not by the arc length.

Lemma 5.13. *Let $\gamma : S^1 \to M$ be an embedding. Then*

$$K_g\circ\gamma = g(\nabla_{\dot\gamma}\dot\gamma,\nu)/|\dot\gamma|^2$$

[3] This is the classical disc isotopy theorem which we probably want to avoid.

PROOF. By definition

$$\begin{aligned} K_g \circ \gamma &= g(\nabla_{\dot\gamma/|\dot\gamma|}(\dot\gamma/|\dot\gamma|), \nu) = g(\nabla_{\dot\gamma}(\dot\gamma/|\dot\gamma|), \nu)/|\dot\gamma| \\ &= g(1/|\dot\gamma| \cdot \nabla_{\dot\gamma}\dot\gamma + \tfrac{\mathrm{d}}{\mathrm{d}t}(1/|\dot\gamma|) \cdot \dot\gamma, \nu)/|\dot\gamma| \end{aligned}$$

and the proof is finished by observing that $g(\dot\gamma, \nu) = 0$. $\square$

Then we can compute $\int_{C_\varepsilon} K_g(C_\varepsilon)\ \mathrm{d}s$ in the coordinate chart given by φ and using the parametrization $\gamma_\varepsilon : S^1 \to \mathbb{R}^2$, $\gamma(z) = \varepsilon \cdot z$

$$\begin{aligned} \int_{C_\varepsilon} K_g(C_\varepsilon)\ \mathrm{d}s &= \int_{S^1} g(\nabla_{\dot\gamma_\varepsilon}\dot\gamma_\varepsilon, \nu)/|\dot\gamma_\varepsilon|^2 \cdot |\dot\gamma_\varepsilon|\ \mathrm{d}s \\ &= \int_{S^1} g_{ij} \cdot \ddot\gamma_\varepsilon^i/|\dot\gamma_\varepsilon| \cdot \nu^j\ \mathrm{d}s + \int_{S^1} g_{ij}\Gamma^i_{kl}\dot\gamma_\varepsilon^k\dot\gamma_\varepsilon^l\nu^j/|\dot\gamma_\varepsilon|\ \mathrm{d}s \end{aligned}$$

Easily the second term tends to zero while the first tends to the situation where[4] $g_{ij} = \delta_{ij}$ is constant and thus the integrand tends to 1, the limit being 2π. $\square$

We will now interpret geometrically $\int_C K_g\ \mathrm{d}s$. Let $\gamma : [a, b] \to M$ be a path parametrized by the arc length and $(u(t), v(t))$ be a positive orthonormal basis at $\gamma(t)$ obtained by transporting $u(a)$ and $v(a)$ parallelly along $\gamma(t)$. Express $\dot\gamma(t)$ in this basis as

$$\dot\gamma(t) = \cos\varphi(t) \cdot u + \sin\varphi(t) \cdot v$$

Then $\nu(t) = -\sin\varphi(t) \cdot u + \cos\varphi(t) \cdot v$ and we may compute

$$\begin{aligned} \nabla_{\dot\gamma}\dot\gamma &= \nabla_{\dot\gamma}(\cos\varphi(t) \cdot u) + \nabla_{\dot\gamma}(\sin\varphi(t) \cdot v) \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}(\cos\varphi(t)) \cdot u + \cos\varphi(t) \cdot \underbrace{\nabla_{\dot\gamma}u}_{0} + \tfrac{\mathrm{d}}{\mathrm{d}t}(\sin\varphi(t)) \cdot v + \sin\varphi(t) \cdot \underbrace{\nabla_{\dot\gamma}v}_{0} \\ &= \dot\varphi(t) \cdot (-\sin\varphi(t) \cdot u + \cos\varphi(t) \cdot v) = \dot\varphi(t) \cdot \nu \end{aligned}$$

Therefore $K_g = g(\nabla_{\dot\gamma}\dot\gamma, \nu) = \dot\varphi$ and finally

$$\int_C K_g\ \mathrm{d}s = \int_C \dot\gamma\ \mathrm{d}t = \varphi(1) - \varphi(0) = \angle(\dot\gamma(a), \dot\gamma(b))$$

measured by transporting parallelly to any point along γ.

