Possible Applications of Quantales and Module Theory

Ulrich Höhle

Bergische Universität, Wuppertal, Germany

Brno, October 4, 2019

Table of Contents

Some Properties of Quantales

A quantale $\mathfrak{Q}$ is semi-unital, if $\alpha \leq \top * \alpha$ and $\alpha \leq \alpha * \top$ for all $\alpha \in \mathfrak{Q}$.

Some Properties of Quantales

A quantale $\mathfrak{Q}$ is semi-unital, if $\alpha \leq \top * \alpha$ and $\alpha \leq \alpha * \top$ for all $\alpha \in \mathfrak{Q}$.

- The semi-unitalization of every quantale $\mathfrak{Q}$ exists.

Some Properties of Quantales

A quantale $\mathfrak{Q}$ is semi-unital, if $\alpha \leq \top * \alpha$ and $\alpha \leq \alpha * \top$ for all $\alpha \in \mathfrak{Q}$.

- The semi-unitalization of every quantale $\mathfrak{Q}$ exists.
- **Construction**. Let $\omega \notin \mathfrak{Q}$. Then $\overline{\mathfrak{Q}} = \mathfrak{Q} \cup \{\omega\}$, ω is the universal upper bound in $\overline{\mathfrak{Q}}$ and the multiplication $*$ is extended as follows:

Some Properties of Quantales

A quantale $\mathfrak{Q}$ is semi-unital, if $\alpha \leq \top * \alpha$ and $\alpha \leq \alpha * \top$ for all $\alpha \in \mathfrak{Q}$.

- The semi-unitalization of every quantale $\mathfrak{Q}$ exists.

- **Construction**. Let $\omega \notin \mathfrak{Q}$. Then $\overline{\mathfrak{Q}} = \mathfrak{Q} \cup \{\omega\}$, ω is the universal upper bound in $\overline{\mathfrak{Q}}$ and the multiplication $*$ is extended as follows:

$$\alpha \bar{*} \beta = \alpha * \beta, \ \ \alpha \bar{*} \omega = \alpha \vee (\alpha * \top), \ \ \omega \bar{*} \alpha = \alpha \vee (\top * \alpha), \ \ \omega \bar{*} \omega = \omega.$$

Some Properties of Quantales

A quantale $\mathfrak{Q}$ is semi-unital, if $\alpha \leq \top * \alpha$ and $\alpha \leq \alpha * \top$ for all $\alpha \in \mathfrak{Q}$.

- The semi-unitalization of every quantale $\mathfrak{Q}$ exists.

- **Construction**. Let $\omega \notin \mathfrak{Q}$. Then $\overline{\mathfrak{Q}} = \mathfrak{Q} \cup \{\omega\}$, ω is the universal upper bound in $\overline{\mathfrak{Q}}$ and the multiplication $*$ is extended as follows:

$$\alpha \bar{*} \beta = \alpha * \beta, \quad \alpha \bar{*} \omega = \alpha \vee (\alpha * \top), \quad \omega \bar{*} \alpha = \alpha \vee (\top * \alpha), \quad \omega \bar{*} \omega = \omega.$$

- A quantale $\mathfrak{Q}$ is two-sided if and only if its semi-unitalization $\overline{\mathfrak{Q}}$ is integral.

Some Properties of Quantales

A quantale $\mathfrak{Q}$ is semi-unital, if $\alpha \leq \top * \alpha$ and $\alpha \leq \alpha * \top$ for all $\alpha \in \mathfrak{Q}$.

- The semi-unitalization of every quantale $\mathfrak{Q}$ exists.
- **Construction**. Let $\omega \notin \mathfrak{Q}$. Then $\overline{\mathfrak{Q}} = \mathfrak{Q} \cup \{\omega\}$, ω is the universal upper bound in $\overline{\mathfrak{Q}}$ and the multiplication $*$ is extended as follows:

$$\alpha \bar{*} \beta = \alpha * \beta, \;\; \alpha \bar{*} \omega = \alpha \vee (\alpha * \top), \;\; \omega \bar{*} \alpha = \alpha \vee (\top * \alpha), \;\; \omega \bar{*} \omega = \omega.$$

- A quantale $\mathfrak{Q}$ is two-sided if and only if its semi-unitalization $\overline{\mathfrak{Q}}$ is integral.
- The semi-unitalization of a quantale is always a subquantale of its unitalization.

Let $\mathfrak{Q}$ be a quantale such that the subquantale of all two-sided elements has the form $\mathbb{I}(\mathfrak{Q}) = \{\bot, \top\}$ (i.e. there does not exist non-trivial two-sided elements).

If $\mathfrak{Q}$ is semi-unital and left-sided, then the quantale multiplication has the following form:

Let $\mathfrak{Q}$ be a quantale such that the subquantale of all two-sided elements has the form $\mathbb{I}(\mathfrak{Q}) = \{\bot, \top\}$ (i.e. there does not exist non-trivial two-sided elements).

If $\mathfrak{Q}$ is semi-unital and left-sided, then the quantale multiplication has the following form:

- $$\alpha * \beta = \begin{cases} \beta, & \alpha \neq \bot, \\ \bot, & \alpha = \bot. \end{cases}, \qquad \alpha, \beta \in \mathfrak{Q}.$$

Let $\mathfrak{Q}$ be a quantale such that the subquantale of all two-sided elements has the form $\mathbb{I}(\mathfrak{Q}) = \{\bot, \top\}$ (i.e. there does not exist non-trivial two-sided elements).

If $\mathfrak{Q}$ is semi-unital and left-sided, then the quantale multiplication has the following form:

- $$\alpha * \beta = \begin{cases} \beta, & \alpha \neq \bot, \\ \bot, & \alpha = \bot. \end{cases}, \qquad \alpha, \beta \in \mathfrak{Q}.$$
- In particular, $\mathfrak{Q}$ is idempotent.

Let $\mathfrak{Q}$ be a quantale such that the subquantale of all two-sided elements has the form $\mathbb{I}(\mathfrak{Q}) = \{\bot, \top\}$ (i.e. there does not exist non-trivial two-sided elements).

