Magnetic field and stellar structure

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Magnetic field and global stellar structure

Magnetic field significantly influences stellar structure if the magnetic field energy is comparable with gravitational energy, that is,

$$\frac{4}{3}\pi R^3 \frac{B^2}{4\pi} \approx \frac{GM^2}{R}.$$

In solar units,

$$B \approx 10^8\,\mathrm{G} \left(\frac{M}{M_\odot}\right) \left(\frac{R}{R_\odot}\right)^{-2}.$$

This limit is never reached in any star. In nondegenerate stars, the magnetic field is always significantly lower than the above limit. The most strongly magnetized non-degenerate star ever discovered is Babcock star HD 215441, which has $M \approx 2M_\odot$, $R \approx 2R_\odot$, and $B \approx 34\,\mathrm{kG}$, i.e., several orders of magnitude below the limit. Neither this is fulfilled in white dwarfs, where the magnetic field is up to $10^8\,\mathrm{G}$, but $R \approx 10^{-2}R_\odot$. The limit is not reached even in magnetars with B upto $10^{15}\,\mathrm{G}$, but very small radii $R \approx 10^{-5}R_\odot$. Magnetic field field does not influence the global structure of stars, but may influence local processes like convection, angular momentum transport, and magnetosphere.

Magnetic field and convection

The interplay between magnetic field and convection establishes a very complex problem, which shall be treated usign MHD simulations in its general form, coining the term magnetoconvection. At least, an analogue of Schwartzchield stability condition can be formulated as

$$\frac{\mathrm{d}\ln T}{\mathrm{d}\ln p} < \frac{\varkappa - 1}{\varkappa} + \frac{B_\mathrm{v}^2}{B_\mathrm{v}^2 + 8\pi\varkappa p},$$

where B_v is the vertical component of the magnetic field. This means that the magnetic field may stabilize atmosphere against the convection if

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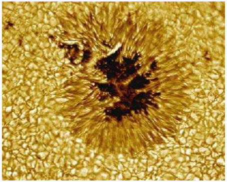
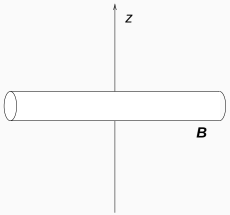
The magnetic field stabilizes the solar atmosphere against convection in solar spots. As a result, the heat transport becomes inefficient and the spot becomes cool.



Magnetic buyonancy

Let us assume the horizontal magnetic flux tubewith internal and external pressure p_i and p_e, respectively. The hydrostatic equilibrium requires that

$$p_i + \frac{B^2}{4\pi} = p_e.$$

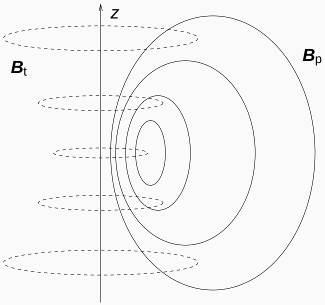


This implies that the density inside the flux tube is lower than outside leading to magnetic buyonancy (Parker & Jensen 1955).

The magnetic flux tubes rise and appear on the stellar surface in the form of Greek letter Ω. This explains the appearance of stellar spots with oposite polarities.

Magnetic field decomposition

Magnetic field decomposition



In many cases, the stars retain some kind of symmetry, for example an axial symmetry around the rotational axis. In such case it becomes convenient to decompose the magnetic field into poloidal and toroidal components,

$$\boldsymbol{B} = \boldsymbol{B}_\mathrm{p} + \boldsymbol{B}_\mathrm{t}.$$

Denoting $\boldsymbol{t}$ unit vector in the azimuthal direction, the components fulfill

$$\boldsymbol{B}_\mathrm{p} \cdot \boldsymbol{t} = 0, \qquad \boldsymbol{B}_\mathrm{t} = B_\varphi \boldsymbol{t}.$$

Introducing scalar P such that the potential of $\boldsymbol{B}_\mathrm{p}$ is $\boldsymbol{A} = A\boldsymbol{t} = -P\boldsymbol{t}/R$, where R is radius in cylindrical coordinates (distance from z axis)

$$\boldsymbol{B}_\mathrm{p} = \mathrm{rot}\,\boldsymbol{A} = -\mathrm{rot}\,\left(\frac{P}{R}\boldsymbol{t}\right) = -\frac{1}{R}\nabla P \times \boldsymbol{t}.$$

Here we used $\mathrm{rot}\,(\psi\boldsymbol{C}) = \psi\mathrm{rot}\,\boldsymbol{C} + \nabla\psi \times \boldsymbol{A}$ and $\mathrm{rot}\,(\boldsymbol{t}/R) = 0$.

