$$\dot{y} = 2y \Rightarrow y = Ce^{2t}$$

$$\dot{y} \equiv \frac{dy}{dt} \qquad \boxed{\dot{y} = 2y} \to y = Ce^{2t}$$

$$\boxed{\dot{x} = 3x} \to x = De^{3t}$$

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x \end{aligned} \Rightarrow \begin{aligned} \dot{x} + \dot{y} &= x + y \\ \dot{x} - \dot{y} &= -(x - y) \end{aligned} \qquad \begin{aligned} u &= x + y \\ v &= x - y \end{aligned}$$

$$\begin{aligned} \dot{u} &= u \\ \dot{v} &= -v \end{aligned} \Rightarrow \begin{aligned} u &= Ce^{t} \\ v &= De^{-t} \end{aligned}$$

$$\begin{aligned} x + y &= Ce^{t} \\ x - y &= De^{-t} \end{aligned} \Rightarrow \boxed{\begin{aligned} x &= Ce^{t} + De^{-t} \\ y &= Ce^{t} - De^{-t} \end{aligned}}$$

$$\begin{aligned} \dot{x} &= 2x + y \\ \dot{y} &= 3x + 4y \end{aligned}$$

DIAGONALISACE

$$\Rightarrow M_{n \times n} \to \begin{pmatrix} a_1 & & \emptyset \\ & a_2 & \\ \emptyset & & \ddots \; a_n \end{pmatrix}$$

$M\vec{v} = \lambda\vec{v} \Leftarrow \vec{v}$ JE VLASTNÍ VEKTOR M

λ JE VLASTNÍ ČÍSLO

n RŮZNÝCH λ, v

$$T = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \cdots & \vec{v}_n \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \to \vec{v}_1 \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \to \vec{v}_2$$

$$M = TDT^{-1}$$

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ & & \lambda_3 & \\ & & & \ddots \; \lambda_n \end{pmatrix}$$

$$M\vec{v} = \lambda\vec{v} \Rightarrow M\vec{v} - \lambda\vec{v} = 0$$

$$\mathbb{1}$$

$$M\vec{v} - \lambda\mathbb{1}\vec{v} = 0$$

$$\underbrace{(M_{m\times m} - \lambda\mathbb{1}_{m\times m})}_{m\times m}\vec{v}_{m} = 0$$

$$\det(M - \lambda\mathbb{1}) = 0$$

$$\begin{pmatrix} M_{11}-\lambda & M_{12} & M_{13} & \cdots \\ M_{21} & M_{22}-\lambda & & \\ \vdots & & \ddots & \\ & & & M_{nn}-\lambda \end{pmatrix}$$

POLYNOM $\lambda = 0$

$$TDT^{-1}$$

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} \dot{x} &= 2x + y \\ \dot{y} &= 3x + 4y \end{aligned} \Rightarrow \dot{\vec{x}} = M \cdot \vec{x} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$$

$$\dot{\vec{x}} = TDT^{-1}\vec{x}$$

$$(T^{-1}\dot{\vec{x}}) = D(T^{-1}\vec{x})$$

$$M = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$$

$$M - \lambda\mathbb{1} = \begin{pmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{pmatrix}$$

$$\frac{6 \pm \sqrt{36-20}}{2} = \frac{6 \pm 4}{2} = 3 \pm 2$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) - 3$$

$$\lambda^2 - 6\lambda + 8 - 3 = \lambda^2 - 6\lambda + 5 = 0$$

$$\lambda = 1; 5 \qquad \vec{v} = (A, B)$$

$$\lambda = 1: \begin{pmatrix} 1 & 1 \\ \cancel{3} & \cancel{3} \end{pmatrix} \Rightarrow \begin{aligned} A + B &= 0 \\ B &= -A \end{aligned}$$