If $\mathfrak{Q}$ is semi-unital and left-sided, then the quantale multiplication has the following form:

- $$\alpha * \beta = \begin{cases} \beta, & \alpha \neq \bot, \\ \bot, & \alpha = \bot. \end{cases}, \qquad \alpha, \beta \in \mathfrak{Q}.$$
- In particular, $\mathfrak{Q}$ is idempotent.
- A prominent example is the ideal multiplication of left ideals of square matrices in finite dimensional vector spaces.

All quantales on the chain $C_3 = \{\bot, a, \top\}$

(1)

$*$	$\bot$	a	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	$\bot$
$\top$	$\bot$	$\bot$	$\bot$

(2)

$*$	$\bot$	a	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	$\bot$
$\top$	$\bot$	$\bot$	a

(6)

$*$	$\bot$	a	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	$\bot$
$\top$	$\bot$	$\bot$	$\top$

(7)

$*$	$\bot$	a	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	a
$\top$	$\bot$	$\bot$	$\top$

(8)

$*$	$\bot$	a	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	$\bot$
$\top$	$\bot$	a	$\top$

(9)

$*$	$\bot$	a	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	a
$\top$	$\bot$	a	$\top$

(15)

$*$	$\bot$	a	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	a
$\top$	$\bot$	a	a

(16)

$*$	$\bot$	a	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	a
$\top$	$\bot$	a	$\top$

(17)

$*$	$\bot$	a	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\top$
$\top$	$\bot$	a	$\top$

(18)

$*$	$\bot$	a	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	a
$\top$	$\bot$	$\top$	$\top$

(19)

$*$	$\bot$	a	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\top$
$\top$	$\bot$	$\top$	$\top$

(20)

$*$	$\bot$	a	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\top$	$\top$
$\top$	$\bot$	$\top$	$\top$

- (9) $=$ *MV*-algebra with three elements is the semi-unitalization of the trivial quantale on $\{0, 1\}$.

- (9) = *MV-algebra* with three elements is the semi-unitalization of the trivial quantale on $\{0, 1\}$.
- (16)= binary meet on $\{\perp, a, \top\}$ is the semi-unitalization of the unique unital quantale $\mathbf{2}$ on $\{0, 1\}$.

- (9) $=$ *MV*-algebra with three elements is the semi-unitalization of the trivial quantale on $\{0, 1\}$.
- (16)$=$ binary meet on $\{\bot, a, \top\}$ is the semi-unitalization of the unique unital quantale $\mathbf{2}$ on $\{0, 1\}$.
- (19) is a commutative Girard quantale, in which the unit coincides with the dualizing and cyclic element.

- (9) = *MV*-algebra with three elements is the semi-unitalization of the trivial quantale on $\{0, 1\}$.
- (16)= binary meet on $\{\bot, a, \top\}$ is the semi-unitalization of the unique unital quantale **2** on $\{0, 1\}$.
- (19) is a commutative Girard quantale, in which the unit coincides with the dualizing and cyclic element.
- On $\{\bot, a, \top\}$ all unital quantales are commutative.

- (9) $=$ *MV*-algebra with three elements is the semi-unitalization of the trivial quantale on $\{0, 1\}$.
- (16)$=$ binary meet on $\{\bot, a, \top\}$ is the semi-unitalization of the unique unital quantale $\mathbf{2}$ on $\{0, 1\}$.
- (19) is a commutative Girard quantale, in which the unit coincides with the dualizing and cyclic element.
- On $\{\bot, a, \top\}$ all unital quantales are commutative.
- 2 quantales are non-commutative and two-sided — namely (7) is strictly right-sided and not semi-unital, while (8) is strictly left-sided and not semi-unital.

- On the chain $C_4 = \{\perp, a, b, \top\}$ consisting of 4 elements (i.e. $\perp < a < b < \top$) there exists exactly two non-commutative and integral quantale structures.

- On the chain $C_4 = \{\perp, a, b, \top\}$ consisting of 4 elements (i.e. $\perp < a < b < \top$) there exists exactly two non-commutative and integral quantale structures.
- There exists exactly two further non-commutative, non-integral and unital quantale structures on C_4:

- On the chain $C_4 = \{\bot, a, b, \top\}$ consisting of 4 elements (i.e. $\bot < a < b < \top$) there exists exactly two non-commutative and integral quantale structures.
- There exists exactly two further non-commutative, non-integral and unital quantale structures on C_4:
- b is the unit $a * a = a$, $a * \top = a$, $\top * a = \top$ and

- On the chain $C_4 = \{\bot, a, b, \top\}$ consisting of 4 elements (i.e. $\bot < a < b < \top$) there exists exactly two non-commutative and integral quantale structures.
- There exists exactly two further non-commutative, non-integral and unital quantale structures on C_4:
- b is the unit $a * a = a, \quad a * \top = a, \quad \top * a = \top$ and
- b is the unit $a * a = a, \quad a * \top = \top, \quad \top * a = a.$

- On the chain $C_4 = \{\bot, a, b, \top\}$ consisting of 4 elements (i.e. $\bot < a < b < \top$) there exists exactly two non-commutative and integral quantale structures.

- There exists exactly two further non-commutative, non-integral and unital quantale structures on C_4:
- b is the unit $a * a = a, \quad a * \top = a, \quad \top * a = \top$ and
- b is the unit $a * a = a, \quad a * \top = \top, \quad \top * a = a.$

- All 4 non-commutative quantales on C_4 cannot be provided with an order-preserving involution, which is an quantale anti-homomorphism.

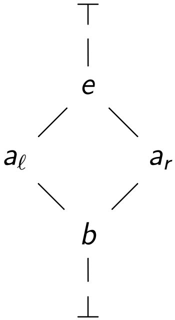
- On the chain $C_4 = \{\bot, a, b, \top\}$ consisting of 4 elements (i.e. $\bot < a < b < \top$) there exists exactly two non-commutative and integral quantale structures.
- There exists exactly two further non-commutative, non-integral and unital quantale structures on C_4:
- b is the unit $a * a = a, \quad a * \top = a, \quad \top * a = \top$ and
- b is the unit $a * a = a, \quad a * \top = \top, \quad \top * a = a$.
- All 4 non-commutative quantales on C_4 cannot be provided with an order-preserving involution, which is an quantale anti-homomorphism.