Magnetic field decomposition: the current

As a result of axial symmetry, from the Ampére's law (e.g., writing in components)

$$\boldsymbol{j}_{\mathrm{t}} = \frac{c}{4\pi} \mathrm{rot}\, \boldsymbol{B}_{\mathrm{p}},$$

$$\boldsymbol{j}_{\mathrm{p}} = \frac{c}{4\pi} \mathrm{rot}\, \boldsymbol{B}_{\mathrm{t}} = \frac{c}{4\pi} \mathrm{rot} \left(R B_{\varphi} \frac{\boldsymbol{t}}{R} \right) = \frac{c}{4\pi R} \nabla (R B_{\varphi}) \times \boldsymbol{t}.$$

This means that poloidal magnetic field creates toroidal currents and vice versa. The Lorentz force density is

$$\frac{1}{c} \boldsymbol{j} \times \boldsymbol{B} = \underbrace{\frac{1}{c} \boldsymbol{j}_{\mathrm{p}} \times \boldsymbol{B}_{\mathrm{t}} + \frac{1}{c} \boldsymbol{j}_{\mathrm{t}} \times \boldsymbol{B}_{\mathrm{p}}}_{\text{poloidal}} + \underbrace{\frac{1}{c} \boldsymbol{j}_{\mathrm{p}} \times \boldsymbol{B}_{\mathrm{p}}}_{\text{toroidal}}.$$

From symmetry, the Gauss's law for magnetism in the differential form is

$$\mathrm{div}\, \boldsymbol{B} = \mathrm{div}\, \boldsymbol{B}_{\mathrm{p}} + \mathrm{div}\, \boldsymbol{B}_{\mathrm{t}} = \mathrm{div}\, \boldsymbol{B}_{\mathrm{p}} = 0.$$

Ferraro isorotation law

Induction equation in the rotating star

Let us study rotation of spherically symmetric star with rotational velocity

$$\boldsymbol{v}_\mathrm{t} = R\Omega\boldsymbol{t},$$

where R is the radius in cylindrical coordinates, Ω is angular frequency, and $\boldsymbol{t}$ is unit toroidal vector. Separating the magnetic field into poloidal and toroidal components, $\boldsymbol{B} = \boldsymbol{B}_\mathrm{p} + \boldsymbol{B}_\mathrm{t}$, the induction equation is

$$\frac{\partial \boldsymbol{B}}{\partial t} = \mathrm{rot}\left(\boldsymbol{v}_\mathrm{t} \times \boldsymbol{B}\right) = \mathrm{rot}\left[R\Omega\boldsymbol{t} \times (\boldsymbol{B}_\mathrm{p} + \boldsymbol{B}_\mathrm{t})\right] = \mathrm{rot}\left(R\Omega\boldsymbol{t} \times \boldsymbol{B}_\mathrm{p}\right).$$

Evaluating this in the cylindrical component, the φ component of induction equation is

$$\frac{\partial B_\varphi}{\partial t} = R\left(\boldsymbol{B}_\mathrm{p} \cdot \nabla\right)\Omega.$$

If angular velocity changes along $\boldsymbol{B}_\mathrm{p}$, then the toroidal field is generated from the original poloidal field.

Equation of motion in rotating star

The equation of motion in the direction of φ is

$$\rho\frac{\partial v_\varphi}{\partial t} = \rho R\frac{\partial\Omega}{\partial t} = \frac{1}{c}\left[\left(\boldsymbol{j}_{\rm t}+\boldsymbol{j}_{\rm p}\right)\times\left(\boldsymbol{B}_{\rm p}+\boldsymbol{B}_{\rm t}\right)\right]\boldsymbol{t} = \frac{1}{c}\left(\boldsymbol{j}_{\rm p}\times\boldsymbol{B}_{\rm p}\right)\boldsymbol{t}.$$

Using the previously derived relations and $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B})$

$$\rho R^2\frac{\partial\Omega}{\partial t} = \frac{R}{c}\frac{c}{4\pi}\left[\left(\nabla(RB_\varphi)\times\frac{\boldsymbol{t}}{R}\right)\times\boldsymbol{B}_{\rm p}\right]\cdot\boldsymbol{t} = \frac{1}{4\pi}\boldsymbol{B}_{\rm p}\cdot\nabla(RB_\varphi).$$

Torsional waves

Combining the induction and momentum equations and neglecting the changes of $\boldsymbol{B}_\mathrm{p}$ we arrive at the wave equation of torsional waves

$$\rho R^2 \frac{\partial^2 \Omega}{\partial t^2} = \frac{1}{4\pi} \boldsymbol{B}_\mathrm{p} \cdot \nabla \left[R^2 \left(\boldsymbol{B}_\mathrm{p} \cdot \nabla \right) \Omega \right],$$

$$\frac{\partial^2 B_\varphi}{\partial t^2} = R \left(\boldsymbol{B}_\mathrm{p} \cdot \nabla \right) \left[\frac{1}{4\pi \rho R^2} \boldsymbol{B}_\mathrm{p} \cdot \nabla (R B_\varphi) \right].$$

Taking into account small variations of $\boldsymbol{B}_\mathrm{p}$ and R

$$\frac{\partial^2 \Omega}{\partial t^2} = \frac{B_\mathrm{p}^2}{4\pi\rho} \frac{\partial^2 \Omega}{\partial s^2} = v_\mathrm{A}^2 \frac{\partial^2 \Omega}{\partial s^2},$$

where s the element length along $\boldsymbol{B}_\mathrm{p}$. The torsional waves propagate with Alfvén speed relatively quicky through the star (within $10^2 - 10^4$ yr).

