The preservation of projective right modules in Sup under duality

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Abstract. Starting from enriched order-theoretic properties of right modules over a unital quantale in the category Sup, this paper presents the following theorem. If the underlying quantale is unital and involutive with a designated element, then the duality of right modules preserves projectivity if and only if the underlying quantale has a dualizing element.

1. Introduction

It follows from the work by I. Stubbe (cf. [7, 8]) that projective right modules over a unital quantale $\mathfrak{Q}$ (projective right $\mathfrak{Q}$-modules for short) are enriched completely distributive and enriched join-complete $\mathfrak{Q}$-preordered sets. Hence the question arises under which necessary and sufficient condition the projectivity of right $\mathfrak{Q}$-modules is preserved under duality. In the topos-theoretic context we have a positive answer, and the necessary and sufficient condition can be expressed by the requirement that the order-theoretic structure of the subobject classifier is Boolean (cf. [9]). A continuation of these investigations has been carried out by H. Lai and L. Shen. In the context of integral and commutative quantales they proved that the duality preserves projective right $\mathfrak{Q}$-modules if and only if the underlying quantale is given by an integral and commutative Girard quantale (cf. [4, Theorem 8.2]).

In this paper we give a rather complete answer to this dualization problem. In the general framework of unital and involutive quantales with a designated element we show that the duality in the category of right $\mathfrak{Q}$-modules preserves projective right $\mathfrak{Q}$-modules if and only if the underlying quantale has a dualizing element (cf. Section 5). We prepare this result by a short survey on enriched order-theoretic properties of right $\mathfrak{Q}$-modules (cf. Section 3).

2. Preliminaries

In order to fix notation we first recall some basic facts of the theory of quantales and quantale-valued preordered sets. For more details the reader is referred to [2].

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Let Sup be the symmetric and monoidal closed category of complete lattices and join-preserving maps. A unital quantale $\mathfrak{Q} = (\mathfrak{Q}, *, 1)$ is a monoid in Sup. Due to the universal property of the tensor product $\otimes$ in Sup the multiplication in $\mathfrak{Q}$ can be identified with a binary map $\mathfrak{Q} \times \mathfrak{Q} \xrightarrow{*} \mathfrak{Q}$ such that $(\mathfrak{Q}, *, 1)$ is a monoid in Set and $*$ is join-preserving in each variable separately — i.e. the following relations hold for $\alpha, \beta \in \mathfrak{Q}$ and $A, B \subseteq M$:

$$\alpha * (\bigvee B) = \bigvee_{\beta \in B} (\alpha * \beta), \qquad (\bigvee A) * \beta = \bigvee_{\alpha \in A} (\alpha * \beta).$$

A unital quantale $\mathfrak{Q}$ is *integral* if the universal upper bound coincides with the unit of $\mathfrak{Q}$, and $\mathfrak{Q}$ is *involutive* if there exists an order-preserving involution $'$ on $\mathfrak{Q}$, which is also an anti-homomorphisms — i.e.

$$(\alpha * \beta)' = \beta' * \alpha', \qquad \alpha, \beta \in \mathfrak{Q}.$$

The right and the left implications of a quantale $\mathfrak{Q}$ are given by:

$$\alpha \searrow \beta = \bigvee \{\gamma \in \mathfrak{Q} \mid \alpha * \gamma \leqslant \beta\}, \qquad \beta \swarrow \alpha = \bigvee \{\gamma \in \mathfrak{Q} \mid \gamma * \alpha \leqslant \beta\},$$

where $\alpha, \beta \in \mathfrak{Q}$. An element δ of $\mathfrak{Q}$ is *dualizing*, if δ satisfies the following property:

$$(\delta \swarrow \alpha) \searrow \delta = \alpha = \delta \swarrow (\alpha \searrow \delta), \qquad \alpha \in \mathfrak{Q}.$$

For every dualizing element δ the following relation holds:

$$(\delta \swarrow \alpha) \searrow \beta = \alpha \swarrow (\beta \searrow \delta), \qquad \alpha, \beta \in \mathfrak{Q}. \tag{2.1}$$

Let $\mathfrak{Q}$ be a unital quantale and X be a set. A $\mathfrak{Q}$-*preorder* on X is map $X \times X \xrightarrow{p} \mathfrak{Q}$ satisfying the following axioms:

(P1) $1 \leqslant p(x, x), \quad x \in X,$ (Reflexivity)

(P2) $p(x, y) * p(y, z) \leqslant p(x, z), \quad x, y, z \in X.$ (Transitivity)

If p is $\mathfrak{Q}$-preorder on X, then the pair (X, p) is called a $\mathfrak{Q}$-*preordered set*.

Since every unital quantale $\mathfrak{Q}$ can be viewed as a biclosed monoidal category, $\mathfrak{Q}$-enriched category theory based on $\mathfrak{Q}$ is available. In this context $\mathfrak{Q}$-preordered sets are equivalent to $\mathfrak{Q}$-enriched categories. The reflexivity axiom (P1) corresponds to the axiom of $\mathfrak{Q}$-enriched identities and the axiom (P2) is equivalent to the $\mathfrak{Q}$-enriched composition law. Since every $\mathfrak{Q}$-enriched category has an underlying ordinary category, the underlying preorder $\leqslant_p$ of a $\mathfrak{Q}$-preorder p has the following form:

$$\leqslant_p = \{(x, z) \in X \times X \mid 1 \leqslant p(x, z)\}.$$

A $\mathfrak{Q}$-preorder p on X is antisymmetric (or, the corresponding $\mathfrak{Q}$-enriched category is skeletal), if the underlying preorder $\leqslant_q$ is antisymmetric. If $\mathfrak{Q} = (\mathfrak{Q}, *, 1, ')$ is a unital and involutive quantale, then also the *dual* $\mathfrak{Q}$-*preorder* p^{op} exists and is given by:

$$p^{op}(x, z) = p(z, x)', \qquad x, z \in X.$$

Morphisms between $\mathfrak{Q}$-preordered sets (X,p) and (Y,q) are $\mathfrak{Q}$-*homomorphisms* — these are maps $X \xrightarrow{\varphi} Y$ satisfying the condition

$$p(x_1,x_2) \leqslant q(\varphi(x_1),\varphi(x_2)), \qquad x_1,x_2 \in X$$

The related category is denoted by $\mathsf{Preord}(\mathfrak{Q})$ and is complete and cocomplete (cf. [2, p. 259]).