  The formation of C_4-enriched meets and C_4-enriched joins are not dual concepts.

Let $\mathfrak{Q}_6$ be the unital quantale of all join-preserving self-mappings of $C_3 = \{\bot, a, \top\}$. Then $\mathfrak{Q}_6$ has six elements.

Let $\mathfrak{Q}_6$ be the unital quantale of all join-preserving self-mappings of $C_3 = \{\bot, a, \top\}$. Then $\mathfrak{Q}_6$ has six elements.

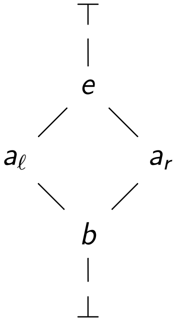
- The Hasse diagram and the multiplication table of $\mathfrak{Q}_6$ is given by:



$*$	$\bot$	b	a_ℓ	a_r	1	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
b	$\bot$	$\bot$	b	$\bot$	b	a_r
a_ℓ	$\bot$	$\bot$	a_ℓ	$\bot$	a_ℓ	$\top$
a_r	$\bot$	b	b	a_r	a_r	a_r
1	$\bot$	b	a_ℓ	a_r	1	$\top$
$\top$	$\bot$	a_ℓ	a_ℓ	$\top$	$\top$	$\top$

Let $\mathfrak{Q}_6$ be the unital quantale of all join-preserving self-mappings of $C_3 = \{\bot, a, \top\}$. Then $\mathfrak{Q}_6$ has six elements.

- The Hasse diagram and the multiplication table of $\mathfrak{Q}_6$ is given by:



$*$	$\bot$	b	a_ℓ	a_r	1	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
b	$\bot$	$\bot$	b	$\bot$	b	a_r
a_ℓ	$\bot$	$\bot$	a_ℓ	$\bot$	a_ℓ	$\top$
a_r	$\bot$	b	b	a_r	a_r	a_r
1	$\bot$	b	a_ℓ	a_r	1	$\top$
$\top$	$\bot$	a_ℓ	a_ℓ	$\top$	$\top$	$\top$

- Since C_3 has a unique order reversing involution, there exists an order preserving involution on $\mathfrak{Q}_6$, which is a quantale antihomomorphism:

$$\top' = \top, \quad 1' = 1, a_\ell' = a_r, \quad a_r' = a_\ell, \quad b' = b, \quad \bot' = \bot.$$

Theorem. Every unital quantale $\mathfrak{Q}$ can be embedded into a unital and *involutive* quantale.

Theorem. Every unital quantale $\mathfrak{Q}$ can be embedded into a unital and *involutive* quantale.

- On the complete lattice $L = \mathfrak{Q} \times \mathfrak{Q}^{op}$ there exists an order reversing involution ^{o} defined by

$$(\alpha, \beta)^o = (\beta, \alpha), \qquad (\alpha, \beta \in \mathfrak{Q}.$$

Theorem. Every unital quantale $\mathfrak{Q}$ can be embedded into a unital and *involutive* quantale.

- On the complete lattice $L = \mathfrak{Q} \times \mathfrak{Q}^{op}$ there exists an order reversing involution ^{o} defined by

$$(\alpha, \beta)^o = (\beta, \alpha), \qquad (\alpha, \beta \in \mathfrak{Q}.$$

- On the unital quantale $[L, L]$ of all join-preserving self maps $L \xrightarrow{f} L$ there exists an involution $'$ determined by:

$$f'(\alpha, \beta) = \big(f^{\vdash}((\alpha, \beta)^0)\big)^0, \qquad (\alpha, \beta) \in L.$$

Theorem. Every unital quantale $\mathfrak{Q}$ can be embedded into a unital and *involutive* quantale.

- On the complete lattice $L = \mathfrak{Q} \times \mathfrak{Q}^{op}$ there exists an order reversing involution ^{o} defined by

$$(\alpha, \beta)^o = (\beta, \alpha), \qquad (\alpha, \beta \in \mathfrak{Q}.$$

- On the unital quantale $[L, L]$ of all join-preserving self maps $L \xrightarrow{f} L$ there exists an involution $'$ determined by:

$$f'(\alpha, \beta) = \big(f^{\vdash}((\alpha, \beta)^0))\big)^0, \qquad (\alpha, \beta) \in L.$$

- $\mathfrak{Q} \xrightarrow{\Phi} [L, L]$ defined by:

$$\big(\Phi(\varkappa)\big)(\alpha, \beta) = \big((\varkappa * \alpha), (\beta \swarrow \varkappa)\big) \qquad \varkappa \in \mathfrak{Q}$$

is a unital quantale monomorphism.

Basic issue of SLP

$X \subset \mathbb{R}^n$ is a convex and compact subset. The components of $x = (x_1, \ldots, x_n) \in X$ are called decision variables.

Basic issue of SLP

$X \subset \mathbb{R}^n$ is a convex and compact subset. The components of $x = (x_1, \ldots, x_n) \in X$ are called decision variables.

- The goal function is map $X \xrightarrow{g} [0, +\infty]$ having the form $g(x) = \sum_{i=1}^{n} c_i \cdot x_i$.

Basic issue of SLP

$X \subset \mathbb{R}^n$ is a convex and compact subset. The components of $x = (x_1, \ldots, x_n) \in X$ are called decision variables.

- The goal function is map $X \xrightarrow{g} [0, +\infty]$ having the form $g(x) = \sum_{i=1}^{n} c_i \cdot x_i$.
- Main Task:
  Optimize the restriction of the goal function g to a finite family of non-deterministic constraints which can be expresed as follows:

Basic issue of SLP

$X \subset \mathbb{R}^n$ is a convex and compact subset. The components of $x = (x_1, \ldots, x_n) \in X$ are called decision variables.