Ferraro isorotation law

The torsional waves dampen down during the stellar lifetime, $\partial B_\varphi/\partial t = 0$, what implies $(\boldsymbol{B}_\mathrm{p}\cdot\nabla)\,\Omega = 0$. For $\boldsymbol{B}_\mathrm{p} = -\frac{1}{R}\nabla P \times \boldsymbol{t}$ we have $(\nabla P \times \boldsymbol{t})\cdot\nabla\Omega = 0$. Again using the identity $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B})$ we have $\boldsymbol{t}\cdot(\nabla P\times\nabla\Omega) = 0$. As a result of symmetry around the z axis, the term in the bracket should have a component just in the direction of $\boldsymbol{t}$. This implies $\nabla P \times \nabla\Omega = 0$, or

$$\Omega = \Omega(P).$$

This is Ferraro isorotation law, which says that the angular rotation frequency is constant along the magnetic field line. In most cases, this means solid body rotation for magnetic stars.

From the equation of motion in the stationary state $\partial\Omega/\partial t = 0$ and therefore $\boldsymbol{B}_\mathrm{p}\cdot\nabla(RB_\varphi) = 0$. Again using $\boldsymbol{B}_\mathrm{p} = -\frac{1}{R}\nabla P\times\boldsymbol{t}$ we have $RB_\varphi = f(P)$. Therefore the current

$$\boldsymbol{j}_\mathrm{p} = \frac{c}{4\pi}\nabla(RB_\varphi)\times\boldsymbol{t} = \frac{c}{4\pi R}f'\nabla P\times\boldsymbol{t} = \frac{c}{4\pi}\boldsymbol{B}_\mathrm{p}$$

flows along the magnetic field and does not produce any force.

Solar dynamo

Dynamo: creating magnetic field by motion

Many stars show strong magnetic fields. One of the possibilities how to explain such fields is by a stellar dynamo, that is, by creating magnetic fields by flow motion. As we shall see, the dynamos work in cool stars in their convective envelopes and in convective cores of hot stars.

Biermann battery

Let us explore the possibility that the polarization electric field is the source of the magnetic field. The polarization electric field is

$$\boldsymbol{E} = -\frac{m_\mathrm{i}}{2e}\boldsymbol{g}.$$

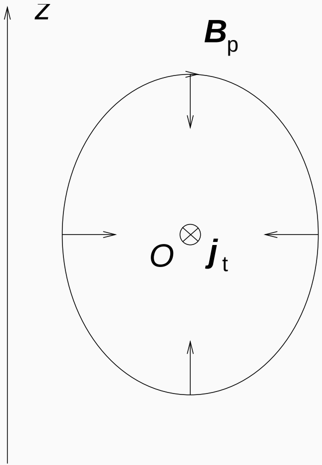
From hydrostatic equilibrium equation $\boldsymbol{g} = \nabla p/\rho$ we have $\boldsymbol{E} = -m_\mathrm{i}\nabla p/\left(2e\rho\right)$. For $n_\mathrm{i} = n_\mathrm{e}$ we have $p = 2p_\mathrm{e}$ and therefore

$$\boldsymbol{E} + \frac{1}{n_\mathrm{e} e}\nabla p_\mathrm{e} = 0.$$

The Faraday's law of induction $-1/c\,\partial\boldsymbol{B}/\partial t = \mathrm{rot}\,\boldsymbol{E}$ provides a possibility to create magnetic field if $\mathrm{rot}\left(\nabla p_\mathrm{e}/n_\mathrm{e}\right) \neq 0$.

In a spherically symmetric nonrotating star $\mathrm{rot}\,\boldsymbol{E} = 0$ and therefore there is no induced magnetic field. However, in a rotating star where $\Omega = \Omega(z)$ from the hydrostatic equilibrium equation $\mathrm{rot}\,\boldsymbol{E} \neq 0$ providing a posibility to induce toroidal magnetic field. This is called Biermann battery. However, the analysis shows that the field is very weak, but it may serve as a seed field for the dynamos.

Cowling antidynamo theorem: the essence



Let us assume axial symmetry. The toroidal magnetic field $\boldsymbol{B}_{\mathrm{p}}$ creates from the Ampére's law toroidal currents $\boldsymbol{j}_{\mathrm{t}}$. This results in a flow velocity $\boldsymbol{j}_{\mathrm{t}} \times \boldsymbol{B}_{\mathrm{p}}$ in the direction to the neutral point. To prevent this, we would need a flow source in the neutral point O, which is impossible due to the conservation of mass.

This means that any axisymmetric magnetic field cannot be sustained against the Ohmic decay via axisymmetric flow of matter.

This is Cowling antidynamo theorem. As a result, we need nonaxisymmtric flow to obtain stable dynamo.