$$(1, -1) \qquad \vec{v} = (A, -A)$$

$$\lambda = 5: \begin{pmatrix} -3 & 1 \\ \cancel{3} & \cancel{-1} \end{pmatrix} \Rightarrow \begin{aligned} -3A + 1B &= 0 \\ B &= 3A \end{aligned}$$

$$\vec{v} = (1, 3)$$

$$T = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \qquad \begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array}$$

$$T^{-1} = \begin{pmatrix} 3/4 & -1/4 \\ 1/4 & 1/4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 4 & 1 & 1 \end{array}$$

$$\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/4 & 1/4 \end{array}$$

$$\dot{x} = 2x + y \qquad \begin{array}{cc|cc} 1 & 0 & 3/4 & -1/4 \\ 0 & 1 & 1/4 & 1/4 \end{array}$$

$$\dot{y} = 3x + 4y$$

$$\hookrightarrow \frac{3}{4}\dot{x} - \frac{1}{4}\dot{y} = \frac{6}{4}x + \frac{3}{4}y - \frac{3}{4}x - y =$$

$$\frac{3}{4}x - \frac{1}{4}y = u \qquad = \frac{3}{4}x - \frac{1}{4}y$$

$$\frac{1}{4}\dot{x} + \frac{1}{4}\dot{y} = \frac{1}{4}(5x + 5y) = 5\left(\frac{1}{4}x + \frac{1}{4}y\right)$$

$$\frac{1}{4}x + \frac{1}{4}y = v$$

$$\dot{u} = u \qquad \Rightarrow \qquad u = Ce^{t}$$
$$\dot{v} = 5v \qquad \qquad v = De^{5t}$$

$$x = Ce^{t} + De^{5t}$$
$$-y = u - 3v = Ce^{t} - 3De^{5t}$$

$$A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + \dots + A_n e^{\lambda_n t}$$

$$\dot{x} = x - y$$

$$M = \begin{pmatrix} 1 & -1 \\ -4 & 1 \end{pmatrix}$$

$$\dot{y} = y - 4x = -4x + y$$

$$0 = \begin{vmatrix} 1-\lambda & -1 \\ -4 & 1-\lambda \end{vmatrix} = (\lambda - 1)^2 - \underset{2^2}{4} =$$

$$= (\lambda - 1 + 2)(\lambda - 1 - 2) = (\lambda - 3)(\lambda + 1)$$

$$\lambda = 3_{i} \ -1$$

$$\lambda = 3: \quad e^{=}$$

$$x = Ae^{3t} \qquad \qquad 3A\cancel{e^{3t}} = (A - B)\cancel{e^{3t}}$$
$$y = Be^{3t} \qquad \Rightarrow \qquad 3B\cancel{e^{3t}} = (B - 4A)\cancel{e^{3t}}$$