- The goal function is map $X \xrightarrow{g} [0, +\infty]$ having the form $g(x) = \sum_{i=1}^{n} c_i \cdot x_i$.

- Main Task:
  Optimize the restriction of the goal function g to a finite family of non-deterministic constraints which can be expresed as follows:
  - there exist a probability space $(\Omega, \mathfrak{A}, \pi)$,

Basic issue of SLP

$X \subset \mathbb{R}^n$ is a convex and compact subset. The components of $x = (x_1, \ldots, x_n) \in X$ are called decision variables.

- The goal function is map $X \xrightarrow{g} [0, +\infty]$ having the form $g(x) = \sum_{i=1}^{n} c_i \cdot x_i$.

- Main Task:
  Optimize the restriction of the goal function g to a finite family of non-deterministic constraints which can be expresed as follows:
  - there exist a probability space $(\Omega, \mathfrak{A}, \pi)$,
  - s-dimensional random vectors h and T_j, $j = 1 \ldots, n$ such that such that $x \in X$ is subjected to the following linear random inequality system:

Basic issue of SLP

$X \subset \mathbb{R}^n$ is a convex and compact subset. The components of $x = (x_1, \ldots, x_n) \in X$ are called decision variables.

- The goal function is map $X \xrightarrow{g} [0, +\infty]$ having the form
  $g(x) = \sum_{i=1}^{n} c_i \cdot x_i$.

- Main Task:
  Optimize the restriction of the goal function g to a finite family of non-deterministic constraints which can be expresed as follows:

– there exist a probability space $(\Omega, \mathfrak{A}, \pi)$,

– s-dimensional random vectors h and T_j, $j = 1 \ldots, n$ such that such that $x \in X$ is subjected to the following linear random inequality system:

– $\zeta(x, \omega) := \sum_{j=1}^{n} T_j(\omega) \cdot x_j - h(\omega) \leq 0, \quad \omega \in \Omega.$

Goal in SLP

- The goal in stochastic linear programming is to achieve a probability distribution of $\zeta(x, \omega)$ with advantageous properties, where x is considered as deterministic.
  (see P. Kall and J. Mayer, Stochastic Linear Programming, Springer-Verlag 2011).

Goal in SLP

- The goal in stochastic linear programming is to achieve a probability distribution of $\zeta(x, \omega)$ with advantageous properties, where x is considered as deterministic.
  (see P. Kall and J. Mayer, Stochastic Linear Programming, Springer-Verlag 2011).
- In order to give a quantitative meaning to the term advantageous we find various approaches in SLP — e.g.

Goal in SLP

- The goal in stochastic linear programming is to achieve a probability distribution of $\zeta(x, \omega)$ with advantageous properties, where x is considered as deterministic.
  (see P. Kall and J. Mayer, Stochastic Linear Programming, Springer-Verlag 2011).
- In order to give a quantitative meaning to the term advantageous we find various approaches in SLP — e.g.
- The introduction of an evaluation function $L^0(\mathbb{R}^s) \xrightarrow{\varrho} \mathbb{R}$ for s-dimensional random vectors, which transforms the given non-deterministic restrictions into deterministic ones by:

$$V(x) = \varrho(\zeta(x, \omega)) \leq 0.$$

Goal in SLP

- The goal in stochastic linear programming is to achieve a probability distribution of $\zeta(x,\omega)$ with advantageous properties, where x is considered as deterministic.
  (see P. Kall and J. Mayer, Stochastic Linear Programming, Springer-Verlag 2011).
- In order to give a quantitative meaning to the term advantageous we find various approaches in SLP — e.g.
- The introduction of an evaluation function $L^0(\mathbb{R}^s) \xrightarrow{\varrho} \mathbb{R}$ for s-dimensional random vectors, which transforms the given non-deterministic restrictions into deterministic ones by:

$$V(x) = \varrho(\zeta(x,\omega)) \leq 0.$$

- The the following deterministic optimization problem is considered:

$$\text{max.} \quad X \xrightarrow{g} [0,+\infty] \quad \text{under} \quad V(x) = \varrho(\zeta(x,\omega)) \leq 0.$$

Observations

- The non-deterministic constrains give rise to a random set on the space X of decision variables S:

$$\omega \longmapsto S(\omega) = \{x \in X \mid \zeta(x, \omega) \leq 0\}.$$

Observations

- The non-deterministic constrains give rise to a random set on the space X of decision variables S:

$$\omega \longmapsto S(\omega) = \{x \in X \mid \zeta(x,\omega) \leq 0\}.$$

- The distribution of the random set S induces a map $X \xrightarrow{f} [0,1]$ by

$$f(x) = \pi\big(\{\omega \in \Omega \mid x \in S(\omega)\}\big), \qquad x \in X.$$

Observations

- The non-deterministic constrains give rise to a random set on the space X of decision variables S:

$$\omega \longmapsto S(\omega) = \{x \in X \mid \zeta(x,\omega) \leq 0\}.$$

- The distribution of the random set S induces a map $X \xrightarrow{f} [0,1]$ by

$$f(x) = \pi\big(\{\omega \in \Omega \mid x \in S(\omega)\}\big), \qquad x \in X.$$

- Problems:

Observations

- The non-deterministic constrains give rise to a random set on the space X of decision variables S:

$$\omega \longmapsto S(\omega) = \{x \in X \mid \zeta(x, \omega) \leq 0\}.$$

- The distribution of the random set S induces a map $X \xrightarrow{f} [0, 1]$ by

$$f(x) = \pi\big(\{\omega \in \Omega \mid x \in S(\omega)\}\big), \qquad x \in X.$$

- Problems:

– What is the restriction of the goal function g to f or what is the image of f under g?