Cowling antidynamo theorem: detailed analysis

Let us search for a stable dynamo assuming axial symmetry. The toroidal component of the Ohm's law is

$$\frac{\boldsymbol{j}_\mathrm{t}}{\sigma} = \boldsymbol{E}_\mathrm{t} + \frac{1}{c}\boldsymbol{v}_\mathrm{p} \times \boldsymbol{B}_\mathrm{p}.$$

Assuming the vector potential in the form of $\boldsymbol{A} = -P\boldsymbol{t}/R$, we have the poloidal magnetic field

$$\boldsymbol{B}_\mathrm{p} = -\frac{1}{R}\nabla P \times \boldsymbol{t}.$$

Decaying axially symmetric magnetic field creates toroidal electric field $\boldsymbol{E}_\mathrm{t} = -1/c\partial \boldsymbol{A}/\partial t$ (as a result of axial symmetry, there is not contribution from $\nabla\varphi$. In a stationary state, $\boldsymbol{E}_\mathrm{t} = 0$ and the toroidal current is

$$\boldsymbol{j}_\mathrm{t} = \frac{c}{4\pi}\mathrm{rot}\,\boldsymbol{B} = \frac{c}{4\pi}\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{A} = \frac{c}{4\pi}\mathrm{rot}\left(\frac{1}{R}\nabla P \times \boldsymbol{t}\right) =$$

$$\frac{c}{4\pi}\left[\frac{\boldsymbol{t}}{R}\,\mathrm{div}\,\nabla P + \nabla P\;\underbrace{\mathrm{div}\left(\frac{\boldsymbol{t}}{R}\right)}_{=0,\ \text{axisym.}} + \underbrace{(\nabla P\cdot\nabla)\left(\frac{\boldsymbol{t}}{R}\right)}_{-1/R\,\partial P/\partial R \mathfrak{t}} - \underbrace{\left(\frac{\boldsymbol{t}}{R}\cdot\nabla\right)\nabla P}_{=0,\ \text{axisym.}}\right].$$

Cowling antidynamo theorem: detailed analysis

The right hand side of the Ohm's law is

$$\frac{1}{c}\boldsymbol{v}_{\mathrm{p}} \times \boldsymbol{B}_{\mathrm{p}} = -\frac{1}{cR}\boldsymbol{v}_{\mathrm{p}} \times (\nabla P \times \boldsymbol{t}) = -\frac{1}{cR}\underbrace{(\boldsymbol{v}_{\mathrm{p}} \cdot \boldsymbol{t})}_{=0} \cdot \nabla P + \frac{1}{cR}(\boldsymbol{v}_{\mathrm{p}} \cdot \nabla P) \cdot \boldsymbol{t}.$$

Combining the last two equations we have the Ohm's law as

$$\boldsymbol{v}_{\mathrm{p}} \cdot \nabla P = \frac{c^2}{4\pi\sigma}\left(\mathrm{div}\,\nabla P - \frac{1}{R}\frac{\partial P}{\partial R}\right).$$

In the neutral point O we have $\nabla P = 0$ $(\partial P/\partial R = 0)$, but $\mathrm{div}\,\nabla P \neq 0$. This means that the above condition is fullfilled either just for $\sigma \to \infty$ (for the frozen field) or in the vicinity of O we shall have $\boldsymbol{v}_{\mathrm{p}} \to \infty$, which does not make a sense. Consequently, any axisymmetric magnetic field cannot be sustained against the Ohmic decay via axisymmetric flow of matter.

Towards the working dynamo: model due to Babcock & Parker

The Cowling antidynamo theorem shows that to sustain a working dynamo, one needs to employ the toroidal field component. The dynamo model is based on oscillarory process:

- The process starts with a dipolar field. As a result of differential rotation, the magnetic field is wind up. Therefore, the differential rotation creates toroidal field from initial poloidal field.
- In a second step, the poloidal field should be recreated from initial toroidal one. This happends due to the magnetic buyonancy and convection, due to which the magnetic flux tubes move upward. The convective bubles expand, what creates the the Coriolis force, which winds us the toroidal field creating the poloidal field.

These steps lead to reversal of the magnetic poles. Therefore the real period of the dynamo is twice that given by these processes.

Parker's dynamo: poloidal component of induction equation

We shall assume general magnetic field that contains both poloidal and toroidal components induced by stellar rotation

$$\boldsymbol{B}_{\mathrm{p}} = \mathrm{rot}\,(A\boldsymbol{t})\,, \qquad \boldsymbol{B}_{\mathrm{t}} = B_\varphi \boldsymbol{t}, \qquad \boldsymbol{v} = \boldsymbol{v}_{\mathrm{p}} + \Omega R \boldsymbol{t}.$$

The general induction equation including the resistivity λ

$$\frac{\partial \boldsymbol{B}}{\partial t} = \mathrm{rot}\,(\boldsymbol{v}\times\boldsymbol{B}) - \lambda\,\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{B} = \mathrm{rot}\,(\boldsymbol{v}\times\boldsymbol{B}) - \lambda\,\Delta\boldsymbol{B}.$$

