Module Theory in Sup and Enriched Order Theory

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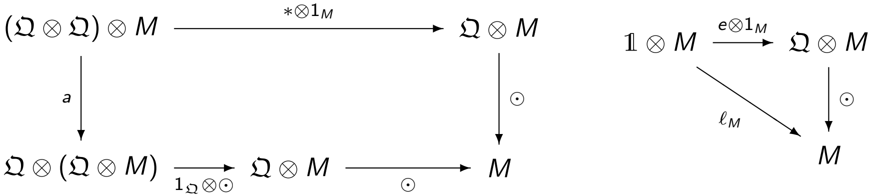

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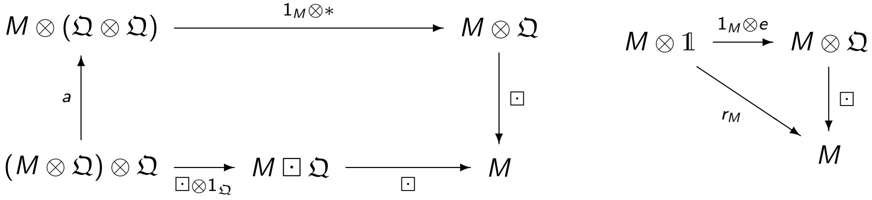

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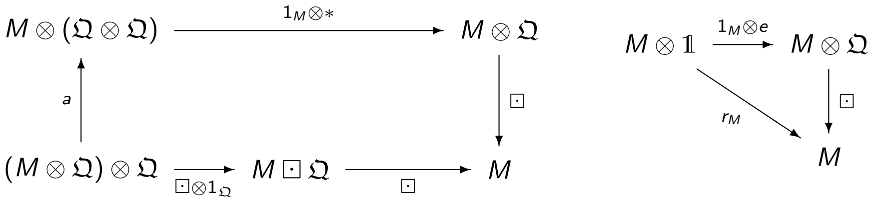

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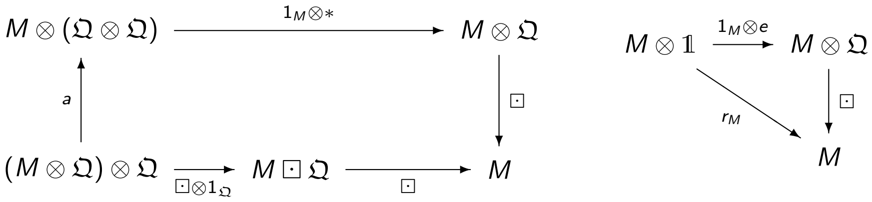

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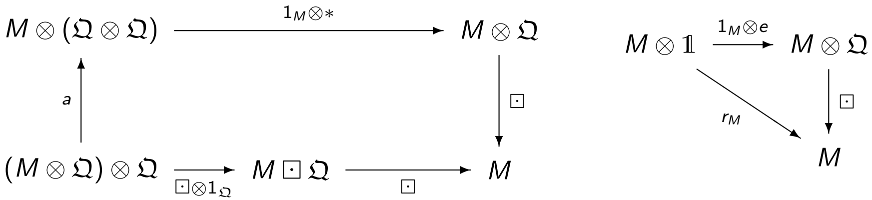

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- $\mathcal{F}(M) = \mathfrak{Q} \otimes M$,

$$\begin{array}{ccc} \mathfrak{Q} \otimes (\mathfrak{Q} \otimes M) & & \\ {\scriptstyle a^{-1}}\downarrow & \searrow^{\odot} & \\ (\mathfrak{Q} \otimes \mathfrak{Q}) \otimes M & \xrightarrow[* \otimes 1_M]{} & \mathfrak{Q} \otimes M \end{array}$$

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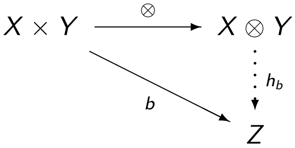

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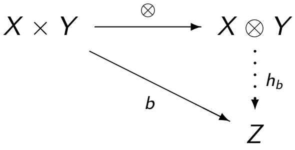

- The previous property is a universal property and $\otimes$ is the universal bimorphism.