Let us consider now a curved triangle. We can use Gauss-Bonet formula after smoothing the corners to obtain

$$\int_{C_1} K_g\ \mathrm{d}s + (\pi - \alpha_3) + \int_{C_2} K_g\ \mathrm{d}s + (\pi - \alpha_1) + \int_{C_3} K_g\ \mathrm{d}s + (\pi - \alpha_2) = 2\pi - \iint_D K\ \mathrm{d}\sigma$$

the terms $\pi - \alpha_i$ being exactly the angle differences (in limit). We obtain

THEOREM 5.14. $\int_{\partial\Delta} K_g\ \mathrm{d}s = (\alpha_1 + \alpha_2 + \alpha_3 - \pi) - \iint_\Delta K\ \mathrm{d}\sigma$.

When all the sides C_i of the triangle are geodesic then $K_g = 0$ and we obtain

THEOREM 5.15. *The sum of the internal angles in a geodesic triangle is*

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi + \iint_\Delta K\ \mathrm{d}\sigma.$$

When the curvature is constant the defect $(\alpha_1 + \alpha_2 + \alpha_3 - \pi)$ is proportional to the area of the triangle. For the Euclidean geometry $K = 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = \pi$. For $K = 1$ we have triangles with defect up to 4π.

LEMMA 5.16. *Let $\gamma : [a, b] \to M$ be a piecewise smooth path such that $\ell\gamma = d(\gamma(a), \gamma(b))$. Then γ is a reparametrization of a geodesic path.*

PROOF. We have proved this when $\gamma(a)$ is sufficiently close to $\gamma(b)$. For an arbitrary γ the statement holds locally. But geodesics are described locally thus γ must be itself a reparametrization of a geodesic. $\square$

[4]Thus it is convenient to assume that the derivative φ_{*0} at zero is an isometry.

THEOREM 5.17 (Hopf-Rinow). *Let (M, g) be a connected geodesically complete Riemannian space. Then arbitrary $x, y \in M$ can be joined by a geodesic path γ satisfying $\ell(\gamma) = d(x, y)$. Such paths are called minimal geodesics.*

PROOF. Let us define the "shell" $\mathrm{Sh}(x, r) = \exp_x(S(x, r))$ where $S(x, r)$ is a sphere in T_xM cetred at 0_x and of radius r. We choose r small enough so that $\exp_x$ is a diffeomorphism on the closed ball of radius r. Since $\mathrm{Sh}(x, r)$ is compact there exists $p \in \mathrm{Sh}(x, r)$ such that $d(p, y)$ is minimal. Then $p = \exp_x(r \cdot v)$ with $|v| = 1$. We will show that $y = \exp_x(d \cdot v)$ where $d = d(x, y)$. This will prove the theorem. But first observe that $d(p, y)$ equals exactly $d(x, y) - r$ for it cannot be smaller as that would give

$$d(x, y) \le d(x, p) + d(p, y) < r + (d(x, y) - r)$$

and it cannot be bigger either as that would contradict the minimality of $d(x, p)$.

Now we will prove that the set

$$T = \{t_0 \in [0, d] \mid \forall 0 \le t \le t_0 : d(\exp_x(t \cdot v), y) = d - t\}$$

equals $[0, d]$. Clearly T is closed in $[0, d]$ and contains 0. It remains to show that it is open by connectedness. Therefore let $t_0 \in T$, $p_0 = \exp_x(t_0 \cdot v)$ and again let p_1 be the closest to y of the points from $\mathrm{Sh}(p_0, r_0)$. We have shown in the first paragraph that $d(p_1, y) = d(p_0, y) - r_0 = d - t_0 - r_0$ and thus the concatenation of the geodesic from x to p_0 and that from p_0 to p_1 is a path having the minimal length $t_0 + r_0 = \ell(x, p_1)$. By the previous lemma it must be a geodesic and in particular $p_1 = \exp_x((t_0 + r_0) \cdot v)$. Since r_0 was arbitrary (small) $t_0 + r_0 \in T$. □

REMARK. For a simply connected geodesically complete Riemannian space of non-positive sectional curvature the minimal geodesic is unique and the exponential map $\exp_x : T_xM \to M$ is a diffeomorphism.

