Topolgization of Semi-Unital Quantales with Applications to C^*-Algebras

Ulrich Höhle

Bergische Universität, Wuppertal, Germany

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- $\mathtt{Sup}$ is the categorical framework for traditional topologies.
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- If $\mathcal{T}$ is a $\mathfrak{Q}$-enriched topology on X, then $(X, \mathcal{T})$ is called a $\mathfrak{Q}$-topological space.

A map $(X, \mathcal{T}) \xrightarrow{\varphi} (Y, \mathcal{S})$ is $\mathfrak{Q}$-continuous if $g \circ \varphi \in \mathcal{T}$ for all $g \in \mathcal{S}$.

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The category $\mathtt{Top}(\mathfrak{Q})$ of $\mathfrak{Q}$-topological spaces and $\mathfrak{Q}$-continuous maps is topological over $\mathtt{Set}$.

- The indiscrete $\mathfrak{Q}$-topology is given by all constant maps taking their value in the subquantale $\mathbb{L}(\mathfrak{Q})$ of all left-sided elements $\mathfrak{Q}$. Hence $\mathtt{Top}(\mathfrak{Q})$ is in general not well-fibred.

Prime elements of quantales

Let $(\mathfrak{Q}, *)$ be a not necessarily unital quantale. An element $p \in \mathfrak{Q}$ is prime, if $p \neq \top$ and the following implication holds:

$$\alpha * \beta \le p \quad \Rightarrow \quad (\alpha * \top \le p \quad \text{or} \quad \top * \beta \le p).$$

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- A quantale $\mathfrak{Q}$ is semi-integral, if $\alpha * \top * \beta \leq \alpha * \beta$ for all $\alpha, \beta \in \mathfrak{Q}$.
- If $\mathfrak{Q}$ is semi-integral, then maximal left-sided (right-sided) elements are prime.

Representation of left-sided and idempotent quantales

- Let $C_3 = \{\bot, a, \top\}$ be the three-chain and $C_3^\ell = (C_3, *_\ell)$ be the leftsided, idempotent and non-commutative quantale. The quantale multiplication $*_\ell$ is uniquely determined by:

$$a *_\ell = a, \quad \top *_\ell a = a, \quad a *_\ell \top = \top.$$

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If $\mathfrak{Q} \xrightarrow{h} C_3^\ell$ is a strong quantale homomorphism, then

$$p = \bigvee \{\alpha \in \mathfrak{Q} \mid h(\alpha) \leq a\}$$

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- If $\mathfrak{Q}$ is left-sided and idempotent and p is a prime element of $\mathfrak{Q}$, then there exists a unique strong quantale homomorphism $\mathfrak{Q} \xrightarrow{h_p} C_3^\ell$ such that

$$p = \bigvee\{\alpha \in \mathfrak{Q} \mid h_p(\alpha) \leq a\}.$$

Construction.

$$h_p(\alpha) = \left\{ \begin{array}{rl} \bot, & \alpha * \top \leq p \\ a, & \alpha \leq p, \alpha * \top \nleq p, \\ \top, & \alpha \nleq p. \end{array} \right\}, \qquad \alpha \in \mathfrak{Q}.$$

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  Theorem. A left-sided and idempotent quantale $\mathfrak{Q}$ is spatial if and only if strong quantale homomorphisms $\mathfrak{Q} \to C_3^\ell$ separate points in $\mathfrak{Q}$.

The topologization of the spectrum

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- $\bigvee_{i \in I} \mathbb{A}_{\alpha_i} = \mathbb{A}_{\bigvee_{i \in I} \alpha_i}$,
- $\mathbb{A}_\top = \underline{\top}$.

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  $\widehat{C_3^\ell} = C_3^\ell \times \{0, 1\}$ and we use the following notation:

$$\alpha = (\alpha, 0), \quad \widehat{\alpha} = (\alpha, 1), \quad e = (\perp, 1), \quad \widehat{\alpha} = \alpha \vee e.$$

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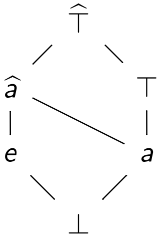
$$\alpha = (\alpha, 0), \quad \widehat{\alpha} = (\alpha, 1), \quad e = (\bot, 1), \quad \widehat{\alpha} = \alpha \vee e.$$

- The Hasse diagram and the multiplication table are given by:



$*$	$\bot$	a	$\top$	e	$\widehat{a}$	$\widehat{\top}$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\top$	a	a	$\top$
$\top$	$\bot$	a	$\top$	$\top$	$\top$	$\top$
e	$\bot$	a	$\top$	e	$\widehat{a}$	$\widehat{\top}$
$\widehat{a}$	$\bot$	a	$\top$	$\widehat{a}$	$\widehat{a}$	$\widehat{\top}$
$\widehat{\top}$	$\bot$	a	$\top$	$\widehat{\top}$	$\widehat{\top}$	$\widehat{\top}$

It follows from the multiplication table of the quantale multiplication $*$ that

$$\mathcal{T}_{\mathfrak{Q}} = \{\mathbb{A}_\alpha \vee (\mathbb{A}_\beta *_\ell a) \mid \alpha, \beta \in \mathfrak{Q}\} \cup \{\widehat{\underline{\top}}\}$$

is a $\widehat{C_3^\ell}$-enriched topology on the spectrum $\sigma(\mathfrak{Q})$ of $\mathfrak{Q}$.

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- $\{\mathbb{A}_\alpha \mid \alpha \in \mathfrak{Q}\} \cup \{\widehat{\underline{\top}}\}$ is a subbase of $\mathcal{T}_{\mathfrak{Q}}$.

- The previous construction leads to a topologization of the quantale of all closed left ideals of a non-commutative C^*-algebra.

The general case of semi-unital quantales

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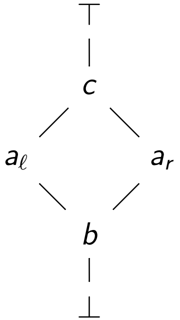
  One is the left-sided non-commutative and idempotent quantale C_3^ℓ, and the other one is the right-sided non-commutative and idempotent quantale C_3^r.

- Their tensor product $\mathfrak{Q}_\mathbf{2} = C_3^\ell \otimes C_3^r$ is the coproduct of C_3^ℓ and C_3^r in the category of balanced and bisymmetric quantales with strong quantale homomorphisms. Therefore $\mathfrak{Q}_\mathbf{2}$ is also called the quantization of $\mathbf{2}$.

The Hasse diagram and the multiplication table of the quantization of **2** are given by:



$*$	$\bot$	b	a_ℓ	a_r	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
b	$\bot$	b	b	a_r	a_r	a_r
a_ℓ	$\bot$	a_ℓ	a_ℓ	$\top$	$\top$	$\top$
a_r	$\bot$	b	b	a_r	a_r	a_r
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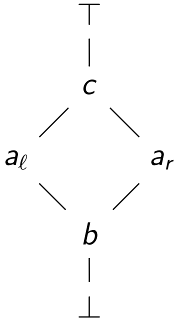

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a_r	$\bot$	b	b	a_r	a_r	a_r
c	$\bot$	a_ℓ	a_ℓ	$\top$	$\top$	$\top$
$\top$	$\bot$	a_ℓ	a_ℓ	$\top$	$\top$	$\top$

- Every prime element p of a semi-unital quantale $\mathfrak{Q}$ can be identified with a strong quantale homomorphism $\mathfrak{Q} \xrightarrow{h_p} \mathfrak{Q}_{\mathbf{2}}$ satisfying the condition:

$$p = \bigvee \{\alpha \in \mathfrak{Q} \mid h_p(\alpha) \leq c\}$$

Construction.

$$h_p(\alpha) = \begin{cases} \bot, & \top * \alpha * \top \leq p, \\ b, & \top * \alpha * \top \nleq p,\ \alpha * \top \leq p \text{ and } \top * \alpha \leq p, \\ a_\ell, & \alpha * \top \nleq p \text{ and } \top * \alpha \leq p, \\ a_r, & \alpha * \top \leq p \text{ and } \top * \alpha \nleq p, \\ c, & \alpha \leq p,\ \alpha * \top \nleq p \text{ and } \top * \alpha \nleq p, \\ \top, & \alpha \nleq p. \end{cases}$$

where $\alpha \in \mathfrak{Q}$.

Construction.

$$h_p(\alpha) = \begin{cases} \bot, & \top * \alpha * \top \le p, \\ b, & \top * \alpha * \top \not\le p,\ \alpha * \top \le p \text{ and } \top * \alpha \le p, \\ a_\ell, & \alpha * \top \not\le p \text{ and } \top * \alpha \le p, \\ a_r, & \alpha * \top \le p \text{ and } \top * \alpha \not\le p, \\ c, & \alpha \le p,\ \alpha * \top \not\le p \text{ and } \top * \alpha \not\le p, \\ \top, & \alpha \not\le p. \end{cases}$$

where $\alpha \in \mathfrak{Q}$.