$3A = A - B$ $\quad 2A + B = 0 \quad \Rightarrow B = -2A$

$3B = B - 4A \Rightarrow$ $\cancel{4A+2B=0}$

$\dot{x} = x - y$

$\dot{y} = y - 4x =$

$= -4x + y$

$\lambda = -1: \quad x = Ce^{-t}$

$y = De^{-t}$

$\Rightarrow \quad -C\cancel{e^{-t}} = C\cancel{e^{-t}} - D\cancel{e^{-t}}$

$-D\cancel{e^{-t}} = D\cancel{e^{-t}} - 4C\cancel{e^{-t}}$

$2C = D$

$\Rightarrow \quad \cancel{4C=2D}$

$x = Ae^{3t} + Ce^{-t}$

$y = \underset{-2A}{B}e^{3t} + \underset{2C}{D}e^{-t}$

$\Rightarrow \quad x = Ae^{3t} + Ce^{-t}$

$y = -2Ae^{3t} + 2Ce^{-t}$

---

1) NAJÍT λ

2) PRO KAŽDÉ λ NAPÍŠEME $A_1 e^{\lambda t}$

A SPOČÍTÁME, JAK SOUVISÍ $A_1, A_2 \ldots$

3) DÁME KUSY DOHROMADY

$\dot{x} = x - 3y$

$\dot{y} = 3x + y$

$M = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$ $\quad i^2 = -1$

$\begin{vmatrix} 1-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} = 0$

$= (\lambda - 1)^2 + 9 = 0$

$= -9$

$= -3^2 \cdot i^2$

$(\lambda-1)^2 - (3i)^2 =$

$= (\lambda - 1 + 3i)(\lambda - 1 - 3i) = 0$

$\lambda = 1 \pm 3i$

$\lambda = \underline{\underline{1 + 3i}}$ $\quad Ce^{(1+3i)t}$

$$x = Ae^{(1+3i)t} \qquad \dot{x} = x - 3y$$
$$y = Be^{(1+3i)t} \qquad \dot{y} = 3x + y$$

$$\hookrightarrow \quad (\cancel{1}+3i)\,A\cancel{e^{(1+3i)t}} = \cancel{A}\cancel{e^{(1+3i)t}} - 3B\cancel{e^{(1+3i)t}}$$
$$(\cancel{1}+3i)\,B\cancel{e^{(1+3i)t}} = 3A\cancel{e^{(1+3i)t}} + \cancel{B}\cancel{e^{(1+3i)t}}$$

$$\hookrightarrow \quad \cancel{3}iA = -\cancel{3}B \Rightarrow$$

$$A = -\frac{B}{i} = iB$$

$$\lambda = \underline{\underline{1-3i}} \quad \rightarrow x = Ce^{(1-3i)t} \Rightarrow \quad C(\cancel{1}-3i) = \cancel{C} - 3D$$
$$y = De^{(1-3i)t} \qquad D(\cancel{1}-3i) = 3C + \cancel{D}$$

$$-3iC = -3D$$
$$D = iC$$

$$x = \overset{iB}{A}e^{(1+3i)t} + Ce^{(1-3i)t}$$
$$y = Be^{(1+3i)t} + \underset{iC}{D}e^{(1-3i)t}$$

$$x = iBe^{(1+3it)} + Ce^{(1-3i)t} =$$
$$= e^t\Big(iB\underset{(\cos 3t + i\sin 3t)}{e^{3it}} + C\underset{(\cos 3t - i\sin 3t)}{e^{-3it}}\Big)$$
$$= e^t\big(iB\cos 3t - B\sin 3t + C\cos 3t - $$
$$- iC\sin 3t\big)$$
$$y = B\cdot e^{(1+3i)t} + iCe^{(1-3i)t}$$

$$\dot{x} = 3x - y \qquad M = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$$
$$\dot{y} = 4x - y$$

$$0 = \begin{vmatrix} 3-\lambda & -1 \\ 4 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) + 4 =$$
$$(\lambda-3)(\lambda+1)$$

$$= \underline{\lambda^2 - 2\lambda + 1} = (\lambda-1)^2$$

$$\underline{\lambda = 1}: \quad \vec{v_1} \quad \vec{v_1}$$
$$\vec{v_2} \quad \times$$

JORDAN

$$\begin{pmatrix} \lambda_1 & & \\ & \boxed{\begin{matrix} \lambda_2 & 1 \\ & \lambda_2 \end{matrix}} \end{pmatrix}$$

$x = Ae^t$

$y = Be^t$

e^t

$\underline{te^t}$

$\dot{x} = 3x - y$

$\dot{y} = 4x - y$

$\Rightarrow$

$x = (At+B)\,e^t$

$y = (Ct+D)\,e^t$

$\Rightarrow$

$(A + At + B)\cancel{e^t} = 3(At+B)\cancel{e^t} - (Ct+D)\cancel{e^t}$

$(C + Ct + D)\cancel{e^t} = 4(At+B)\cancel{e^t} - (Ct+D)\cancel{e^t}$

$A = 3A - C$

$C = 4A - C$ $\Rightarrow$ $2A - C = 0$ $\Rightarrow$ $\underline{C = 2A}$

$\cancel{4A - 2C = 0}$

$A+B = 3B - D$

$2A+D = C+D = 4B - D$ $\Rightarrow$ $2B - D = A$ $\Rightarrow$ $\underline{D = 2B + A}$