Adjointness between $\mathfrak{Q}$-homomorphisms can be expressed as follows. $(X,p) \xrightarrow{\varphi} (Y,q)$ is left adjoint to $(Y,q) \xrightarrow{\psi} (X,p)$ (resp. ψ is right adjoint to φ) if $q(\varphi(x),y) = p(x,\psi(y))$ for all $x,y \in X$.

Let (X,p) be a $\mathfrak{Q}$-preordered set. A map $X \xrightarrow{f} \mathfrak{Q}$ is called:

(a) a *contravariant* $\mathfrak{Q}$-*presheaf* on (X,p) if f is left-extensional w.r.t. p — i.e.

$$p(z,x) * f(x) \leqslant f(z), \qquad x,z \in X.$$

(b) a *covariant* $\mathfrak{Q}$-*presheaf* on (X,p), if f is right-extensional w.r.t. p — i.e.

$$f(x) * p(x,z) \leqslant f(z), \qquad z,x \in X.$$

The complete lattices $\mathbb{P}(X,p)$ and $\mathbb{P}^\dagger(X,p)$ of all contravariant and covariant $\mathfrak{Q}$-presheaves on (X,p) are pointwisely ordered and provided with the following $\mathfrak{Q}$-preorders d and $d^\dagger$:

$$d(f_1,f_2) = \bigwedge_{x\in X} (f_1(x) \searrow f_2(x)), \qquad f_1,f_2 \in \mathbb{P}(X,p),$$
$$d^\dagger(g_1,g_2) = \bigwedge_{x\in X} (g_1(x) \swarrow g_2(x)), \qquad g_1,g_2 \in \mathbb{P}^\dagger(X,p)$$

The underlying preorder $\leqslant_d$ coincides with the given order on $\mathbb{P}(X,p)$, while the underlying preorder $\leqslant_{d^\dagger}$ coincides with the opposite order on $\mathbb{P}^\dagger(X,p)$.

Finally, the $\mathfrak{Q}$-Yoneda embedding $(X,p) \xrightarrow{\eta_X} \mathbb{P}(X,p)$ is given by:

$$(\eta_X(x))(z) = \widetilde{x}(z) = p(z,x), \qquad x,z \in X.$$

Then (X,p) is $\mathfrak{Q}$-enriched join-complete if and only if η_X has a left adjoint $\mathfrak{Q}$-homomorphism, which is also call the formation of $\mathfrak{Q}$-enriched joins. On the basis of the axiom of choice, **2**-enriched join-completeness is equivalent to join-completeness in the traditional sense.

3. Right $\mathfrak{Q}$-modules in Sup and their enriched order structure

From a historical perspective it should be noted that for the first time module theory in Sup appeared in [3]. Here we are primarily interested in non-commutative and unital quantales $\mathfrak{Q}$. Hence a *right* $\mathfrak{Q}$-*module* $M = (M,\boxdot)$ is a complete lattice M provided with a right action $M \otimes \mathfrak{Q} \xrightarrow{\boxdot} M$ in the sense of Sup (for the general concept of of right actions in symmetric monoidal categories the reader is referred to [5]). Due to the universal property of the tensor product a right action $\boxdot$ in Sup can be identified with a map $M \times \mathfrak{Q} \xrightarrow{\boxdot} M$, which is join-preserving in each variable separately and satisfies the following axioms for all $t \in M$ and $\alpha,\beta \in \mathfrak{Q}$:

(M1) $t \boxdot 1 = t$, (Unity axiom)
(M2) $(t \boxdot \alpha) \boxdot \beta = t \boxdot (\alpha * \beta)$, (Associativity axiom)

In the following considerations we apply the subsequent notation as far as it is reasonable. Elements of the underlying unital quantale $\mathfrak{Q}$ are denoted by α, β, …, right $\mathfrak{Q}$-modules by M and N and their elements by t, s, …. If the context is clear and no confusion is possible, we will write $t\alpha$ instead of $t \boxdot \alpha$.

A right $\mathfrak{Q}$-module homomorphism is a join-preserving map $M \xrightarrow{h} N$ such that

$$h(t\alpha) = h(t)\alpha, \qquad t \in M,\ \alpha \in \mathfrak{Q}.$$

The related category is denoted by $\mathsf{Mod}_r(\mathfrak{Q})$.

Let M be a right $\mathfrak{Q}$-module. Since Sup has a self-duality given by the construction of right adjoint maps, we can compute the right adjoint map

$$M^{op} \xrightarrow{\boxdot^{\vdash}} (M \otimes \mathfrak{Q})^{op}$$

of the right action $\boxdot$ (see also [2, Theorem 2.2.10] in the special case $M = \mathfrak{Q}$), and it follows immediately from the unit and associativity axiom of the right action $\boxdot$ that the map $M \times M \xrightarrow{p} \mathfrak{Q}$ defined by:

$$p(s,t) = \big(\boxdot^{\vdash}(t)\big)(s) = \bigvee\{\alpha \in \mathfrak{Q} \mid s\alpha \leqslant t\}, \qquad s,t \in M \tag{3.1}$$

is a $\mathfrak{Q}$-preorder on M, which we will call the *intrinsic $\mathfrak{Q}$-preorder* of M. Hence *right $\mathfrak{Q}$-modules in* Sup carry always the structure of a *$\mathfrak{Q}$-preordered set*, and every right $\mathfrak{Q}$-module homomorphism is always a $\mathfrak{Q}$-homomorphism w.r.t. the respective intrinsic $\mathfrak{Q}$-preorders. Thus there exists a natural forgetful functor from $\mathsf{Mod}_r(\mathfrak{Q})$ to $\mathsf{Preord}(\mathfrak{Q})$, which has a left adjoint functor.