The poloidal component has the form of

$$\frac{\partial\,(\mathrm{rot}\,(A\boldsymbol{t}))}{\partial t} = \mathrm{rot}\,(\boldsymbol{v}\times\boldsymbol{B}) - \lambda\,\mathrm{rot}\,\mathrm{rot}\,\mathrm{rot}\,(A\boldsymbol{t})\,,$$

what gives for A after integration with $\boldsymbol{B}_{\mathrm{p}} = \mathrm{rot}\,(A\boldsymbol{t}) = -\boldsymbol{t}\times\nabla A$

$$\frac{\partial\,(A\boldsymbol{t})}{\partial t} = -\boldsymbol{v}\times(\boldsymbol{t}\times\nabla A) - \lambda\,\mathrm{rot}\,\mathrm{rot}\,(A\boldsymbol{t})\,.$$

With $\mathrm{rot}\,\mathrm{rot}\,(A\boldsymbol{t}) = \nabla\underbrace{\mathrm{div}\,(A\boldsymbol{t})}_{=0\,(\mathrm{sym.})} - \Delta\,(A\boldsymbol{t}) = -\Delta A - A/R^2$ we have

$$\frac{\partial A}{\partial t} + \boldsymbol{v}_{\mathrm{p}}\cdot\nabla A = \lambda\left(\Delta - \frac{1}{R^2}\right)A.$$

Parker's dynamo: toroidal component of induction equation

The toroidal component of the induction equation is

$$\boldsymbol{t}\frac{\partial B_\varphi}{\partial t} = \mathrm{rot}\left(\Omega R \boldsymbol{t} \times \boldsymbol{B}_\mathrm{p} + \boldsymbol{v}_\mathrm{p} \times \left(B_\varphi \boldsymbol{t}\right)\right) - \Delta\left(B_\varphi \boldsymbol{t}\right).$$

The first right hand term is after some manipulation
$\mathrm{rot}\left(\Omega R \boldsymbol{t} \times \boldsymbol{B}_\mathrm{p}\right) = \mathrm{rot}\left(\boldsymbol{t} \times \boldsymbol{B}_\mathrm{p}\right)\Omega R - \left(\boldsymbol{t} \times \boldsymbol{B}_\mathrm{p}\right) \times \nabla\left(\Omega R\right) =$
$\boldsymbol{t}\underbrace{\mathrm{div}\,\boldsymbol{B}_\mathrm{p}}_{=0} + \boldsymbol{t}\boldsymbol{B}_\mathrm{p} \cdot \nabla\left(R\Omega\right) - \underbrace{\nabla\left(R\Omega\right) \cdot \boldsymbol{t}}_{=0}\boldsymbol{B}_\mathrm{p}.$

After evaluating in components after some math the second term is
$\mathrm{rot}\left(\boldsymbol{v}_\mathrm{p} \times \boldsymbol{t} B_\varphi\right) = \mathrm{rot}\left(\boldsymbol{v}_\mathrm{p} \times \boldsymbol{t}\right) B_\varphi - \left(\boldsymbol{v}_\mathrm{p} \times \boldsymbol{t}\right) \times \nabla B_\varphi =$
$-\boldsymbol{t} R\,\mathrm{div}\left(\boldsymbol{v}_\mathrm{p}/R\right) B_\varphi - \boldsymbol{t}\,\boldsymbol{v}_\mathrm{p} \cdot \nabla B_\varphi = -\boldsymbol{t} R\,\mathrm{div}\left(\boldsymbol{v}_\mathrm{p} B_\varphi / R\right).$

As a result, the toroidal component of the induction equation is

$$\frac{\partial B_\varphi}{\partial t} + R\,\mathrm{div}\left(\frac{1}{R} B_\varphi \boldsymbol{v}_\mathrm{p}\right) = R\boldsymbol{B}_\mathrm{p} \cdot \nabla\Omega + \lambda\left(\Delta - \frac{1}{R^2}\right) B_\varphi.$$

Dynamo equations

Let us summarize that we obtained the following system of equations:

$$\frac{\partial A}{\partial t} + \boldsymbol{v}_{\mathrm{p}} \cdot \nabla A = \lambda \left(\Delta - \frac{1}{R^2} \right) A,$$

$$\frac{\partial B_\varphi}{\partial t} + R \mathrm{div} \left(\frac{1}{R} B_\varphi \boldsymbol{v}_{\mathrm{p}} \right) = R \boldsymbol{B}_{\mathrm{p}} \cdot \nabla \Omega + \lambda \left(\Delta - \frac{1}{R^2} \right) B_\varphi$$

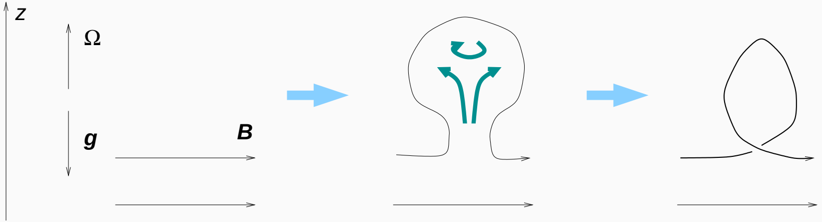
for toroidal and poloidal components of the magnetic field. The term $R\boldsymbol{B}_{\mathrm{p}} \cdot \nabla \Omega$ describes the generation of toroidal magnetic field from the poloidal field due to differential rotation (winding up of the field lines). This is the first step of the dynamo.