$\cancel{4B - 2D = 2A}$

$x = (At+B)e^t$

$y = (2At + 2B + A)e^t$

$\dot{x} = 2x - y - z$

$\dot{y} = 2x - y - 2z$

$\dot{z} = -x + y + 2z$

$$0 = \left| \begin{matrix} 2-\lambda & -1 & -1 \\ 2 & -1-\lambda & -2 \\ -1 & 1 & 2-\lambda \end{matrix} \right| \begin{matrix} 2-\lambda & -1 \\ 2 & -1-\lambda \\ -1 & 1 \end{matrix}$$

$= (2-\lambda)^2(-1-\lambda) - \cancel{2} - \cancel{2} + \cancel{1} + \lambda +$

$+ 2(\cancel{2} - \lambda) + 2(\cancel{2} - \lambda) =$

$= -(\lambda-2)^2(\lambda+1) + 5 - 3\lambda =$

$(\lambda^2 - 4\lambda + 4)$

$= -[\lambda^3 - 3\lambda^2 + 4] + 5 - 3\lambda =$

$= -[\lambda^3 - 3\lambda^2 + 4 + 3\lambda - 5] =$

$= -[\lambda^3 - 3\lambda^2 + 3\lambda - 1]$

$= -(\lambda - 1)^3$

$1 \quad -4 \quad 4$

$\quad\; 1 \quad 1$

$1 \quad -4 \quad 4$

$1 \quad -4 \quad 4$

$1 \quad -3 \quad 0 \quad 4$ ✓

$$x = (At^2 + Bt + C)e^t$$
$$y = (Dt^2 + Et + F)e^t$$
$$z = (Gt^2 + Ht + I)e^t$$

$$\begin{cases} \dot{x} = y + 2e^t \\ \dot{y} = x + t^2 \end{cases}$$

$$\frac{d}{dt}(x+y) = x+y$$
$$\frac{d}{dt}(x-y) = -(x-y)$$

$$x+y = Ae^t$$
$$x-y = Be^{-t}$$

$$\boxed{\begin{aligned} x &= Ae^t + Be^{-t} \\ y &= Ae^t - Be^{-t} \end{aligned}}$$

$A(t)\ B(t)$

$$\dot{x} = \dot{A}e^t + Ae^t + \dot{B}e^{-t} - Be^{-t}$$
$$\dot{y} = \dot{A}e^t + Ae^t - \dot{B}e^{-t} + Be^{-t}$$

$$\dot{A}e^t + \dot{B}e^{-t} + \cancel{Ae^t} - \cancel{Be^{-t}} = \cancel{Ae^t} - \cancel{Be^{-t}} + 2e^t$$
$$\dot{A}e^t - \dot{B}e^{-t} + \cancel{Ae^t + Be^t} = \cancel{Ae^t + Be^t} + t^2$$

$$2\dot{A}e^t = 2e^t + t^2 \Rightarrow \dot{A} = 1 + \tfrac{1}{2}\underline{t^2e^{-t}}$$
$$2\dot{B}e^{-t} = 2e^t - t^2 \Rightarrow \dot{B} = e^{2t} - \tfrac{1}{2}t^2e^{-t}$$

$$\int t^2e^{-t}\,dt = \begin{Vmatrix} t^2 & 2t \\ e^{-t} & -e^{-t} \end{Vmatrix} = -t^2e^{-t} + 2\int te^{-t}\,dt$$
$$= \begin{Vmatrix} t & 1 \\ e^{-t} & -e^{-t} \end{Vmatrix} = \underline{-t^2e^{-t}} - 2te^{-t} - 2e^{-t} + C$$