Example 3.1. Let (X,p) be a $\mathfrak{Q}$-preordered set. On $\mathbb{P}(X,p)$ and on the dual lattice $\big(\mathbb{P}^{\dagger}(X,p)\big)^{op}$ we introduce right actions determined as follows:

$$(f \boxdot \alpha)(x) = f(x) * \alpha, \qquad f \in \mathbb{P}(X,p),\ x \in X,\ \alpha \in \mathfrak{Q},$$
$$(g \boxdot \alpha)(x) = \alpha \searrow g(x), \qquad g \in \mathbb{P}^{\dagger}(X,p),\ x \in X,\ \alpha \in \mathfrak{Q}.$$

In this way, $\mathbb{P}(X,p)$ is the *free right $\mathfrak{Q}$-module* generated by (X,p), and the intrinsic $\mathfrak{Q}$-preorder of $\mathbb{P}(X,p)$ coincides with d. Similarly, the intrinsic $\mathfrak{Q}$-preorder of the right $\mathfrak{Q}$-module $\big(\mathbb{P}^{\dagger}(X,p)\big)^{op}$ coincides with $d^{\dagger}$.

The intrinsic $\mathfrak{Q}$-preorder p of a right $\mathfrak{Q}$-module M satisfies the following properties:

(1) The underlying preorder $\leqslant_p$ coincides with the order on M.
(2) For $s,t \in M$ and $S,T \subseteq M$ the following relations hold:

$$\bigwedge_{s\in S} p(s,t) = p(\bigvee S, t) \quad \text{and} \quad \bigwedge_{t\in T} p(s,t) = p(s, \bigwedge T).$$

(3) $\alpha \searrow p(s,t) = p(s\alpha, t)$ for all $\alpha \in \mathfrak{Q}$ and $s,t \in M$.

Moreover, the right $\mathfrak{Q}$-module homomorphism $\mathbb{P}(M,p) \xrightarrow{\mathsf{sup}_M} M$ defined by

$$\mathsf{sup}_M f = \bigvee_{t\in M} t \boxdot f(t), \qquad f \in \mathbb{P}(M,p) \tag{3.2}$$

is left adjoint to the $\mathfrak{Q}$-Yoneda embedding $(M,p) \xrightarrow{\eta_M} \big(\mathbb{P}(M,p),d\big)$. In fact, the following relation holds for all $t \in M$ and $f \in \mathbb{P}(M,p)$:

$$p\big(\mathsf{sup}_M f, t\big) = \bigwedge_{s\in M} p\big(\big((s \boxdot f(s)), t\big) = \bigwedge_{s\in M} \big(f(s) \searrow p(s,t)\big) = d(f,\widetilde{t}).$$

Hence the *underlying $\mathfrak{Q}$-preordered set* (M,p) of a given right $\mathfrak{Q}$-module M is always *$\mathfrak{Q}$-enriched join-complete.*

The next theorem characterizes those $\mathfrak{Q}$-homomorphisms which can be lifted to a right $\mathfrak{Q}$-module homomorphism.

Theorem 3.2. *Let M and N be right $\mathfrak{Q}$-modules with the respective intrinsic preorders p and q. Further, let $(M,p) \xrightarrow{h} (N,q)$ be a $\mathfrak{Q}$-homomorphism. Then $M \xrightarrow{h} N$ is a right $\mathfrak{Q}$-module homomorphism if and only if h has a right adjoint $\mathfrak{Q}$-homomorphism.*

Proof. (a) If $M \xrightarrow{h} N$ is a right $\mathfrak{Q}$-module homomorphism, then its right adjoint $\mathfrak{Q}$-homomorphism $N \xrightarrow{h^{\vdash}} M$ is defined by:

$$h^{\vdash}(s) = \mathsf{sup}_M q(h(\_),s) = \bigvee_{t\in M} t \boxdot q(h(t),s), \qquad s \in N.$$

Since h is a right $\mathfrak{Q}$-module homomorphism, the relation $h \circ h^{\vdash} \leqslant \mathrm{id}_N$ holds. Hence the relation

$$q(h(t),s) \leqslant p(t, t \boxdot q(h(t),s)) \leqslant p(t, h^{\vdash}(s)) \leqslant q(h(t),s)$$

follows for all $t \in M$ and $s \in N$. Hence $h^{\vdash}$ is right adjoint to h.

(b) If h is a $\mathfrak{Q}$-homomorphism, then $h(t)\alpha \leqslant h(t\alpha)$ holds for all $\alpha \in \mathfrak{Q}$ and $t \in M$ (cf. [2, Proposition 3.3.23]). Now let $h^{\vdash}$ be the right adjoint $\mathfrak{Q}$-homomorphism of h. Then we obtain for $\alpha \in \mathfrak{Q}$ and $t \in M$:

$$\alpha \leqslant q(h(t), h(t)\alpha) = p\big(t, h^{\vdash}(h(t)\alpha)\big).$$

Hence the relation $h(t\alpha) \leqslant h(h^{\vdash}(h(t)\alpha)) \leqslant h(t)\alpha$ follows, and so h is a right $\mathfrak{Q}$-module homomorphism. □

Remark 3.3 (Self-duality of $\mathsf{Mod}_r(\mathfrak{Q})$)**.** Let $\mathfrak{Q}$ be a unital and involutive quantale and M be a right $\mathfrak{Q}$-module with its intrinsic $\mathfrak{Q}$-preorder p. On M^{op} there exists a right action $\boxdot^{op}$ determined by:

$$t \boxdot^{op} \alpha = \bigvee \{s \in M \mid s\alpha' \leqslant t\}, \qquad t \in M,\, \alpha \in \mathfrak{Q}. \tag{3.3}$$

The right $\mathfrak{Q}$-module $(M^{op}, \boxdot^{op})$ is also called the *dual right $\mathfrak{Q}$-module* of M. In particular, the intrinsic $\mathfrak{Q}$-preorder of $(M^{op}, \boxdot^{op})$ coincides with the *dual $\mathfrak{Q}$-preorder* p^{op} of p.