COROLLARY 5.18. *A geodesically complete Riemannian space is complete as a metric space.*

PROOF. Pick a point $x \in M$ and let x_n be a Cauchy sequence. The set $d(x, x_n)$ is necessarily bounded by some r and hence x_n lie in a compact subspace $\exp_x(B(x, r))$ which implies the convergence. □

6. Geodesic variations

Let F be a vector field along f as in

$$\begin{tikzcd} f^*TM \arrow[r, "\phi"] \arrow[d] & TM \arrow[d] \\ N \arrow[r, "f"'] \arrow[ur, "F"] & M \end{tikzcd}$$

and write ∇F for the covariant derivative using the induced connection $f^*\nabla$. We will now compute the torsion $T(f_*A, f_*B)$ in terms of the covariant derivative on f^*TM. In terms of the equivariant maps we have

$$\begin{aligned} D\theta(\widetilde{f_*A}, \widetilde{f_*B}) &= \mathrm{d}\theta(\widetilde{f_*A}, \widetilde{f_*B}) = \mathrm{d}\theta(\phi_*\tilde{A}, \phi_*\tilde{B}) = \mathrm{d}(\phi^*\theta)(\tilde{A}, \tilde{B}) \\ &= \tilde{A}(\phi^*\theta(\tilde{B})) - \tilde{B}(\phi^*\theta(\tilde{A})) - \phi^*\theta[\tilde{A}, \tilde{B}] \\ &= \tilde{A}(\theta(\phi_*\tilde{B})) - \tilde{B}(\theta(\phi_*\tilde{A})) - \phi^*\theta\widetilde{[A, B]} \\ &= \tilde{A}(\theta(\widetilde{f_*B})) - \tilde{B}(\theta(\widetilde{f_*A})) - \theta(\widetilde{f_*[A, B]}) \end{aligned}$$

which corresponds back to $\nabla_A f_*B - \nabla_B f_*A - f_*[A, B]$. We conclude that

$$0 = T(f_*A, f_*B) = \nabla_A f_*B - \nabla_B f_*A - f_*[A, B].$$

Analogously we obtain

$$R(f_*A, f_*B)F = \nabla_A \nabla_B F - \nabla_B \nabla_A F - \nabla_{[A,B]} F$$

DEFINITION 6.1. Consider a path $\gamma : [a, b] \to M$ and let $I \subseteq \mathbb{R}$ be an open interval containing zero. By a *variation* of γ we understand a smooth map $V : [a, b] \times I \to M$ satisfying $V(t, 0) = \gamma(t)$.

DEFINITION 6.2. A *geodesic variation* of a geodesic path γ is a variation V such that $V(-, s)$ is geodesic for each $s \in I$.

On $[a, b]$ we use parameter t and on I parameter s. On the product $[a, b] \times I$ we have vector fields $\frac{\partial}{\partial t}, \frac{\partial}{\partial s}$. We denote

$$V_* \tfrac{\partial}{\partial t} = \partial_t V \qquad V_* \tfrac{\partial}{\partial s} = \partial_s V$$

For a vector field $F : [a, b] \times I \to RM$ along V we denote

$$\nabla_{\frac{\partial}{\partial t}} F = D_t F \qquad \nabla_{\frac{\partial}{\partial s}} F = D_s F$$

Our formula for torsion for vactor fields $\frac{\partial}{\partial t} \frac{\partial}{\partial s}$ can be written as

$$D_t \partial_s V - D_s \partial_t V = 0$$

since $[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}] = 0$. For a geodesic variation we compute

$$D_t^2 \partial_s V = D_t D_t \partial_s V = D_t D_s \partial_t V = D_s D_t \partial_t V + R(\partial_t V, \partial_s V) \partial_t V$$