Observations

- The non-deterministic constrains give rise to a random set on the space X of decision variables S:

$$\omega \longmapsto S(\omega) = \{x \in X \mid \zeta(x,\omega) \leq 0\}.$$

- The distribution of the random set S induces a map $X \xrightarrow{f} [0,1]$ by

$$f(x) = \pi\big(\{\omega \in \Omega \mid x \in S(\omega)\}\big), \qquad x \in X.$$

- Problems:

– What is the restriction of the goal function g to f or what is the image of f under g?

– What is the supremmum of the image of f under g?

Reformulation and the perspective of solution

- The map f can be considered as a fuzzy subset of X.

Reformulation and the perspective of solution

- The map f can be considered as a fuzzy subset of X.
- The image of f under the goal function g is obviously a fuzzy subset of $[0, +\infty]$.

Reformulation and the perspective of solution

- The map f can be considered as a fuzzy subset of X.
- The image of f under the goal function g is obviously a fuzzy subset of $[0, +\infty]$.
- What is the supremum of a fuzzy subset of $[0, +\infty]$?

Reformulation and the perspective of solution

- The map f can be considered as a fuzzy subset of X.
- The image of f under the goal function g is obviously a fuzzy subset of $[0, +\infty]$.
- What is the supremum of a fuzzy subset of $[0, +\infty]$?
- The solution is given by module theory in the category $\mathtt{Sup}$ of complete lattices and join-preserving maps.

Reformulation and the perspective of solution

- The map f can be considered as a fuzzy subset of X.
- The image of f under the goal function g is obviously a fuzzy subset of $[0, +\infty]$.
- What is the supremum of a fuzzy subset of $[0, +\infty]$?

- The solution is given by module theory in the category Sup of complete lattices and join-preserving maps.
- As a monoid in Sup we choose the real unit interval $[0, 1]$ provided with Łukasiewicz arithmetic conjunction:

$$\alpha * \beta = \max(\alpha + \beta - 1, 0), \qquad \alpha, \beta \in [0, 1].$$

Join-completeness

Let $(X, \leq)$ be a preordered set. A subset $A \subseteq X$ is *downclosed* if $z \leq x$ and $x \in A$ implies $z \in A$.

Join-completeness

Let $(X, \leq)$ be a preordered set. A subset $A \subseteq X$ is *downclosed* if $z \leq x$ and $x \in A$ implies $z \in A$.

- $\mathrm{Dwn}(X)$ is the set of all downclosed subsets of X ordered by set-inclusion.

Join-completeness

Let $(X, \leq)$ be a preordered set. A subset $A \subseteq X$ is *downclosed* if $z \leq x$ and $x \in A$ implies $z \in A$.

- $\mathrm{Dwn}(X)$ is the set of all downclosed subsets of X ordered by set-inclusion.
- There is an isotone map $X \xrightarrow{\eta_X} \mathrm{Dwn}(X)$ defined by

$$\eta_X(x) = {\downarrow}x = \{z \in X \mid z \leq x\}. \qquad x \in X.$$

Join-completeness

Let $(X, \leq)$ be a preordered set. A subset $A \subseteq X$ is *downclosed* if $z \leq x$ and $x \in A$ implies $z \in A$.

- $\mathrm{Dwn}(X)$ is the set of all downclosed subsets of X ordered by set-inclusion.
- There is an isotone map $X \xrightarrow{\eta_X} \mathrm{Dwn}(X)$ defined by

$$\eta_X(x) = {\downarrow}x = \{z \in X \mid z \leq x\}. \qquad x \in X.$$

- C.J. Mikkelsen 1976:

  A preordered set $(X, \leq)$ is *join-complete* (i.e. every subset has a join) if an only if $X \xrightarrow{\eta_X} \mathrm{Dwn}(X)$ has a left adjoint map $\mathrm{Dwn}(X) \xrightarrow{\sup} X$.

Join-completeness

Let $(X, \leq)$ be a preordered set. A subset $A \subseteq X$ is *downclosed* if $z \leq x$ and $x \in A$ implies $z \in A$.

- $\mathrm{Dwn}(X)$ is the set of all downclosed subsets of X ordered by set-inclusion.
- There is an isotone map $X \xrightarrow{\eta_X} \mathrm{Dwn}(X)$ defined by

$$\eta_X(x) = {\downarrow}x = \{z \in X \mid z \leq x\}. \qquad x \in X.$$

- C.J. Mikkelsen 1976:

  A preordered set $(X, \leq)$ is *join-complete* (i.e. every subset has a join) if an only if $X \xrightarrow{\eta_X} \mathrm{Dwn}(X)$ has a left adjoint map $\mathrm{Dwn}(X) \xrightarrow{\sup} X$.

  The proof requires the Axiom of Choice.

Join-completeness

Let $(X, \leq)$ be a preordered set. A subset $A \subseteq X$ is *downclosed* if $z \leq x$ and $x \in A$ implies $z \in A$.

- $\mathsf{Dwn}(X)$ is the set of all downclosed subsets of X ordered by set-inclusion.
- There is an isotone map $X \xrightarrow{\eta_X} \mathsf{Dwn}(X)$ defined by

$$\eta_X(x) = \downarrow x = \{z \in X \mid z \leq x\}. \qquad x \in X.$$

- C.J. Mikkelsen 1976:

  A preordered set $(X, \leq)$ is *join-complete* (i.e. every subset has a join) if an only if $X \xrightarrow{\eta_X} \mathsf{Dwn}(X)$ has a left adjoint map $\mathsf{Dwn}(X) \xrightarrow{\sup} X$.

  The proof requires the Axiom of Choice.
- Replacement of $\mathbf{2}$ by the unital commutative quantale $([0,1], *)$.