The object function $(M, \boxdot) \longmapsto (M^{op}, \boxdot^{op})$ can be completed to a contravariant endofunctor $\mathcal{S}$ of $\mathsf{Mod}_r(\mathfrak{Q})$ as follows:

$$M \xrightarrow{h} N, \quad N^{op} \xrightarrow{\mathcal{S}(h)} M^{op}, \quad \mathcal{S}(h) = h^{\vdash},$$

where $N^{op} \xrightarrow{h^{\vdash}} M^{op}$ is the right adjoint map of h. It is not difficult to show that $h^{\vdash}$ is a right $\mathfrak{Q}$-module homomorphism. Since $\mathcal{S} \circ \mathcal{S} = \mathrm{id}_{\mathsf{Mod}_r(\mathfrak{Q})}$ holds, $\mathcal{S}$ is a *self-duality* of $\mathsf{Mod}_r(\mathfrak{Q})$.

We finish this section with the introduction of the $\mathfrak{Q}$-valued totally below relation. Therefore we fix a right $\mathfrak{Q}$-module M with its intrinsic $\mathfrak{Q}$-preorder p. Then the $\mathfrak{Q}$-valued totally below relation $M \times M \xrightarrow{\lhd} \mathfrak{Q}$ is defined by:

$$\lhd(s,t) = \bigwedge_{f \in \mathbb{P}(M,p)} (f(s) \swarrow p(t, \mathsf{sup}_M f)), \qquad s, t \in M.$$

It is not difficult to show that $\lhd$ satisfies the following properties for all $r, s, t, u \in M$:

(1) $p(r,s) * \lhd(s,t) \leqslant \lhd(r,t)$.
(2) $\lhd(s,t) * p(t,u) \leqslant \lhd(s,u)$.

Hence $\lhd(\_, t)$ is a contravariant $\mathfrak{Q}$-presheaf on (M,p), and the correspondence $t \longmapsto \lhd(\_, t)$ is a $\mathfrak{Q}$-homomorphism from (M,p) to $(\mathbb{P}(M,p), d)$.

Since $\bigwedge_{r \in M}(\widetilde{r}(s) \swarrow p(t, \mathsf{sup}_M \widetilde{r})) = p(s,t)$, the definition of $\lhd$ implies: $\lhd(s,t) \leqslant p(s,t)$, and consequently we have:

$$\mathsf{sup}_M \lhd(\_, t) \leqslant t, \qquad t \in M.$$

Referring again to the definition of $\lhd$ we obtain for $t \in M$ and $f \in \mathbb{P}(M,p)$:

$$p(t, \mathsf{sup}_M f) \leqslant \bigwedge_{s \in M} (\lhd(s,t) \searrow f(s)) = d(\lhd(\_, t), f).$$

Hence $t \longmapsto \lhd(\_, t)$ is left adjoint to sup_M if and only if

$$d(\lhd(\_, t), f) \leqslant p(t, \mathsf{sup}_M f), \qquad t \in M,\ f \in \mathbb{P}(M,p),$$

if and only if $t \leqslant \mathsf{sup}_M \lhd(\_, t)$ for all $t \in M$.

This observation is a motivation to introduce the following terminology. The $\mathfrak{Q}$-valued totally below relation $\lhd$ is *approximating*, if

$$t = \mathsf{sup}_M \lhd(\_, t), \qquad t \in M.$$

Hence we can summarize the previous results as follows. The correspondence $t \longmapsto \lhd(\_, t)$ is left adjoint to sup_M if and only if $\lhd$ is approximating.

Finally, if we replace $\mathfrak{Q}$ by the unique unital quantale $\mathbf{2}$ on $\{0,1\}$, then $\lhd$ is the characteristic function of the traditional totally below relation of complete lattices. Further, it is well know that a complete lattice is completely distributive if and only if $t \longmapsto \{s \in M \mid s \lhd t\}$ is left adjoint to sup (cf. [6]). Hence the previous results suggest to call a right $\mathfrak{Q}$-module $\mathfrak{Q}$-*enriched completely distributive*, if its $\mathfrak{Q}$-valued totally below relation is approximating.

4. A survey on the characterization of projective right $\mathfrak{Q}$-modules

First we recall that surjective right $\mathfrak{Q}$-module homomorphisms, epimorphisms and regular epimorphisms are equivalent concepts in $\mathsf{Mod}_r(\mathfrak{Q})$ (see also [2, Fact II on p. 218]). Moreover, a retract of a projective object in $\mathsf{Mod}_r(\mathfrak{Q})$ is again projective (cf. [1, Proposition 4.6.4]).

The following results has been obtain by I. Stubbe in a more general context given by quantaloid enriched categories. Here we recall these results (cf. [8, Lemma 3.2, Proposition 4.1 and Proposition 5.4]) in a quantale-setting.

Lemma 4.1. *The free right $\mathfrak{Q}$-module $\mathbb{P}(X,p)$ of all contravariant $\mathfrak{Q}$-presheaves on (X,p) is projective.*

Theorem 4.2. *Let M be a right $\mathfrak{Q}$-module. Then the following assertions are equivalent*:

(i) *M is a projective object in $\mathsf{Mod}_r(\mathfrak{Q})$.*
(ii) *sup_M has a left adjoint $\mathfrak{Q}$-homomorphism.*
(iii) *The $\mathfrak{Q}$-valued totally below relation $\lhd$ is approximating.*

Hence projective right $\mathfrak{Q}$-modules and $\mathfrak{Q}$-enriched completely distributive right $\mathfrak{Q}$-modules are equivalent concepts.