Writing $\dot\gamma_t = \partial_t V$ we see that $D_t \partial_t V = \nabla_{\dot\gamma_t} \dot\gamma_t = 0$ and finally

$$D_t^2 \partial_s V = R(\dot\gamma_t, \partial_s V)\dot\gamma_t.$$

DEFINITION 6.3. A vector field X along a geodesic path γ is called a *Jacobi field* if $\nabla_{\dot\gamma}^2 X = R(\dot\gamma, X)\dot\gamma$.

The condition on a Jacobi field is a second order linear differential equation. Thus a solution is determined uniquely by $X(a)$ and $\nabla_{\dot\gamma} X(a)$. We have shown above that for every geodesic variation V of γ the vector field $\partial_s V(t, 0)$ is a Jacobi field. In the opposite direction we have.

THEOREM 6.4. *For every Jacobi field X along γ there exists a geodesic variation V of γ such that $\partial_s V(t, 0) = X(t)$.*

PROOF. We assume $a = 0$ for simplicity. Let $\beta : I \to M$ be any path with $\dot\beta(0) = X(0)$. Put

$$\gamma(s) = \mathrm{Pt}_\beta(\dot\gamma(0) + s \cdot (\nabla_{\dot\gamma} X)(0), s)$$

and $V(t, s) = \exp_{\beta(s)}(t \cdot \gamma(s))$. Since V is a geodesic variation of γ the derivative $\partial_s V(t, 0)$ is a Jacobi field along γ and we will now show that it equals $X(t)$. But the initial conditions for $\partial_s V(t, 0)$ are

$$\begin{aligned} \partial_s V(0, 0) &= \tfrac{\partial}{\partial s}\big|_{s=0} \beta(s) = X(0) \\ (D_t \partial_s V)(0, 0) &= (D_s \partial_t V)(0, 0) = (D_s \gamma)(0) \\ &= \underbrace{(\nabla_{\dot\beta} \mathrm{Pt}_\beta(\dot\gamma(0), s))}_{0}(0) + \nabla_{\dot\beta}(s \cdot \mathrm{Pt}_\beta((\nabla_{\dot\gamma} X)(0), s))(0) = (\nabla_{\dot\gamma} X)(0) \end{aligned}$$

i.e. the same as that for X and thus the vector fields must also agree. □

EXAMPLE 6.5. Let $\gamma : [a, b] \to M$ be a geodesic path. Then both

$$\gamma(t + s) \qquad \text{and} \qquad \gamma((1 + s)t)$$

are geodesic variations (for each s they are affine reparametrizations of γ). The corresponding Jacobi fields are

$$\begin{aligned} \partial_s \gamma(t + s)\big|_{s=0} &= \dot\gamma(t) \\ \partial_s \gamma((1 + s)t)\big|_{s=0} &= t \cdot \dot\gamma(t) = \hat\gamma(t). \end{aligned}$$

LEMMA 6.6. *For each Jacobi field X along γ it holds*

$$\tfrac{\mathrm{d}^2}{\mathrm{d}t^2} g(X, \dot\gamma) = 0.$$

PROOF. We compute

$$\begin{aligned}\tfrac{\mathrm{d}^2}{\mathrm{d}t^2} g(X,\dot\gamma) &= \tfrac{\mathrm{d}}{\mathrm{d}t}(g(\nabla_{\dot\gamma}X,\dot\gamma) + g(X,\underbrace{\nabla_{\dot\gamma}\dot\gamma}_{0})) = g(\nabla^2_{\dot\gamma}X,\dot\gamma) + g(\nabla_{\dot\gamma}X,\underbrace{\nabla_{\dot\gamma}\dot\gamma}_{0})\\ &= g(R(\dot\gamma,X)\dot\gamma,\dot\gamma) = 0\end{aligned}$$