Enriched join-completeness based on $([0,1],*)$

A [0, 1]-*preorder* on a set X is a map $X \times X \xrightarrow{p} [0,1]$ satisfying the conditions:

- $1 = p(x,x), \quad x \in X,$
- $p(x,y) + p(y,z) - 1 \leq p(x,z), \quad x,y,z \in X.$

Enriched join-completeness based on $([0,1], *)$

A *[0, 1]-preorder* on a set X is a map $X \times X \xrightarrow{p} [0,1]$ satisfying the conditions:

- $1 = p(x,x), \quad x \in X,$
- $p(x,y) + p(y,z) - 1 \leq p(x,z), \quad x,y,z \in X.$

- A $[0,1]$-preorder is *antisymmetric* if the following property holds:

$$1 = p(x,y) \wedge p(y,x)) \quad \Rightarrow \quad x = y.$$

Enriched join-completeness based on $([0,1],*)$

A [0, 1]-*preorder* on a set X is a map $X \times X \xrightarrow{p} [0,1]$ satisfying the conditions:

- $1 = p(x,x), \quad x \in X,$
- $p(x,y) + p(y,z) - 1 \leq p(x,z), \quad x,y,z \in X.$

- A $[0,1]$-preorder is *antisymmetric* if the following property holds:

$$1 = p(x,y) \wedge p(y,x)) \quad \Rightarrow \quad x = y.$$

- Let (X,p) be a $[0,1]$-preordered set. A map $X \xrightarrow{f} [0,1]$ is a [0, 1]-enriched presheaf if f is *left-extensional* w.r.t. p — i.e.

$$p(z,x) + f(x) - 1 \leq f(z), \quad x,z \in X.$$

Enriched join-completeness based on $([0,1], *)$

A [0, 1]-*preorder* on a set X is a map $X \times X \xrightarrow{p} [0,1]$ satisfying the conditions:

- $1 = p(x,x), \quad x \in X,$
- $p(x,y) + p(y,z) - 1 \le p(x,z), \quad x, y, z \in X.$

- A [0, 1]-preorder is *antisymmetric* if the following property holds:

$$1 = p(x,y) \wedge p(y,x)) \quad \Rightarrow \quad x = y.$$

- Let (X, p) be a [0, 1]-preordered set. A map $X \xrightarrow{f} [0,1]$ is a [0, 1]-enriched presheaf if f is *left-extensional* w.r.t. p — i.e.

$$p(z,x) + f(x) - 1 \le f(z), \quad x, z \in X.$$

- The [0, 1]-preorder d on the set $\mathbb{P}(X,p)$ of all [0, 1]-enriched presheaves on $(X.p)$ is given by:

$$d(f,g) = \inf_{x \in X} \min(1 - f(x) + g(x), 1), \quad f, g \in \mathbb{P}(X,p).$$

$[0,1]$-Enriched join-completeness

The $[0,1]$-enriched Yoneda embedding $(X,p) \xrightarrow{\eta_{(X,p)}} \mathbb{P}(X,p)$ is given by:

$$\big(\eta_{(X,p)}(x)\big)(z) = p(z,x), \qquad x, y \in X.$$

$[0,1]$-Enriched join-completeness

The $[0,1]$-enriched Yoneda embedding $(X,p) \xrightarrow{\eta_{(X,p)}} \mathbb{P}(X,p)$ is given by:

$$\big(\eta_{(X,p)}(x)\big)(z) = p(z,x), \qquad x,y \in X.$$

- (X,p) is $[0,1]$-enriched join-complete if and only if $\eta_{(X,p)}$ has a left adjoint map $\mathbb{P}(X,p) \xrightarrow{\sup_{(X,p)}} X$ — i.e.

$$p(\sup_{(X,p)}(f), y) = d\big(f, p(\_, z)\big), \quad y \in X,\ f \in \mathbb{P}(X,p).$$

[0, 1]-Enriched join-completeness

The [0, 1]-enriched Yoneda embedding $(X, p) \xrightarrow{\eta_{(X,p)}} \mathbb{P}(X, p)$ is given by:

$$\big(\eta_{(X,p)}(x)\big)(z) = p(z, x), \qquad x, y \in X.$$

- (X, p) is [0, 1]-enriched join-complete if and only if $\eta_{(X,p)}$ has a left adjoint map $\mathbb{P}(X, p) \xrightarrow{\sup_{(X,p)}} X$ — i.e.

  $$p(\sup_{(X,p)}(f), y) = d\big(f, p(\_, z)\big), \quad y \in X,\ f \in \mathbb{P}(X, p).$$

- **Example**. The space $\big(\mathbb{P}(X, p), d\big)$ is [0, 1]-enriched join-complete. The left adjoint map $\sup_{\mathbb{P}(X,p)}$ is given by:

  $$\Big(\sup_{\big(\mathbb{P}(X,p),d\big)}(F)\Big)(x) = \sup_{f \in \mathbb{P}(X,p)} \max(f(x) + F(f) - 1, 0)$$

  where $F \in \mathbb{P}\big(\mathbb{P}(X, p), d\big)$ and $x \in X$.

A $[0,1]$-enriched join-complete lattice is a right $[0,1]$-module X in $\mathtt{Sup}$.

A $[0,1]$-enriched join-complete lattice is a right $[0,1]$-module X in Sup.

- Due to the universal property of the tensor product a right action on a complete lattice X can always be identified with a map $X \times [0,1] \xrightarrow{\boxdot} X$, which is join-preserving in each variable separately and satisfies the following conditions for all $x \in X$, $\alpha, \beta \in [0,1]$:

$$x \boxdot 1 = x \quad \text{and} \quad (x \boxdot \alpha) \boxdot \beta = x \boxdot \big(\max(\alpha + \beta - 1, 0)\big), \quad .$$

A $[0,1]$-enriched join-complete lattice is a right $[0,1]$-module X in Sup.

- Due to the universal property of the tensor product a right action on a complete lattice X can always be identified with a map $X \times [0,1] \xrightarrow{\boxdot} X$, which is join-preserving in each variable separately and satisfies the following conditions for all $x \in X$, $\alpha, \beta \in [0,1]$:

$$x \boxdot 1 = x \quad \text{and} \quad (x \boxdot \alpha) \boxdot \beta = x \boxdot \big(\max(\alpha + \beta - 1, 0)\big), \quad .$$

- If $(X, \boxdot)$ is a right $[0,1]$-module, then the corresponding intrinsic $[0,1]$-preorder p and the formation of $[0,1]$-enriched joins are given by:

$$p(x,y) = \sup\{\alpha \in [0,1] \mid x \boxdot \alpha \le y\}, \quad x, y \in X,$$

$$\sup_{(X,p)}(f) = \bigvee_{x \in X} x \boxdot f(x), \quad f \in \mathbb{P}(X,p).$$

$[0,+\infty]$ as a right $[0,1]$-module

Let $\boxdot$ be a right action on $[0,+\infty]$ with the corresponding $[0,1]$-preorder p.