5. Projective right $\mathfrak{Q}$-modules and the self-duality in $\mathsf{Mod}(\mathfrak{Q})_r$

The next theorem presents a sufficient condition that projective right $\mathfrak{Q}$-modules are preserved under duality in $\mathsf{Mod}_r(\mathfrak{Q})$.

Theorem 5.1. *Let $\mathfrak{Q}$ be a unital and involutive quantale and M be a projective right $\mathfrak{Q}$-module. If $\mathfrak{Q}$ has a dualizing element, then the dual right $\mathfrak{Q}$-module M^{op} is again projective.*

The proof of Theorem 5.1 is based on a sequence of further results. First we fix a $\mathfrak{Q}$-preordered set (X,p) and examine the structure of the right $\mathfrak{Q}$-module $\big(\mathbb{P}^\dagger(X,p)\big)^{op}$ (cf. Example 3.1).

Theorem 5.2. *Let $\mathfrak{Q}$ be a unital quantale with a dualizing element and (X,p) be a $\mathfrak{Q}$-preordered set. Then the right $\mathfrak{Q}$-module $\big(\mathbb{P}^\dagger(X,p)\big)^{op}$ is projective.*

Proof. Let δ be a dualizing element in $\mathfrak{Q}$, and $d^\dagger$ be the intrinsic $\mathfrak{Q}$-preorder of the right $\mathfrak{Q}$-module $\big(\mathbb{P}^\dagger(X,p)\big)^{op}$. For $x \in X$ and $f \in \mathbb{P}^\dagger(X,p)$ we observe:

$$\begin{aligned}
d^\dagger(\delta \swarrow p(_,x), f) &= \bigwedge_{y\in X} ((\delta \swarrow p(y,x)) \swarrow f(y)) \\
&= \bigwedge_{y\in X} (\delta \swarrow (f(y) * p(y,x))) \\
&= \delta \swarrow \Big(\bigvee_{y\in X} (f(y) * p(y,x))\Big) \\
&= \delta \swarrow f(x).
\end{aligned}$$

The previous property motivates the following notation:

$$\overline{x} = \delta \swarrow p(\_, x), \qquad x \in X.$$

Since δ is a dualizing element in $\mathfrak{Q}$, we can express the $\mathfrak{Q}$-enriched join of any contravariant $\mathfrak{Q}$-presheaf F on $(\mathbb{P}^\dagger(X,p), d^\dagger)$ as follows:

$$\begin{aligned}
\big(\mathsf{sup}_{(\mathbb{P}^\dagger(X,p))^{op}} F\big)(x) &= \bigvee_{f\in\mathbb{P}^\dagger(X,p)}^{op} (F(f) \searrow f(x)) \\
&= \bigwedge_{f\in\mathbb{P}^\dagger(X,p)} \big(((\delta \swarrow f(x)) * F(f)\big) \searrow \delta\big) \\
&= \big(\bigvee_{f\in\mathbb{P}^\dagger(X,p)} (d^\dagger(\overline{x}, f) * F(f))\big) \searrow \delta \\
&= F(\overline{x}) \searrow \delta.
\end{aligned}$$

This motivates the following construction. Referring to (2.1) we first observe for each $f \in \mathbb{P}^\dagger(X,p)$ and $F \in \mathbb{P}\big(\mathbb{P}^\dagger(X,p), d^\dagger\big)$:

$$\begin{aligned}
d^\dagger\big(f, \mathsf{sup}_{(\mathbb{P}^\dagger(X,p))^{op}} F\big) &= \bigwedge_{x\in X} (f(x) \swarrow (F(\overline{x}) \searrow \delta)) \\
&= \bigwedge_{x\in X} ((\delta \swarrow f(x)) \searrow F(\overline{x})) \\
&= \bigwedge_{x\in X} (d^\dagger(\overline{x}, f) \searrow F(\overline{x})) \\
&= \bigwedge_{x\in X} \big(d^\dagger(\overline{x}, f) \searrow \big(\bigwedge_{g\in\mathbb{P}^\dagger(X,p)} (d^\dagger(g, \overline{x}) \searrow F(g))\big)\big) \\
&= \bigwedge_{g\in\mathbb{P}^\dagger(X,p)} \big(\big(\bigvee_{x\in X} d^\dagger(g, \overline{x}) * d^\dagger(\overline{x}, f)\big) \searrow F(g)\big).
\end{aligned}$$

Hence the $\mathfrak{Q}$-homomorphism $(\mathbb{P}^\dagger(X,p), d^\dagger) \xrightarrow{\psi} \big(\mathbb{P}(\mathbb{P}^\dagger(X,p), d^\dagger), d\big)$ given by

$$\big(\psi(f)\big)(g) = \bigvee_{x\in X} (d^\dagger(g, \overline{x}) * d^\dagger(\overline{x}, f)), \qquad f, g \in \mathbb{P}^\dagger(X,p)$$

is left adjoint to $\mathsf{sup}_{(\mathbb{P}^\dagger(X,p))^{op}}$. Consequently we conclude from Theorem 4.2 that $(\mathbb{P}^\dagger(X,p))^{op}$ is projective. $\square$

Remark 5.3. Let $\mathfrak{Q}$ be a unital quantale and $\{\cdot\}$ be a singleton provided with the discrete $\mathfrak{Q}$-preorder $p_0(\cdot,\cdot) = 1$. It is easily seen that the right $\mathfrak{Q}$-module $\big(\mathbb{P}^\dagger(\{\cdot\}, p_0)\big)^{op}$ is isomorphic to the right $\mathfrak{Q}$-module $(\mathfrak{Q}^{op}, \boxdot)$ where the right action is determined by $\alpha \boxdot \beta = \beta \searrow \alpha$. Referring to Example 3.1 the intrinsic $\mathfrak{Q}$-preorder p of $(\mathfrak{Q}^{op}, \boxdot)$ has the form

$$p(\alpha, \beta) = \bigvee\{\gamma \in \mathfrak{Q} \mid \alpha \boxdot \gamma \leqslant^{op} \beta\} = \bigvee\{\gamma \in \mathfrak{Q} \mid \gamma \searrow \alpha \geq \beta\} = \alpha \swarrow \beta$$

and the formation of $\mathfrak{Q}$-enriched joins in $(\mathfrak{Q}^{op}, \boxdot)$ is given by

$$\mathsf{sup}_{\mathfrak{Q}^{op}} f = \bigwedge_{\beta\in\mathfrak{Q}} \big(f(\beta) \searrow \beta\big), \qquad f \in \mathbb{P}(\mathfrak{Q}^{op}, p).$$