However, due to topological difference between these field components, there is no such term in the equation for the poloidal component. Therefore, there has to be a term propotional to B_φ to close the dynamo loop. In such case the poloidal equation reads

$$\frac{\partial A}{\partial t} + \boldsymbol{v}_{\mathrm{p}} \cdot \nabla A = \alpha B_\alpha + \lambda \left(\Delta - \frac{1}{R^2} \right) A.$$

The magic with α

The possibility how to create poloidal field from the toroidal field is the followind one. As a result of the convective motion, hot buble moves upward. The buble expands as it moves up. This leas the circular movement of the gas in the bubble due to to the action of the Coriolis force. The magnetic field follows the matter in the plume, creating the poloidal field from the toroidal one.



This does not violate the Cowling antidynamo theorem, because the motion due to the convection is not axisymmetric.

Convenient model of dynamo

Let us for simplicity study the dynamo in cartesian coordinates with axis y in toroidal direction and axis z corresponding to the rotation. We shall impose the axial symmetry $\partial/\partial z = 0$ and the vector of the differential rotation velocity $\boldsymbol{v} = (0, v(z), 0)$. In such a case

$$\boldsymbol{B} = \left(-\frac{\partial A}{\partial z}, B_y, \frac{\partial A}{\partial x}\right)$$

fulfills $\operatorname{div} \boldsymbol{B} = 0$ constraint. The induction equations then read

$$\frac{\partial B_y}{\partial t} = v' \frac{\partial A}{\partial x} + \eta \Delta B_y,$$

$$\frac{\partial A}{\partial t} = \alpha B_y + \eta \Delta A,$$

where $v' = \mathrm{d}v/\mathrm{d}z$.

Oscillatory dynamo

We will search the solution in the form of $\exp(ikx + pt)$ with constants k and p. This yields the system of algebraic equations

$$pB_y = ikv'A - \eta k^2 B_y,$$

$$pA = \alpha B_y - \eta k^2 A.$$

From the second equation we have $B_y = A/\alpha\left(p + \eta k^2\right)$, which inserting into the first equation yields $(p + \eta k^2)^2 = i\alpha v' k$, or

$$p + \eta k^2 = (1 \pm i)\sqrt{|D|}\eta k^2, \qquad \text{with } D = \frac{\alpha v'}{2\eta^2 k^3},$$

and $\pm$ depending on the sign of D. The solution has the form of

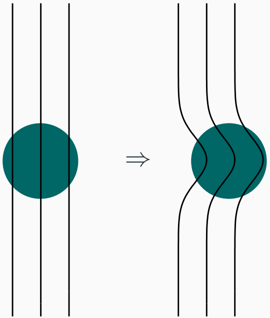
$$A \sim e^{\left[\eta k^2\left(\sqrt{|D|}-1\right)t\right]} e^{i\left[kx \pm \eta k^2 \sqrt{|D|}t\right]}.$$

Therefore, for $|D| > 1$ we obtain exponentially growing wave (with amplitude constrained due to damping) with period $2\pi/\left(\eta k^2\sqrt{|D|}\right)$. For $D > 0$ ($v' > 0$) it disseminates towards the poles, while for $D < 0$ ($v' < 0$) it disseminates towards the equator, explaining the cyclical movement of solar spots.

Magneto-rotational instability

Magneto-rotational instability: principle

The magneto-rotational instability is a possible source of anomalous viscosity in accretion disks. Let us study the stability of the material in the disk imersed in magnetic field.



The magnetic field follows the density perturbation. For a strong field ($\beta \lesssim 1$): magnetic field returns the blob to its original position. On the other hand, for weak field ($\beta \gtrsim 1$) the centrifugal force wins and the material is further accelerated leading to instability. Therefore, the magnetorotational isntability is a weak field instability.

Magneto-rotational instability: perturbations

We will study the magneto-rotational instability (MRI) in so-called Boussinesq approximation assuming constant density and introducing just density variations in the buyonancy term. We will study the perturbations in the disk with radially variable angular velocity $\Omega(R)$. We will assume that the stationary magnetic field is homogeneous and has nonzero component just in the direction perpendicular to the disk. Thus, the velocity in cylindrical coordinates is

$$\boldsymbol{v} = (\delta v_R, \Omega R + \delta v_\varphi, \delta v_z)$$

and the magnetic field

$$\boldsymbol{B} = (\delta B_R, \delta B_\varphi, B_z + \delta B_z).$$

We will search for the axisymmetric perturbations in the form of waves,

$$\delta x \sim e^{i(k_R R + k_z z - \omega t)}.$$

MRI: continuity and induction equations

Withinh our assumptions, the continuity equation $\text{div}\, \boldsymbol{v} = 0$ is