since the curvature tensor is antisymmetric in its last two variables. □

From the lemma it follows that $g(X,\dot\gamma) = \alpha + \beta t$. Assuming for simplicity that $|\dot\gamma| = 1$ we have $g(\hat\gamma,\dot\gamma) = t$. Therefore

$$g(X - \alpha\dot\gamma - \beta\hat\gamma,\dot\gamma) = 0$$

We have proved

THEOREM 6.7. *Every Jacobi field X along a geodesic γ can be uniquely decomposed as*

$$X = \alpha\dot\gamma + \beta\hat\gamma + Y$$

where Y is a Jacobi field perpendicular to $\dot\gamma$. □

We are now in the position to prove Gauss lemma asserting that

$$g(\exp_{x*}(v,v),\exp_{x*}(v,w)) = g^0(v,w)$$

for all $v,w \in T_xM$.

PROOF. Consider the geodesic variation $\exp_x(t(v+sw))$ and its Jacobi field

$$X(t) = \partial_s \exp_x(t(v+sw))|_{s=0} = \exp_{x*}(tv,tw)$$

With $\gamma(t) = \exp_x(tv)$ the last lemma says that

$$g(X(t),\dot\gamma(t)) = g(\exp_{x*}(tv,tw),\exp_{x*}(tv,v)) = t\cdot g(\exp_{x*}(tv,w),\exp_{x*}(tv,v))$$

should be linear in t. Therefore $g(\exp_{x*}(tv,w),\exp_{x*}(tv,v))$ must be constant and

$$g(\exp_{x*}(v,w),\exp_{x*}(v,v)) = g((0,w),(0,v)) = g^0(w,v)$$

□

REMARK. The above Jacobi field is the only one with $X(0) = 0$.

DEFINITION 6.8. We say that two points $\gamma(\alpha),\gamma(\beta)$ are *conjugate* if there exists a nonzero Jacobi filed X satisfying $X(\alpha) = 0 = X(\beta)$.

DEFINITION 6.9. For $x \in M$ consider $\exp_x : U_x \to M$. A point $y \in U_x$ (i.e. a small vector in T_xM) is said to be *conjugate* to x if the rank of $\exp_{x*}$ at y is less than $\dim M$.

THEOREM 6.10. *A point $y \in U_x$ is conjugate to x if and only if $x = \exp_x 0$ and $z = \exp_x y$ are conjugate points of the geodesic* $\exp_x(ty), t \in [0,1]$.

PROOF. For the implication "$\Rightarrow$" let $w \in \ker \exp_{x*y}$. Then the Jacobi field $\exp_{x*}(ty,tw)$ of the geodesic variation $\exp_x t(y+sw)$ has zeroes for $t = 0,1$.

For the reverse implication let X be a nonzero Jacobi field along $\exp_x ty$ satisfying $X(0) = 0 = X(1)$. There exists a geodesic variation of the form $\exp_x(t\cdot y(s))$, with $y(0) = y$, generating X. Then

$$X(t) = \tfrac{\partial}{\partial s}\big|_{s=0} \exp_x(t\cdot y(s)) = \exp_{x*}(ty,t\dot y(0))$$

and $0 = X(1) = \exp_{x*}(y,\dot y(0))$. Moreover $\dot y(0) \neq 0$ as that would imply $X \equiv 0$. □

THEOREM 6.11. *If $-g(R(\dot\gamma,Y)\dot\gamma,Y) \le 0$ for any vector field Y along γ then no points of γ are conjugate. In particular if $K(p) \le 0$ then $\exp_x$ is a local diffeomorphism (on its domain).*

PROOF. We start with a computation

$$\tfrac{\mathrm{d}}{\mathrm{d}t} g(\nabla_{\dot\gamma} X, X) = g(\nabla_{\dot\gamma} X, \nabla_{\dot\gamma} X) + g(\nabla^2_{\dot\gamma} X, X) = |\nabla_{\dot\gamma} X|^2 + g(R(\dot\gamma, X)\dot\gamma, X) \geq 0$$