If $\mathfrak{Q}$ has additionally a dualizing element δ, then we conclude from the proof of Theorem 5.2 that the $\mathfrak{Q}$-valued totally below relation of $(\mathfrak{Q}^{op}, \boxdot)$ has the following form

$$\lhd(\beta,\alpha) = (\beta \swarrow \delta) * (\delta \swarrow \alpha), \qquad \alpha, \beta \in \mathfrak{Q}.$$

Theorem 5.4. *Let $\mathfrak{Q}$ be a unital and involutive quantale. Then the involution $'$ is a right $\mathfrak{Q}$-module isomorphism from $\big(\mathbb{P}(X,p)\big)^{op}$ to $\big(\mathbb{P}^{\dagger}(X,p^{op})\big)^{op}$.*

Proof. First we notice that $f \in \mathbb{P}(X,p)$ if and only if $f' \in \mathbb{P}^{\dagger}(X,p^{op})$. Then for $f \in \mathbb{P}(X,p)$ and $\alpha \in \mathfrak{Q}$ the following relation holds:

$$\big(f \boxdot^{op} \alpha\big)' = \big(f \swarrow \alpha'\big)' = \alpha \searrow f' = f'\alpha. \qquad \square$$

Proof of Theorem 5.1. Let M be a projective right $\mathfrak{Q}$-module with its intrinsic $\mathfrak{Q}$-preorder p. Then we conclude from Theorem 4.2 that the $\mathfrak{Q}$-valued totally below relation is approximating. Hence the $\mathfrak{Q}$-homomorphism $(M,p) \xrightarrow{h} (\mathbb{P}(M,p),d)$ determined by

$$h(t) = \lhd(\_,t), \qquad t \in M$$

is left adjoint to sup_M and is consequently a right $\mathfrak{Q}$-module homomorphism (cf. Theorem 3.2).
Now we apply the self-duality in $\mathsf{Mod}_r(\mathfrak{Q})$ (cf. Remark 3.3), and obtain the following situation. Since h is a section of sup_M, then $(\mathsf{sup}_M)^{\vdash}$ is a section of $h^{\vdash}$ and consequently

$$\mathbb{P}(M,p)^{op} \xrightarrow{h^{\vdash}} M^{op}$$

is a retraction. Since $\mathbb{P}(M,p)^{op}$ is projective (cf. Theorems 5.2 and 5.4), M^{op} is a retraction of a projective right $\mathfrak{Q}$-module, and is therefore also projective. $\square$

In what follows we investigate the question whether the existence of a dualizing element in the underlying unital and involutive quantale is also necessary for the property that the self-duality preserves projective right $\mathfrak{Q}$-modules. First we need some more terminology. Motivated by the previous results we make the following

Standing Assumption: For the remaining part of this section every quantale is *unital and involutive.*

Definition 5.5. Let $\lhd$ be the $\mathfrak{Q}$-valued totally below relation of the right $\mathfrak{Q}$-module $(\mathfrak{Q}^{op}, \boxdot)$ (cf. Remark 5.3). An element $\delta \in \mathfrak{Q}$ is called *designated* if the relation

$$\lhd(\beta,\alpha) \leqslant \big((\beta \swarrow \delta) * (\delta \swarrow \alpha)\big) \wedge \big((\beta \swarrow \delta') * (\delta' \swarrow \alpha)\big)$$

holds for all $\alpha, \beta \in \mathfrak{Q}$.

If δ is dualizing, then also δ' is dualizing. Hence we infer from Remark 5.3 that *every* dualizing element of a unital and involutive quantale is designated. However, the converse is not always true, as the next lemma demonstrates.

Lemma 5.6. *The universal lower bound of every integral and involutive quantale is designated.*

Proof. By Remark 5.3 we have that $\mathsf{sup}_{\mathfrak{Q}^{op}}\underline{\perp} = \perp \searrow \perp = \top$, and since $\top$ is the unit in $\mathfrak{Q}$, $p(\alpha,\top) = \alpha \swarrow \top = \alpha$. Hence

$$\lhd(\beta,\alpha) \leqslant \underline{\perp}(\beta) \swarrow p(\alpha, \mathsf{sup}_M \underline{\perp}) = \perp \swarrow \alpha = (\beta \swarrow \perp) * (\perp \swarrow \alpha)$$

for all $\alpha, \beta \in \mathfrak{Q}$. Since $\perp$ is hermitian — i.e. $\perp = \perp'$, $\perp$ is designated. □

We recall that the product of two unital (involutive) quantales is again unital (involutive) relative to the componentwise ordering and multiplication.

Proposition 5.7. *Let $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ be quantales such that $\mathfrak{Q}_1$ is integral, $\perp$ is the universal lower bound in $\mathfrak{Q}_1$ and δ is a dualizing element in $\mathfrak{Q}_2$. Then $(\perp,\delta)$ is a designated element of $\mathfrak{Q}_1 \times \mathfrak{Q}_2$.*

Proof. Let $*_1$ and $*_2$ be the respective multiplications in $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$. The left implication in $\mathfrak{Q}_1 \times \mathfrak{Q}_2$ will be also denoted by $\swarrow$. If p is the intrinsic $\mathfrak{Q}_1 \times \mathfrak{Q}_2$-preorder of $\big((\mathfrak{Q}_1 \times \mathfrak{Q}_2)^{op}, \boxdot\big)$, then for $\alpha_i, \beta_i \in \mathfrak{Q}_i$ (i=1,2) we have the following simple relation:

$$p\big((\alpha_1,\alpha_2),(\beta_1,\beta_2)\big) = (\alpha_1,\alpha_2) \swarrow (\beta_1,\beta_2) = (\alpha_1 \swarrow \beta_1, \alpha_2 \swarrow \beta_2).$$

