Preservation of Projective Right Modules in Sup under Duality

Ulrich Höhle

Brno, October 7, 2019

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The $\mathfrak{Q}$-valued totally below relation

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Hence $\lhd(\_, m)$ is a contravariant $\mathfrak{Q}$-presheaf on (M, p) and the correspondence $m \longmapsto \lhd(\_, m)$ is a $\mathfrak{Q}$-homomorphism $(M, p) \to \mathbb{P}(M, p)$.

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- Further for all $f \in \mathbb{P}(M, p)$ the definition of $\lhd$ implies:

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- Hence $m \longmapsto \lhd(\_, m)$ is left adjoint to $\sup_M$ if and only if

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- Further, it is well know that a complete lattice is completely distributive if and only if $m \longmapsto \{n \in M \mid n \lhd m\}$ is left adjoint to sup (see Raney 1953).

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- **Questions**. Do $\mathfrak{Q}$-enriched completely distributive right $\mathfrak{Q}$-modules exist?

Projective right $\mathfrak{Q}$-modules

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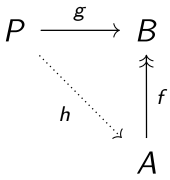

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$$\begin{array}{ccc} P & \xrightarrow{g} & B \\ & \overset{h}{\searrow} & \uparrow f \\ & & A \end{array}$$

- A retract of a projective object in $\mathtt{Mod}_r(\mathfrak{Q})$ is again projective.

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- Hence $f \circ h \circ \eta_{(X,p)} = g \circ \eta_{(X,p)}$. Since the extension to a free right $\mathfrak{Q}$-module is unique, the relation $f \circ h = g$ follows. Q.E.D.

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- ((ii) $\Rightarrow$ (i)) follows from the previous Lemma.

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- Then $(M^{op}, \boxdot^{op})$ is a right $\mathfrak{Q}$-module and is called the dual right $\mathfrak{Q}$-module of M. The intrinsic $\mathfrak{Q}$-preorder of M^{op} coincides with the dual $\mathfrak{Q}$-preorder p^{op}.

Selfduality in $\mathtt{Mod}_r(\mathfrak{Q})$

The object function $M \longmapsto M^{op}$ can be extended to a contravariant endofunctor $\mathcal{S}$ of $\mathtt{Mod}_r(\mathfrak{Q})$ as follows:

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Main Theorem 2 (Gutiérrez García, Hö, Kubiak 2019)

If the involutive and unital quantale $\mathfrak{Q}$ has a dualizing element, then the self-duality preserves the projectivity — i.e. if M is projective, then M^{op} is projective.

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- On the dual lattice $\left(\mathbb{P}^{\dagger}(X,p)\right)^{op}$ of $\mathbb{P}^{\dagger}(X,p)$ we introduce a right action $\boxdot$, which is determined as follows:

$$(f \boxdot \alpha)(x) = \alpha \searrow f(x), \qquad x \in X,\ \alpha \in \mathfrak{Q},\ f \in \mathbb{P}^{\dagger}(X,p).$$

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- **Theorem** 1. If $\mathfrak{Q}$ has a dualizing element, then $\mathbb{P}^{\dagger}(X,p)^{op}$ is projective.

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- $\mathfrak{Q}$-enriched join of any contravariant $\mathfrak{Q}$-presheaf F on $(\mathbb{P}^{\dagger}(X, p), d^{\dagger})$ can be expressed as follows:

$$
\begin{aligned}
\big(\sup_{(\mathbb{P}^{\dagger}(X,p),d^{\dagger})}(F)\big)(x) &= \overset{op}{\bigvee_{f \in \mathbb{P}^{\dagger}(X,p)}} \big(f \boxdot F(f)\big)(x) \\
&= \bigwedge_{f \in \mathbb{P}^{\dagger}(X,p)} \big(F(f) \searrow f(x)\big) \\
&= \bigwedge_{f \in \mathbb{P}^{\dagger}(X,p)} \Big(\big((\delta \swarrow f(x)) * F(f)\big) \searrow \delta\Big) \\
&= \Big(\bigvee_{f \in \mathbb{P}^{\dagger}(X,p)} \big(d^{\dagger}(\overline{x}, f) * F(f)\big)\Big) \searrow \delta \\
&= F(\overline{x}) \searrow \delta.
\end{aligned}
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- A $\mathfrak{Q}$-homomorphism $(\mathbb{P}^{\dagger}(X,p), d^{\dagger}) \xrightarrow{\psi} \big(\mathbb{P}(\mathbb{P}^{\dagger}(X,p), d^{\dagger}), d\big)$ is defined by:

$$\big(\psi(f)\big)(g) = \bigvee_{x \in X} (d^{\dagger}(g,\overline{x}) * d^{\dagger}(\overline{x}, f)), \qquad f, g \in \mathbb{P}^{\dagger}(X,p).$$

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- M^{op} is a retraction of a projective object (cf. Theorem 1, Theorem 2), and hence also projective.

The next theorem is an improvement of a result obtained by H. Lai and L. Shen 2018

Theorem 3. Let $\mathfrak{Q}$ be an involutive and integral quantale. The self-duality in $\mathtt{Mod}_r(\mathfrak{Q})$ preserves projective objects of $\mathtt{Mod}_r(\mathfrak{Q})$ if and only if $\mathfrak{Q}$ has a dualizing element, which is necessarily the bottom element of $\mathfrak{Q}$.

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- Since the right $\mathfrak{Q}$-module $\mathfrak{Q}$ is projective, $\mathfrak{Q}^{op}$ must be projective and hence the isomorphic right $\mathfrak{Q}$-module $(Q^{op}, \boxdot)$ with $\beta \boxdot \alpha = \alpha \searrow \beta$ must also be projective.

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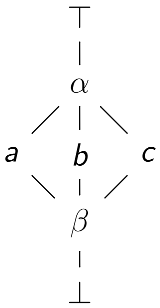
- $(\perp \swarrow \alpha) \searrow \perp) \leq \alpha$. Since $\mathfrak{Q}$ is involutive, $\perp$ is dualizing.

Example of an integral, involutive Frobenius quantale

- The Hasse diagram and the multiplication table of a quantale consisting of 7 elements is given by:



$*$	$\bot$	β	a	b	c	α	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
β	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	β
a	$\bot$	$\bot$	β	$\bot$	β	β	a
b	$\bot$	$\bot$	β	β	$\bot$	β	b
c	$\bot$	$\bot$	$\bot$	β	β	β	c
α	$\bot$	$\bot$	β	β	β	β	α
$\top$	$\bot$	β	a	b	c	α	$\top$

The involution is determined by:

$\top' = \top, \quad \alpha' = \alpha, \quad a' = a, \quad b' = c, \quad c' = b, \quad \beta' = \beta, \quad \bot' = \bot.$

$\bot$ is not cyclic, because $b * c = \bot$, but $c * b = \beta$.

Beyond integral quantales

Let $\mathfrak{Q}$ be an unital and involutive quantale and $\mathfrak{Q}^{op}$ be the right $\mathfrak{Q}$-module provided with the right action determined by:

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- **Theorem** 4. Let $\mathfrak{Q}$ be a unital and involutive quantale with a designated element. The duality in $\mathtt{Mod}_r(\mathfrak{Q})$ preserves projectivity if and only if every designated element of $\mathfrak{Q}$ is dualizing.

Examples

Let $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ be the two integral quantales on $C_3 = \{\bot, a, \top\}$. On $\mathfrak{Q}_1$ we use the binary minimum and on $\mathfrak{Q}_2$ the multiplication of the three-valued *MV*-algebra.

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$\mathfrak{Q}_3 \quad =$

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- The element $(\bot, a)$ of $\mathfrak{Q}_1 \times \mathfrak{Q}_3$ is designated, but not dualizing. Hence the duality in $\mathtt{Mod}_r(\mathfrak{Q}_1 \times \mathfrak{Q}_3)$ does not preserve projectivity.

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- The element $(\bot, a)$ of $\mathfrak{Q}_2 \times \mathfrak{Q}_3$ is designated, but dualizing. Hence the duality in $\mathtt{Mod}_r(\mathfrak{Q}_2 \times \mathfrak{Q}_3)$ preserves projectivity.