$[0, +\infty]$ as a right $[0, 1]$-module

Let $\boxdot$ be a right action on $[0, +\infty]$ with the corresponding $[0, 1]$-preorder p.

- Every set X is understood as the set provided with its discrete $[0, 1]$-preorder p_0 — i.e.

$$p_0(x, x) = 1 \quad \text{and} \quad p(x_1, x_2) = 0 \quad \text{for} \quad x_1 \neq x_2.$$

$[0, +\infty]$ as a right $[0, 1]$-module

Let $\boxdot$ be a right action on $[0, +\infty]$ with the corresponding $[0, 1]$-preorder p.

- Every set X is understood as the set provided with its discrete $[0, 1]$-preorder p_0 — i.e.

$$p_0(x, x) = 1 \quad \text{and} \quad p(x_1, x_2) = 0 \quad \text{for} \quad x_1 \neq x_2.$$

- Hence every fuzzy subset $X \xrightarrow{f} [0, 1]$ of X is a $[0, 1]$-enriched presheaf on (X, p_0).

$[0, +\infty]$ as a right $[0, 1]$-module

Let $\boxdot$ be a right action on $[0, +\infty]$ with the corresponding $[0, 1]$-preorder p.

- Every set X is understood as the set provided with its discrete $[0, 1]$-preorder p_0 — i.e.

$$p_0(x, x) = 1 \quad \text{and} \quad p(x_1, x_2) = 0 \quad \text{for} \quad x_1 \neq x_2.$$

- Hence every fuzzy subset $X \xrightarrow{f} [0, 1]$ of X is a $[0, 1]$-enriched presheaf on (X, p_0).
- The image $g(f)$ of a fuzzy subset f of X under a map $X \xrightarrow{g} [0, +\infty]$ in the sense of the monad of $[0, 1]$-enriched presheaves is given by:

$$g(f)(z) = \sup\{\max(p(z, g(x)) + f(x), 0) \mid x \in X\}, \qquad z \in [0, +\infty].$$

- The $[0,1]$-enriched join of $g(f)$ is given by:

$$\begin{aligned}\sup_{([0,+\infty],p)}\big(g(f)\big) &= \bigvee_{z\in[0,+\infty]} z \boxdot g(f)(z) \\ &= \bigvee_{x\in X} g(x) \boxdot f(x)\end{aligned}.$$

- The $[0,1]$-enriched join of $g(f)$ is given by:

$$\sup_{([0,+\infty],p)}(g(f)) = \bigvee_{z\in[0,+\infty]} z \boxdot g(f)(z)$$
$$= \bigvee_{x\in X} g(x) \boxdot f(x) .$$

- The previous expression is the solution (i.e. optimum) of the stochastic linear programming problem (SLPP).

- The $[0,1]$-enriched join of $g(f)$ is given by:

$$\sup_{([0,+\infty],p)}(g(f)) = \bigvee_{z\in[0,+\infty]} z \boxdot g(f)(z) = \bigvee_{x\in X} g(x) \boxdot f(x).$$

- The previous expression is the solution (i.e. optimum) of the stochastic linear programming problem (SLPP).
- The solution of SLPP depends on the chosen right action on the extended non-negative real line.

- The $[0,1]$-enriched join of $g(f)$ is given by:

$$\sup_{([0,+\infty],p)}(g(f)) = \bigvee_{z\in[0,+\infty]} z \boxdot g(f)(z)$$
$$= \bigvee_{x\in X} g(x) \boxdot f(x)$$

- The previous expression is the solution (i.e. optimum) of the stochastic linear programming problem (SLPP).
- The solution of SLPP depends on the chosen right action on the extended non-negative real line.
- As an illustration we choose now different right actions $\boxdot$ on $[0,+\infty]$.

Different $[0,1]$-enriched versions of suprema for SLPP

(A) Let $[0,+\infty] \times [0,1] \xrightarrow{\boxdot} [0,+\infty]$ be the trivial right action:

$$z \boxdot \alpha = \begin{cases} z, & \alpha = 1, \\ 0, & z \neq 1. \end{cases}$$

Different $[0,1]$-enriched versions of suprema for SLPP

(A) Let $[0,+\infty] \times [0,1] \xrightarrow{\boxdot} [0,+\infty]$ be the trivial right action:

$$z \boxdot \alpha = \begin{cases} z, & \alpha = 1, \\ 0, & z \neq 1. \end{cases}$$

- Then the $[0,1]$-preorder p on $[0,+\infty]$ is the characteristic function of the usual order on $[0,+\infty]$ and

$$\sup_{([0,+\infty],p)}(g(f)) = \sup\{g(x) \mid 1 = f(x)\}.$$

Different $[0,1]$-enriched versions of suprema for SLPP

(A) Let $[0,+\infty]\times[0,1]\xrightarrow{\boxdot}[0,+\infty]$ be the trivial right action:

$$z\boxdot\alpha=\begin{cases} z, & \alpha=1,\\ 0, & z\neq 1.\end{cases}$$

- Then the $[0,1]$-preorder p on $[0,+\infty]$ is the characteristic function of the usual order on $[0,+\infty]$ and

$$\sup_{([0,+\infty],p)}(g(f))=\sup\{g(x)\mid 1=f(x)\}.$$

- Hence the previous formula is the deterministic solution of the stochastic linear programming problem.

(B) The right action $\boxdot$ on $[0, +\infty]$ is induced by the Łukasiewicz implication.

(B) The right action $\boxdot$ on $[0, +\infty]$ is induced by the Łukasiewicz implication.