We fix $(\alpha_1,\alpha_2) \in \mathfrak{Q}_1 \times \mathfrak{Q}_2$ and define a contravariant $\mathfrak{Q}_1 \times \mathfrak{Q}_2$-presheaf f on $\big((\mathfrak{Q}_1 \times \mathfrak{Q}_2)^{op}, p\big)$ as follows:

$$f(\beta_1,\beta_2) = \big(\perp, (\beta_2 \swarrow \delta) *_2 (\delta \swarrow \alpha_2)\big) \qquad (\beta_1,\beta_2) \in \mathfrak{Q}_1 \times \mathfrak{Q}_2$$

Since δ is dualizing in $\mathfrak{Q}_2$, it is easily seen that the relation

$$\begin{aligned} \alpha_2 &= (\delta \swarrow \alpha_2) \searrow \delta = \mathsf{sup}_{(\mathfrak{Q}_2)^{op}}\big((\_ \swarrow \delta) *_2 (\delta \swarrow \alpha_2)\big) \\ &= \bigwedge_{\beta_2 \in \mathfrak{Q}_2} \big(((\beta_2 \swarrow \delta) *_2 (\delta \swarrow \alpha_2)) \searrow \beta_2\big). \end{aligned} \tag{5.1}$$

holds. Hence $\mathsf{sup}_{(\mathfrak{Q}_1 \times \mathfrak{Q}_2)^{op}} f = (\top, \alpha_2)$ follows, where $\top$ is the universal upper bound in $\mathfrak{Q}_1$. Now observe:

$$\begin{aligned} \lhd\big((\beta_1,\beta_2),(\alpha_1,\alpha_2)\big) &\leqslant f(\beta_1,\beta_2) \swarrow p\big((\alpha_1,\alpha_2),(\top,\alpha_2)\big) \\ &= f(\beta_1,\beta_2) \swarrow \big((\alpha_1,\alpha_2) \swarrow (\top,\alpha_2)\big) \\ &\leqslant f(\beta_1,\beta_2) \swarrow (\alpha_1, 1) \\ &= \big((\perp \swarrow \alpha_1), ((\beta_2 \swarrow \delta) *_2 (\delta \swarrow \alpha_2))\big) \\ &= \big((\beta_1 \swarrow \perp) *_1 (\perp \swarrow_1 \alpha_1)), ((\beta_2 \swarrow \delta) *_2 (\delta \swarrow \alpha_2))\big) \\ &= \big((\beta_1,\beta_2) \swarrow (\perp,\delta)\big) * \big((\perp,\delta) \swarrow (\alpha_1,\alpha_2)\big). \end{aligned}$$

Since δ' is also dualizing and $(\perp,\delta)' = (\perp,\delta')$, we also have

$$\lhd\big((\beta_1,\beta_2),(\alpha_1,\alpha_2)\big) \leqslant \big((\beta_1,\beta_2) \swarrow (\perp,\delta)'\big) * \big((\perp,\delta)' \swarrow (\alpha_1,\alpha_2)\big).$$

Thus $(\perp,\delta)$ is a designated value of $\mathfrak{Q}_1 \times \mathfrak{Q}_2$. □

The next example is an illustration of Proposition 5.7.

Example 5.8. Let $C_3 = \{\bot, a, \top\}$ be the chain consisting of three elements. It is well know that on C_3 there exist three unital quantales $\mathfrak{Q}_i = (C_3, *_i)$ (i=1,2,3), (see [2, (9), (16) and (19) in Exercise 2.2.1 and Exercise 2.3.1]), which are all commutative. Thus as involution we choose the identity map on all three of them. The first two of them are integral, and in this context $*_1$ denotes the multiplication of the MV-algebra with three elements, while $*_2$ represents the binary minimum. The third unital quantale is non-integral and consequently the unit coincides with a. It is easy to see that a is the dualizing element in $\mathfrak{Q}_3$. Then we have the following situation:

(i) The unital quantale $\mathfrak{Q}_1 \times \mathfrak{Q}_3$ is non-integral and commutative, and has $(\bot, a)$ as designated element, which *is* dualizing in $\mathfrak{Q}_1 \times \mathfrak{Q}_3$. Hence $\mathfrak{Q}_1 \times \mathfrak{Q}_3$ is a non-integral and commutative Girard quantale (cf. [2]).

(ii) The unital quantale $\mathfrak{Q}_2 \times \mathfrak{Q}_3$ is non-integral and commutative and has $(\bot, a)$ as designated element, which is *not* dualizing in $\mathfrak{Q}_2 \times \mathfrak{Q}_3$.

After this digression on designated elements, we give now an answer to the question to which existent the existence of a dualizing element is necessary for the property that the self-duality in $\mathsf{Mod}_r(\mathfrak{Q})$ preserves projectivity.

Theorem 5.9. *Let $\mathfrak{Q}$ be a unital and involutive quantale. If the self-duality in $\mathsf{Mod}_r(\mathfrak{Q})$ preserves projectivity — i.e. if M is projective, then also M^{op} is projective, then any designated element of $\mathfrak{Q}$ is dualizing.*

Proof. Since $\mathbb{P}(\{\cdot\}, p_0)$ is projective (cf. Lemma 4.1), the dual right $\mathfrak{Q}$-module $\big(\mathbb{P}(X, p_0)\big)^{op}$ is projective. Consequently $(\mathfrak{Q}^{op}, \boxdot)$ is also projective (cf. Theorem 5.4 and Remark 5.3).
Let $\delta \in \mathfrak{Q}$ be a designated element of $\mathfrak{Q}$. For each $\alpha \in \mathfrak{Q}$ we consider the contravariant $\mathfrak{Q}$-presheaves f_α and g_α on $(\mathfrak{Q}^{op}, p)$ defined by

$$f_\alpha(\beta) = (\beta \swarrow \delta) * (\delta \swarrow \alpha) \quad \text{and} \quad g_\alpha(\beta) = (\beta \swarrow \delta') * (\delta' \swarrow \alpha), \qquad \beta \in \mathfrak{Q}.$$