Integrating from a to b we obtain

$$g(\nabla_{\dot\gamma} X(b), X(b)) - g(\nabla_{\dot\gamma} X(a), X(a)) \geq 0$$

and the equality can occur only for a parallel vector field. But if $X(a) = 0 = X(b)$ then both terms are zero and thus necessarily $X \equiv 0$. □

THEOREM 6.12. *If M is a connected complete Riemannian space with non-positive sectional curvature then every $\exp_x : T_xM \to M$ is a covering. In particular when M is simply connected then $\exp_x$ is a global diffeomorphism.*

PROOF. Let $v, w \in T_xM$ and consider the geodesic variation $\exp_x(t(v + sw))$ and its Jacobi field $X(t) = \exp_{x*}(tv, tw)$. In particular $X(1) = \exp_{x*}(v, w)$. We will now study the behaviour of $|X(t)|$ for $t > 0$.

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t} g(X, X)^{1/2} &= \frac{g(\nabla_{\dot\gamma} X, X)}{|X|} \\
\tfrac{\mathrm{d}^2}{\mathrm{d}t^2} g(X, X)^{1/2} &= \frac{|\nabla_{\dot\gamma} X|^2 + g(R(\dot\gamma, X)\dot\gamma, X)}{|X|} - \frac{g(\nabla_{\dot\gamma} X, X)^2}{|X|^3} \\
&= \frac{(|X|^2 |\nabla_{\dot\gamma} X|^2 - g(\nabla_{\dot\gamma} X, X)^2) - |X|^2 R(\dot\gamma, X, \dot\gamma, X)}{|X|^3} \geq 0
\end{aligned}$$

In the numerator the first bracket is non-negative by the Cauchy-Schwarz inequality while the second is non-positive by our assumption on the sectional curvature. For $t \geq 0$ let us denote $f(t) = |X(t)| - t|w|$ and study its Taylor expansion. In local coordinates we can write

$$X(t) = \exp_{x*}(tv, tw) = t \cdot w(t)$$

where w is a curve with $w(0) = w$ which we may assume to be non-zero. Thus

$$|X(t)| = t \cdot |w(t)|$$

is smooth and hence so is f whose value and first derivative at zero are zero. By continuity the second derivative on $[0, \infty)$ must be non-negative and thus the same must be true for the first derivative and finally also for the value. For $t = 1$ this means $|\exp_{x*}(v, w)| = |X(1)| \geq |w|$. In other words $\exp_{x*}$ is non-contracting.

We will now show that $\exp_x : T_xM \to M$ possesses the path-lifting property. Let $\gamma : [a, b] \to M$ be a path with $\gamma(a) = \exp_x y_0$. Denote by

$$T = \{t \in [a, b] \mid \gamma|_{[a,t]} \text{ can be lifted to } \tilde\gamma \text{ with } \tilde\gamma(a) = y_0\}$$

We will show that $T = [a, b]$ by connectedness. Clearly T is nonempty and open since $\exp_x$ is a local diffeomorphism. Let $t_n \to b_0 \leq b$ be a sequence with $t_n \in T$ and denote by $\tilde\gamma : [a, b_0) \to T_xM$ a lift with $\tilde\gamma(a) = y_0$. It exists by the uniqueness of the lifts (thanks to the local diffeomorphism property). Then

$$|\tilde\gamma(t_n) - \tilde\gamma(t_m)| \leq \ell(\tilde\gamma|_{[t_n, t_m]}) \leq \ell(\gamma|_{[t_n, t_m]}) < \varepsilon$$

for $n, m \gg 0$ since $\exp_x$ is non-contracting and $\dot\gamma$ is bounded. Thus $\tilde\gamma(t_n)$ is a Cauchy sequence and converges to some y. As $\exp_x$ is a local diffeomorphism at y the lift $\tilde\gamma$ can be prolonged.