$$k_R \delta v_R + k_z \delta v_z = 0.$$

The induction equation $\partial \boldsymbol{B}/\partial t = \text{rot}\,(\boldsymbol{v} \times \boldsymbol{B})$ gives with for the R, z, and φ components (neglecting second order terms)

$$-i\omega \delta B_R = i k_z B_z \delta v_R,$$

$$-i\omega \delta B_z = -i k_R B_z \delta v_R,$$

$$-i\omega \delta B_\varphi = i k_z B_z \delta v_\varphi + \Omega R i k_z \delta B_z + (\Omega R)' \delta B_R + \Omega R i k_R \delta B_R,$$

where we used the continuity equation in the z-component. The condition $R\text{div}\, \delta \boldsymbol{B} = 0$ gives $i k_R R \delta B_R + \delta B_R + i k_z R \delta B_z = 0$, which cancels several terms in the φ-component equation giving

$$-i\omega \delta B_\varphi = i k_z B_z \delta v_\varphi + \Omega' R \delta B_R.$$

MRI: equation of motion

The perturbations in the equation of motion

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \frac{1}{\rho} \nabla \left(p + \frac{B^2}{4\pi} \right) - \boldsymbol{g} - \frac{1}{4\pi\rho} \boldsymbol{B} \cdot \nabla \boldsymbol{B} = 0$$

give for the R, z, and φ components

$$-i\omega\delta v_R - \underbrace{2\Omega\delta v_\varphi}_{v_\varphi^2/R} + \underbrace{\frac{ik_R}{\rho}\delta p - \frac{\delta\rho}{\rho^2}\frac{\partial p}{\partial R}}_{\text{Boussinesq}} + \frac{ik_R}{4\pi\rho} B_z \delta B_z - \frac{ik_z}{4\pi\rho} B_z \delta B_R = 0,$$

$$-i\omega\delta v_z + \frac{ik_z}{\rho}\delta p - \frac{\delta\rho}{\rho^2}\frac{\partial p}{\partial z} = 0,$$

$$-i\omega\delta v_\varphi + \frac{\kappa^2}{2\Omega}\delta v_R - \frac{ik_z}{4\pi\rho} B_z \delta B_\varphi = 0,$$

where the epicyclic frequency is

$$\kappa^2 = \frac{1}{R^3}\frac{\mathrm{d}\left(\Omega R^2\right)^2}{\mathrm{d}R}.$$

En route we find...

The strategy is to evaluate all perturbations in term of δv_R and insert the terms into the R-component of the momentum equation,

$$-i\omega\delta v_R - \underbrace{2\Omega\delta v_\varphi}_{v_\varphi^2/R} + \frac{ik_R}{\rho}\delta p - \frac{\delta\rho}{\rho^2}\frac{\partial p}{\partial R} + \frac{ik_R}{4\pi\rho}B_z\delta B_z - \frac{ik_z}{4\pi\rho}B_z\delta B_R = 0.$$

From the induction equation we have

$$\delta B_R = \frac{k_z B_z}{\omega}\delta v_R, \qquad \delta B_z = \frac{k_R B_z}{\omega}\delta v_R$$

and from the φ-component of equation of motion (inserting induction equation)

$$-\frac{i}{\omega}\left(\omega^2 - k_z^2 v_{\mathrm{A}z}^2\right)\delta v_\varphi + \frac{\kappa^2}{2\Omega}\delta v_R + \frac{\Omega' R}{\omega}k_z\delta B_R = 0$$

or

$$-\frac{i}{\omega}\left(\omega^2 - k_z^2 v_{\mathrm{A}z}^2\right)\delta v_\varphi + \frac{\kappa^2}{2\Omega}\delta v_R + \frac{\Omega' R}{\omega^2}k_z^2 B_z\delta v_R = 0.$$

Employing the Boussinesq approximation

Assuming the entropy conservation for gas with $\kappa = 5/3$

$$\frac{\mathrm{D} \ln \left(p\rho^{-5/3}\right)}{\mathrm{D}t} = \frac{\partial \ln \left(p\rho^{-5/3}\right)}{\partial t} + \boldsymbol{v} \cdot \nabla \ln \left(p\rho^{-5/3}\right) = 0,$$

which within the Boussinesq approximation gives

$$i\omega \frac{5}{3}\frac{\delta\rho}{\rho} + \delta v_z \frac{\partial \ln \left(p\rho^{-5/3}\right)}{\partial z} + \delta v_R \frac{\partial \ln \left(p\rho^{-5/3}\right)}{\partial R} = 0.$$

This gives from the z-component of the equation of motion

$$\frac{\delta p}{\rho} = \frac{\omega}{k_z}\delta v_z - i\frac{\delta\rho}{k_z \rho^2}\frac{\partial p}{\partial z} =$$

$$= \frac{\omega}{k_z}\delta v_z r + \frac{3}{5}\frac{1}{k_z\rho}\frac{\partial p}{\partial z}\left[\delta v_z \frac{\partial \ln \left(p\rho^{-5/3}\right)}{\partial z} + \delta v_R \frac{\partial \ln \left(p\rho^{-5/3}\right)}{\partial R}\right].$$