- Łukasiewicz implication is a right action on $[0,1]^{op}$ — i.e. $[0,1]$ is provided with its dual order and

$$x \boxdot \alpha = \alpha \to x = \min(1 - \alpha + x, 1), \quad x, \alpha \in [0,1]$$

In fact: $x \boxdot 1 = x, \quad (x \boxdot \alpha) \boxdot \beta = x \boxdot (\max(\alpha + \beta - 1, 0)).$

(B) The right action $\boxdot$ on $[0, +\infty]$ is induced by the Łukasiewicz implication.

- Łukasiewicz implication is a right action on $[0,1]^{op}$ — i.e. $[0,1]$ is provided with its dual order and

$$x \boxdot \alpha = \alpha \to x = \min(1 - \alpha + x, 1), \quad x, \alpha \in [0,1]$$

In fact: $x \boxdot 1 = x, \quad (x \boxdot \alpha) \boxdot \beta = x \boxdot (\max(\alpha + \beta - 1, 0))$.

- $z \longmapsto \exp(-z)$ is an order-isomorphism from $[0, +\infty]$ to $[0,1]^{op}$.

(B) The right action $\boxdot$ on $[0,+\infty]$ is induced by the Łukasiewicz implication.

- Łukasiewicz implication is a right action on $[0,1]^{op}$ — i.e. $[0,1]$ is provided with its dual order and

$$x \boxdot \alpha = \alpha \to x = \min(1-\alpha+x,1), \quad x,\alpha \in [0,1]$$

In fact: $x \boxdot 1 = x, \quad (x \boxdot \alpha) \boxdot \beta = x \boxdot (\max(\alpha+\beta-1,0))$.

- $z \longmapsto \exp(-z)$ is an order-isomorphism from $[0,+\infty]$ to $[0,1]^{op}$.
- Hence the right action $\boxdot$ on $[0,+\infty]$ has the form:

$$z \boxdot \alpha = -\ln\big(\min(1-\alpha+\exp(-z),1)\big), \qquad z \in [0,+\infty],\ \alpha \in [0,1].$$

(B) The right action $\boxdot$ on $[0, +\infty]$ is induced by the Łukasiewicz implication.

- Łukasiewicz implication is a right action on $[0, 1]^{op}$ — i.e. $[0, 1]$ is provided with its dual order and

$$x \boxdot \alpha = \alpha \to x = \min(1 - \alpha + x, 1), \quad x, \alpha \in [0, 1]$$

In fact: $x \boxdot 1 = x, \quad (x \boxdot \alpha) \boxdot \beta = x \boxdot (\max(\alpha + \beta - 1, 0))$.

- $z \longmapsto \exp(-z)$ is an order-isomorphism from $[0, +\infty]$ to $[0, 1]^{op}$.
- Hence the right action $\boxdot$ on $[0, +\infty]$ has the form:

$$z \boxdot \alpha = -\ln\big(\min(1 - \alpha + \exp(-z), 1)\big), \qquad z \in [0, +\infty], \ \alpha \in [0, 1].$$

- The corresponding $[0, 1]$-preorder on $[0, +\infty]$ is given by:

$$p(z_1, z_2) = \min(1 - \exp(-z_2) + \exp(-z_1), 1), \qquad z_1, z_2 \in [0, +\infty].$$

A non-deterministic solution of SLPP

$$\sup_{([0,+\infty],p)}(g(f)) = \sup_{x \in X} -\ln\big((1 - f(x) + \exp(-g(x)), 1)\big).$$

A non-deterministic solution of SLPP

$$\sup_{([0,+\infty],p)}(g(f)) = \sup_{x\in X} -\ln\big((1 - f(x) + \exp(-g(x)), 1)\big).$$

- The expression $1 - f(x)$ is the probability that x violates the constraints.

A non-deterministic solution of SLPP

$$\sup_{([0,+\infty],p)}(g(f)) = \sup_{x\in X} -\ln\big((1 - f(x) + \exp(-g(x)), 1)\big).$$

- The expression $1 - f(x)$ is the probability that x violates the constraints.
- It is interesting to see how this value enters the construction of the $[0,1]$-enriched supremum of the stochastic linear programming problem.

A non-deterministic solution of SLPP

$$\sup_{([0,+\infty],p)}(g(f)) = \sup_{x\in X} -\ln\big((1 - f(x) + \exp(-g(x)), 1)\big).$$

- The expression $1 - f(x)$ is the probability that x violates the constraints.
- It is interesting to see how this value enters the construction of the $[0,1]$-enriched supremum of the stochastic linear programming problem.
- This value can be seen as costs caused by some kind of penalty strategy.

A non-deterministic solution of SLPP

$$\sup_{([0,+\infty],p)}(g(f)) = \sup_{x \in X} -\ln\big((1 - f(x) + \exp(-g(x)), 1)\big).$$

- The expression $1 - f(x)$ is the probability that x violates the constraints.
- It is interesting to see how this value enters the construction of the $[0, 1]$-enriched supremum of the stochastic linear programming problem.
- This value can be seen as costs caused by some kind of penalty strategy.
- The real number of the non-deterministic solution is larger than the result related to the deterministic supremum in (A).

A non-deterministic solution of SLPP

$$\sup_{([0,+\infty],p)}(g(f)) = \sup_{x\in X} -\ln\big((1-f(x)+\exp(-g(x)),1)\big).$$

- The expression $1 - f(x)$ is the probability that x violates the constraints.
- It is interesting to see how this value enters the construction of the $[0,1]$-enriched supremum of the stochastic linear programming problem.
- This value can be seen as costs caused by some kind of penalty strategy.
- The real number of the non-deterministic solution is larger than the result related to the deterministic supremum in (A).
- Non-deterministic solutions depend obviously on an order isomorphism between $[0,+\infty] \to [0,1]^{op}$ and seem to play an interesting role in stochastic linear programming.

Result:

Result:

Right actions on $[0, +\infty]$ and therewith enriched suprema play a significant role in the construction of solutions of the stochastic linear programming problem.