It is a simple matter to deduce that a local diffeomorphism $\exp_x$ is a covering from the path-lifting property. Namely a trivialization is produced from radial rays in a coordinate chart. □

REMARK. If M and N are two simply connected complete Riemannian manifolds of the same dimension and the same constant non-positive sectional curvature then in the diagram

$$\begin{tikzcd}
T_xM \arrow[r, "\text{isometry}", "\cong"'] \arrow[d, "\exp_x"', "\cong"] & T_yN \arrow[d, "\exp_y", "\cong"'] \\
M \arrow[r, dashed, "\cong"'] & N
\end{tikzcd}$$

the dotted arrow is an isometry. The same is true for positive curvature but the vertical arrows are not isomorphisms. We will try to explain the situation by a computation. Let us denote the constant value of the curvature by $K > 0$. We know that

$$R(X,Y)Z = K \cdot (g(Y,Z)X - g(X,Z)Y)$$

If γ is a geodesic parametrized by the arc length and X is a Jacobi field perpendicular to $\dot\gamma$ then

$$\nabla^2_{\dot\gamma} X = R(\dot\gamma, X)\dot\gamma = -K \cdot X$$

If we put $K = \varphi^2$ then the solution of this equation is

$$X(t) = \sin(\varphi t) \cdot \mathrm{Pt}_\gamma(w,t)$$

and we see that $X(\pi/\varphi) = 0$ for all w. Thus the whole sphere $S(x, \pi/\varphi)$ is mapped to a single point and $\exp_x$ induces a map

$$D(x,\pi/\varphi)/S(x,\pi/\varphi) \xrightarrow{\exp_x} M$$

which is a diffeomorphism on the interior of $D(x, \pi/\varphi)$. Its metric properties are the following: it preserves orthogonality of the radial rays to the spheres and preserves the metric on the radial rays while on the sphere of radius r it multiplies it by $\sin(\varphi r)$. The point is that this behaviour only depends on the curvature K and thus for two manifolds S^m and M in the diagram

$$\begin{tikzcd}
D^m(\pi/\varphi)/S^{m-1}(\pi/\varphi) \arrow[r, "\exp_y"] \arrow[d, "\cong"', "\text{isometry}"] & S^m(1/\varphi) \arrow[d, dashed] \\
D(x,\pi/\varphi)/S(x,\pi/\varphi) \arrow[r, "\exp_x"'] & M
\end{tikzcd}$$

the dotted arrow, which is defined on the image of the interior of D^m, preserves the metric. A similar map can be defined using a different point on the sphere and together they provide a local isometry from S^m to M. It is a covering by the proof of the last theorem and thus an isometry.

7. Problems

Problem 7.1. Determine the Levi-Civita connection (or the corresponding covariant derivative) for the Euclidean space $\mathbb{E}_m$ by guessing what it might be and then proving that it indeed is symmetric and metric.

Problem 7.2. For $\mathbb{E}_m$ determine the Christoffel symbols, all curvatures and geodesics.

Problem 7.3. Determine the connection form of the Levi-Civita connection on S^m.

Problem 7.4. Show that the sphere has constant sectional curvature by studying the effect of an orthogonal transformation.

Problem 7.5. Determine the sectional curvature of the unit sphere.

Problem 7.6. In $\mathbb{R}^{m,1} = \mathbb{R}^m \times \mathbb{R}$ we consider the scalar product

$$\langle y, y\rangle = x_1^2 + \cdots + x_m^2 - x_0^2$$

of signature $(m, 1)$ where $y = (x, x_0) = ((x_1, \ldots, x_m), x_0)$. The *hyperbolic space* of dimension m is the subset

$$\mathbb{H}^m = \{y = (x, x_0) \in \mathbb{R}^{m,1} \mid \langle y, y\rangle = -1, x_0 > 0\}$$

Show that with the induced scalar product $\mathbb{H}^m$ is a Riemannian manifold. Show that $\mathbb{H}^m \cong \mathrm{O}^+(m,1)/\,\mathrm{O}(m)$ and has a constant sectional curvature. Determine this sectional curvature.