Dispersion relation

Denoting $\tilde{\omega}^2 = \omega^2 - k_z^2 v_{\mathrm{A}z}^2$ and inserting into the R-component of the momentum equation we finally derive the dispersion relation

$$\tilde{\omega}^4 + \frac{k_z^2}{k_R^2 + k_z^2}\left[\frac{3}{5\rho}\left(\frac{k_R}{k_z}\frac{\partial p}{\partial z} - \frac{\partial p}{\partial R}\right)\left(\frac{k_R}{k_z}\frac{\partial \ln\left(p\rho^{-5/3}\right)}{\partial z} + \frac{\partial \ln\left(p\rho^{-5/3}\right)}{\partial R}\right) - \kappa^2\right]\tilde{\omega}^2 - 4\Omega^2 \frac{k_z^2}{k_R^2 + k_z^2} v_{\mathrm{A}z}^2 = 0.$$

From the relation $p = p(\rho)$ follows

$$\frac{\partial p}{\partial R}\frac{\partial \ln\left(p\rho^{-5/3}\right)}{\partial z} = \frac{\partial p}{\partial z}\frac{\partial \ln\left(p\rho^{-5/3}\right)}{\partial R}.$$

Finally, we introduce pieces of Brunt-Väisälä frequency

$$N_R^2 = -\frac{3}{5\rho}\frac{\partial p}{\partial R}\frac{\partial \ln\left(p\rho^{-5/3}\right)}{\partial R}, \qquad N_z^2 = -\frac{3}{5\rho}\frac{\partial p}{\partial z}\frac{\partial \ln\left(p\rho^{-5/3}\right)}{\partial z}.$$

Dispersion relation

We arrive at the dispersion relation

$$\frac{k_z^2 + k_R^2}{k_z^2}\tilde{\omega}^4 - \left[\kappa^2 + \left(\frac{k_R}{k_z}N_z - N_R\right)^2\right]\tilde{\omega}^2 - 4\Omega^2 k_z^2 v_{\mathrm{A}z}^2 = 0.$$

The discriminant of the dispersion relation is always positive, therefore always $\tilde{\omega}^2$ and also ω^2 are real. Therefore, it is sufficient to investigate the instability around the root $\omega^2 = 0$, where the imaginary part of ω appears. In this case the dispersion relation becomes in terms of k_R

$$k_R^2(k_z^2 v_{\mathrm{A}z}^2 + N_z^2) - 2k_R k_z N_R N_z + k_z^2\left(\frac{\mathrm{d}\Omega^2}{\mathrm{d}\ln R} + N_R^2 + k_z^2 v_{\mathrm{A}z}^2\right) = 0.$$

Magneto-rotational instability

This equation is never fullfilled (and therefore ω^2 never passes through 0) if the discriminant is negative, that is (in terms of k_z^2)

$$k_z^4 v_{\mathrm{A}z}^4 + k_z^2 v_{\mathrm{A}z}^2 \left(N_R^2 + N_z^2 + \frac{\mathrm{d}\Omega^2}{\mathrm{d}\ln R} \right) + N_z^2 \frac{\mathrm{d}\Omega^2}{\mathrm{d}\ln R} > 0.$$

For arbitrary k_z, the condition of stability is fulfilled only if

$$\frac{\mathrm{d}\Omega^2}{\mathrm{d}R} > 0.$$

However, this never happens in real disks where $\mathrm{d}\Omega/\mathrm{d}R < 0$. Therefore, the instability, which is called the magneto-rotational instability, always appears in real disks (Fricke 1969, Balbus & Hawley 1991). The instability leads to turbulence, which is considered to be the main source of anomalous viscosity in disks.

Magneto-rotational instability: conditions

Neglecting the Brunt-Väisälä frequencies, the instability appears (the previous condition is not fulfilled) for small wavenumbers

$$k_z < k_{z,\mathrm{max}} = \frac{1}{v_{\mathrm{A}z}} \left| \frac{\mathrm{d}\Omega^2}{\mathrm{d}\ln R} \right|^{1/2}.$$

For larger wavenumbers the magnetic stresses are so large that they return the blob to its original position.

Denoting a typical vertical disk thickness as $2^{3/2}H$ with $H = aR/v_{\mathrm{K}}$, where v_{K} is the Keplerian rotation velocity and a is the sound speed, only the modes with wavelength lower than $2^{3/2}H$ can exist in the disk, giving the condition for the lowest wavelength leading to instability as $2\pi/k_{z,\mathrm{max}} < 2^{3/2}H$. Inserting the Alfvén speed with the midplane density ρ_0, the conditions for the development of instability gives

$$B_z < \frac{\sqrt{6}}{\pi} \left(\sqrt{\frac{2}{\pi}} \frac{a v_{\mathrm{K}} \dot{M}}{v_R R^2} \right)^{1/2},$$

where the disk mass-loss rate is $\dot{M} = (2\pi)^{3/2}\, v_R \rho_0 R H$.