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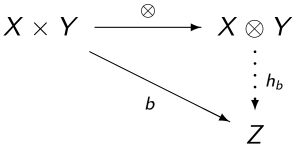

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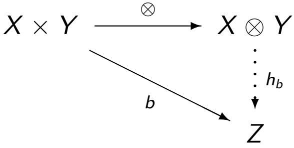

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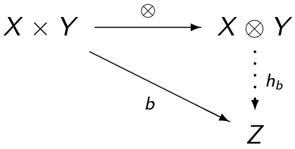

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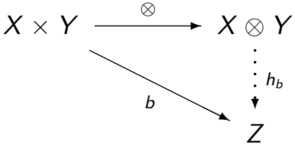

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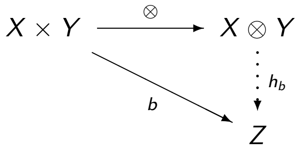

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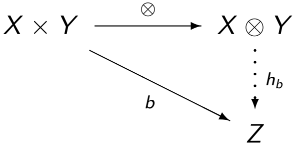

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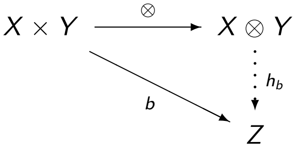
- Sup as star-autonomous category and in particular as symmetric monoidal closed category has been first recognized by M. Barr 1979 (see `Inf` on p. 99 in LNM 752 Springer-Verlag), and later repeated by A. Joyal and M. Tierney 1984, who made an extensive use of its tensor product.

Let $X \times Y \xrightarrow{b} Z$ be a bimorphism and h_b be the unique join-preserving map determined by the tensor product in Sup



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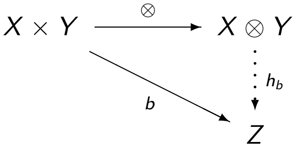

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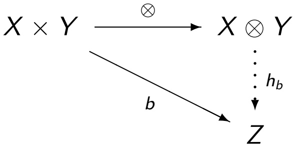

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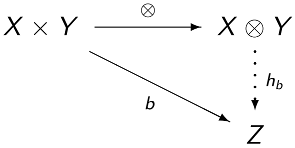

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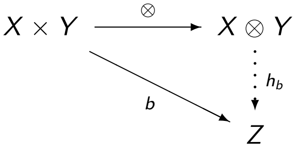

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$$f_z(x) = \bigvee\{y \in Y \mid b(x, y) \leq z\}, \qquad x \in X.$$

- Since $h_b(x \otimes y) = b(x, y) \leq z \quad \Leftrightarrow \quad x \otimes y \leq f_z$, we obtain:

$$h_b^{\vdash}(z) = f_z, \qquad z \in Z.$$

General principles of module theory in Sup

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- Let (X,p) be a $\mathfrak{Q}$-preordered set. A map $X \xrightarrow{f} \mathfrak{Q}$ is called a contravariant $\mathfrak{Q}$-presheaf on (X,p), if f is left-extensional — i.e.

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Enriched order structures on right modules in Sup

- Every right $\mathfrak{Q}$-module with its intrinsic $\mathfrak{Q}$-preorder p carries the structure of a $\mathfrak{Q}$-preorder set. The underlying preorder $\leq_p$ of p coincides with the order given on M. Hence every intrinsic $\mathfrak{Q}$-preorder is antisymmetric.

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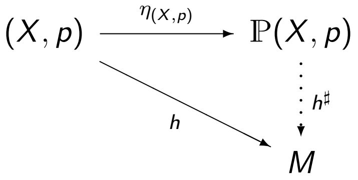
- The intrinsic $\mathfrak{Q}$-preorder of $\mathbb{P}(X,p)$ coincides with d.

Theorem 1. Let (X,p) be a $\mathfrak{Q}$-preordered set and M be a right $\mathfrak{Q}$-module with its intrinsic $\mathfrak{Q}$-preorder q. If $(X,p) \xrightarrow{h} (M,q)$ is a $\mathfrak{Q}$-homomorphism, then there exists a unique right $\mathfrak{Q}$-module homomorphism $\mathbb{P}(X,p) \xrightarrow{h^\sharp} M$ making the following diagram commutative:



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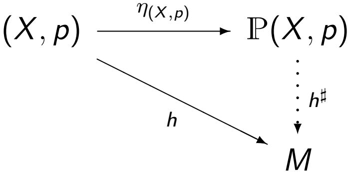

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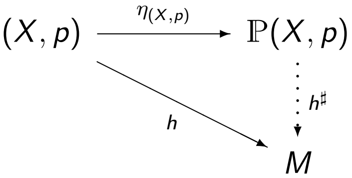

In particular, $h^\sharp(f) = \bigvee_{x \in X} h(x) \boxdot f(x)$ for $f \in \mathbb{P}(X, p)$.