Obviously, the following relation holds (cf. (5.1)):

$$\mathsf{sup}_{\mathfrak{Q}^{op}} f_\alpha = \bigwedge_{\beta \in \mathfrak{Q}} (f_\alpha(\beta) \searrow \beta) = (\delta \swarrow \alpha) \searrow \delta.$$

Since δ is designated we have that $\lhd(\_, \alpha) \leqslant f_\alpha$ and since the $\mathfrak{Q}$-valued totally below relation $\lhd$ in $(\mathfrak{Q}^{op}, \boxdot)$ is approximating, we obtain:

$$\alpha = \mathsf{sup}_{\mathfrak{Q}^{op}} \lhd(\_, \alpha) \leqslant^{op} \mathsf{sup}_{\mathfrak{Q}^{op}} f_\alpha = (\delta \swarrow \alpha) \searrow \delta.$$

Hence $(\delta \swarrow \alpha) \searrow \delta \leqslant \alpha$ follows.
If in the previous argumentation we replace f_α by g_α, then we also obtain $(\delta' \swarrow \alpha') \searrow \delta' \leqslant \alpha'$ — i.e. $\delta \swarrow (\alpha \searrow \delta) \leqslant \alpha$. Thus δ is dualizing in $\mathfrak{Q}$. □

The results of this section can be summarized in the following

Theorem 5.10. *Let $\mathfrak{Q}$ be a unital and involutive quantale with a designated element. The self-duality in $\mathsf{Mod}_r(\mathfrak{Q})$ preserves projectivity if and only if $\mathfrak{Q}$ has a dualizing element.*

In particular by Lemma 5.6 we have:

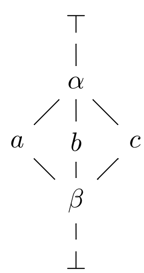
Corollary 5.11. *Let $\mathfrak{Q}$ be an integral and involutive quantale. The self-duality in $\mathsf{Mod}_r(\mathfrak{Q})$ preserves projectivity if and only if $\bot$ is the dualizing element of $\mathfrak{Q}$.*

The previous Theorem and Corollary are a far reaching extension of the equivalence (i) $\Longleftrightarrow$ (ii) in [4, Theorem 8.2] from the class of integral and commutative quantales to the class of unital and involutive (and in general non-integral and non-commutative) quantales with a designated element.

As a confirmation of this statement we first recall that the quantale $\mathfrak{Q}_3$ from Example 5.8 is involutive and unital (but non-integral) and has a dualizing element. Consequently, it follows from Theorem 5.10 that the self-duality in $\mathsf{Mod}_r(\mathfrak{Q}_3)$ preserves projectivity.

Finally, we include two examples of unital, involutive and non-commutative quantales $\mathfrak{Q}$ such that the self-duality in $\mathsf{Mod}_r(\mathfrak{Q})$ preserves projectivity.

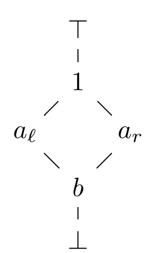
Examples 5.12. (1) (Cf. [2, Exercise 2.6.5]) Let us consider a complete lattice $\mathfrak{Q} = \{\bot, \beta, a, b, c, \alpha, \top\}$ endowed with a multiplication $*$, where the Hasse diagram and the multiplication table of $*$ are given by:



$*$	$\bot$	β	a	b	c	α	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
β	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	β
a	$\bot$	$\bot$	β	$\bot$	β	β	a
b	$\bot$	$\bot$	β	β	$\bot$	β	b
c	$\bot$	$\bot$	$\bot$	β	β	β	c
α	$\bot$	$\bot$	β	β	β	β	α
$\top$	$\bot$	β	a	b	c	α	$\top$

Obviously $\mathfrak{Q}$ is an integral, involutive and non-commutative quantale, where the involution $'$ is defined by $\top' = \top$, $\alpha' = \alpha$, $a' = a$, $b' = c$, $c' = b$, $\beta' = \beta$ and $\bot' = \bot$. Finally, since $\bot$ is dualizing, it follows from Corollary 5.11 that the self-duality in $\mathsf{Mod}_r(\mathfrak{Q})$ preserves projectivity.

(2) (Cf. [2, Exercise 2.6.2]) Let $C_3 = \{\bot, a, \top\}$ be the chain consisting of three elements. Then C_3 has a unique order-reversing involution, and consequently the unital quantale of all join-preserving self-maps of C_3 is an involutive Girard quantale. The Hasse diagram and the multiplication table have following form:



$*$	$\bot$	b	a_ℓ	a_r	1	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
b	$\bot$	$\bot$	b	$\bot$	b	a_r
a_ℓ	$\bot$	$\bot$	a_ℓ	$\bot$	a_ℓ	$\top$
a_r	$\bot$	b	b	a_r	a_r	a_r
1	$\bot$	b	a_ℓ	a_r	1	$\top$
$\top$	$\bot$	a_ℓ	a_ℓ	$\top$	$\top$	$\top$

Obviously $\mathfrak{Q}$ is an unital, involutive and non-integral and non-commutative quantale, where the involution $'$ is defined by $\top' = \top$, $1' = 1$, $a'_\ell = a_r$, $a'_r = a_\ell$, $b' = b$ and $\bot' = \bot$. Finally, since b is dualizing, it follows from Theorem 5.10 that the self-duality in $\mathsf{Mod}_r(\mathfrak{Q})$ preserves projectivity.

The previous construction can be generalized, if we replace C_3 be any completely distributive lattice with an order-reversing involution (cf. Example 2.3.31 on p. 107–108 and Example 2.6.16 on p. 173–175 in [2]).

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