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If $\mathfrak{Q}$ is left-sided, then the construction reduces to the previous construction in the case of left-sided and idempotent quantales.

- **Theorem**. A semi-unital quantale $\mathfrak{Q}$ is spatial if and only if strong homomorphisms $\mathfrak{Q} \to \mathfrak{Q}_2$ separate points in $\mathfrak{Q}$.

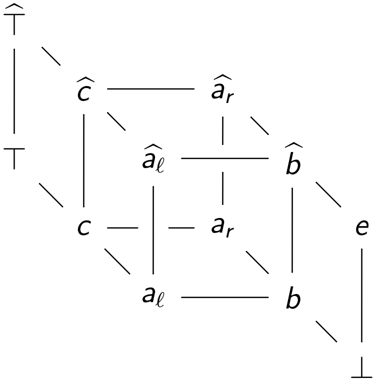
The topologization of the spectrum of spatial semi-unital quantales

Unitalization of $\widehat{\mathfrak{Q}_2}$ **=the quantization of 2** is given by:

The topologization of the spectrum of spatial semi-unital quantales

Unitalization of $\widehat{\mathfrak{Q}_2}$ =the quantization of 2 is given by:

- The Hasse diagram of $\widehat{\mathfrak{Q}_2}$:



and the multiplication table of $\widehat{\mathfrak{Q}_2}$:

$\widehat{*}$	$\bot$	b	a_ℓ	a_r	c	$\top$	e	$\widehat{b}$	$\widehat{a_\ell}$	$\widehat{a_r}$	$\widehat{c}$	$\widehat{\top}$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
b	$\bot$	b	b	a_r	a_r	a_r	b	b	b	a_r	a_r	a_r
a_ℓ	$\bot$	a_ℓ	a_ℓ	$\top$	$\top$	$\top$	a_ℓ	a_ℓ	a_ℓ	$\top$	$\top$	$\top$
a_r	$\bot$	b	b	a_r	a_r	a_r	a_r	a_r	a_r	a_r	a_r	a_r
c	$\bot$	a_ℓ	a_ℓ	$\top$	$\top$	$\top$	c	c	c	$\top$	$\top$	$\top$
$\top$	$\bot$	a_ℓ	a_ℓ	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
e	$\bot$	b	a_ℓ	a_r	c	$\top$	e	$\widehat{b}$	$\widehat{a_\ell}$	$\widehat{a_r}$	$\widehat{c}$	$\widehat{\top}$
$\widehat{b}$	$\bot$	b	a_ℓ	a_r	a	$\top$	$\widehat{b}$	$\widehat{b}$	$\widehat{a_\ell}$	$\widehat{a_r}$	$\widehat{c}$	$\widehat{\top}$
$\widehat{a_\ell}$	$\bot$	a_ℓ	a_ℓ	$\top$	$\top$	$\top$	$\widehat{a_\ell}$	$\widehat{a_\ell}$	$\widehat{a_\ell}$	$\widehat{\top}$	$\widehat{\top}$	$\widehat{\top}$
$\widehat{a_r}$	$\bot$	b	a_ℓ	a_r	c	$\top$	$\widehat{a_r}$	$\widehat{a_r}$	$\widehat{c}$	$\widehat{a_r}$	$\widehat{c}$	$\widehat{\top}$
$\widehat{c}$	$\bot$	a_ℓ	a_ℓ	$\top$	$\top$	$\top$	$\widehat{c}$	$\widehat{c}$	$\widehat{c}$	$\widehat{\top}$	$\widehat{\top}$	$\widehat{\top}$
$\widehat{\top}$	$\bot$	a_ℓ	a_ℓ	$\top$	$\top$	$\top$	$\widehat{\top}$	$\widehat{\top}$	$\widehat{\top}$	$\widehat{\top}$	$\widehat{\top}$	$\widehat{\top}$

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- Hence $\mathcal{T}_\mathfrak{Q} = \{\mathbb{A}_\alpha \vee (\mathbb{A}_\beta * a_\ell) \mid \alpha, \beta \in \mathfrak{Q}\} \cup \{\widehat{\top}\}$

is a $\widehat{\mathfrak{Q}_2}$-enriched topology on the spectrum $\sigma(\mathfrak{Q})$ of $\mathfrak{Q}$.

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- $\{\mathbb{A}_\alpha \mid \alpha \in \mathfrak{Q}\} \cup \{\widehat{\top}\}$ is a subbase of $\mathcal{T}_{\mathfrak{Q}}$.