Hence $\mathbb{P}(X, p)$ is the free right $\mathfrak{Q}$-module generated by (X, p).

Theorem 2. Let M be a right $\mathfrak{Q}$-module and p be its intrinsic $\mathfrak{Q}$-preorder. Then (M, p) is a $\mathfrak{Q}$-enriched join-complete $\mathfrak{Q}$-preordered set. In particular, the formation of $\mathfrak{Q}$-enriched joins $\mathbb{P}(M, p) \xrightarrow{\sup_M} M$ is given by:

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Proof.

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Example. Consider the right $\mathfrak{Q}$-module $\mathfrak{Q}$. Then for $\mathfrak{Q} \xrightarrow{f} \mathfrak{Q}$ we have:

$$\mathsf{sup}_{\mathfrak{Q}}(\downarrow f) = \bigvee_{\alpha \in \mathfrak{Q}} \alpha * f(\alpha).$$

Theorem 3. Let (X, p) be a $\mathfrak{Q}$-enriched join-complete, antisymmetric $\mathfrak{Q}$-preordered set. Then X provided with the underlying partial order is complete in the traditional sense, and there exists a right action $\boxdot$ on X determined by:

$$x \boxdot \alpha = \mathsf{sup}_{(X,p)}(\widetilde{x} * \alpha), \qquad x \in X,\ \alpha \in \mathfrak{Q}.$$

Hence right $\mathfrak{Q}$-modules and $\mathfrak{Q}$-enriched join-complete, antisymmetric $\mathfrak{Q}$-preordered sets are equivalent concepts.

Theorem 3. Let (X, p) be a $\mathfrak{Q}$-enriched join-complete, antisymmetric $\mathfrak{Q}$-preordered set. Then X provided with the underlying partial order is complete in the traditional sense, and there exists a right action $\boxdot$ on X determined by:

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Hence right $\mathfrak{Q}$-modules and $\mathfrak{Q}$-enriched join-complete, antisymmetric $\mathfrak{Q}$-preordered sets are equivalent concepts.

Theorem 4. Let M and N be right $\mathfrak{Q}$-modules with the respective intrinsic $\mathfrak{Q}$-preorders p and q. Further, let $(M, p) \xrightarrow{h} (N, q)$ be a $\mathfrak{Q}$-homomorphism. Then $M \xrightarrow{h} N$ is a right $\mathfrak{Q}$-module homomorphism if and only if h has a right adjoint $\mathfrak{Q}$-homomorphism.

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Corollary (Stubbe 2006 and 2007). The category $\mathtt{Mod}_r(\mathfrak{Q})$ of right $\mathfrak{Q}$-modules is isomorphic to the category of $\mathtt{Sup}(\mathfrak{Q})$ of $\mathfrak{Q}$-enriched join-complete, antisymmetric $\mathfrak{Q}$-preordered sets with $\mathfrak{Q}$-enriched join-preserving $\mathfrak{Q}$-homomorphisms.

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- If $f \in \mathbb{P}^{\dagger}(M, p)$, then

$$\begin{aligned}\inf_{(M,p)}(f) &= \bigvee_{m \in M} m \boxdot \Big(\bigvee_{n \in M} p(m,n) \swarrow f(n) \Big) \\ &= \bigwedge_{m \in M} \Big(\bigvee \{ n \in M \mid n \boxdot f(m) \leq m \} \Big).\end{aligned}$$

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- On the dual lattice M^{op} we introduce a left action $\odot$ determined by:

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- Since $\inf_M(f) = \bigvee^{op}_{m\in M} f(m) \odot m$, the **Answer** is in general negative.

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- Then $(M^{op}, \boxdot^{op})$ is the dual right $\mathfrak{Q}$-module of M, and its intrinsic $\mathfrak{Q}$-preorder of coincides with the dual $\mathfrak{Q}$-preorder of p:

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- $\inf_{(M,p)}(f) = \bigvee^{op}_{m \in M} f(m) \odot m = \bigvee^{op}_{m \in M} m \boxdot^{op} f(m)' = \sup_{M^{op}}(f')$.

Summary

The replacement of the unique unital quantale $\mathbf{2}$ on $\{0, 1\}$ by an arbitrary involutive and unital quantale leads to a complete to algebraization of the theory of complete lattices.