$$A = t - \tfrac{1}{2}t^2e^{-t} - te^{-t} - e^{-t} + C$$
$$B = \tfrac{1}{2}e^{2t} + \tfrac{1}{2}t^2e^{-t} + te^{-t} + e^{-t} + D$$

$$\dot{\vec{x}} = M\vec{x} \qquad \frac{d}{dt}x = Mx \qquad \exp M = \sum_{n=0}^{\infty} \frac{M^n}{n!}$$

$$f(x+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} \cdot h^n$$

$$e^{h\cdot\frac{d}{dx}} \cdot f(x) = \sum_{n=0}^{\infty} \frac{h^n \cdot \frac{d^n f}{dx^n}\Big|_x}{n!}$$

$$e^{t\cdot\frac{d}{dt}} \vec{x}_0 = e^{Mt} \cdot \vec{x}_0$$

$$y'' + y' - 2y = 0 \qquad y = Ae^{-2x} + Be^{x}$$

$$\lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda = -2;\ 1$$

$$\begin{vmatrix} -\lambda & 1 \\ 2 & -1-\lambda \end{vmatrix} = \lambda(\lambda+1) - 2 = \lambda^2 + \lambda - 2$$

$$\boxed{\begin{aligned} r &= y' \\ r' + r - 2y &= 0 \end{aligned}} \Rightarrow \begin{aligned} y' &= r \\ r' &= 2y - r \end{aligned}$$

$$y''' - 4y'' - 4y' + 16y = 0$$

$$\lambda^3 - 4\lambda^2 - 4\lambda + 16 = 0$$

$$\lambda = \frac{p}{q} \qquad p, q \in \mathbb{Z}$$

NESOUDĚLNÁ!

$$\frac{p^3}{q^3} - 4\frac{p^2}{q^2} - 4\frac{p}{q} + 16 = 0 \qquad q = 1$$

$$p^3 - 4p^2q - 4pq^2 + 16q^3 = 0$$

$$p(p^2 - 4pq - 4q^2) = -16q^3$$

$$p = 1, 2, 4, 8, 16$$

$\pm 1, \pm 2, \pm 4, \pm 8, \pm 16$

$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0$

JE-LI $x_0 \in \mathbb{Q}$ ŘEŠENÍ $\quad x_0 = \frac{p}{q}$ $\quad p$ JE DĚLITEL a_0, q JE DĚLITEL a_n

	1	−4	−4	+16
1	1	−3	−7	9
−1	1	−5	1	13
2	1	−2	−8	0

$\lambda^2 - 2\lambda - 8 = 0$

$\frac{2 \pm \sqrt{4+32}}{2} = 1 \pm 3 \to -2,\ 4$

$y = Ae^{2x} + Be^{-2x} + Ce^{4x}$

$y'' + 3y' + 2y = (20x + 29)e^{3x}$

$\lambda^2 + 3\lambda + 2 = 0 \Rightarrow \lambda = -2, -1$

$y_1, y_2 \qquad y = Ae^{-2x} + Be^{-x}$

$y_1 - y_2 = y_H, \Rightarrow y_1 = y_H + y_2$

$Q(x)e^{\alpha x} \qquad P(x)e^{\alpha x}$

$y_T = (Ax + B)e^{3x}$

$y_T' = (A + 3Ax + 3B)e^{3x} = (3Ax + 3B + A)e^{3x}$

$y_T'' = (3A + 3A + 9Ax + 9B)e^{3x} = (9Ax + 6A + 9B) \cdot e^{3x}$

$$(9Ax+6A+9B)\cancel{e^{3x}}+3(3Ax+3B+A)\cancel{e^{3x}}+$$

$$+2(Ax+B)\cancel{e^{3x}}=(20x+29)\cancel{e^{3x}}$$

$$(\underbrace{9A+9A+2A}_{20})x+(\overbrace{6A+9B+9B+3A+2B}^{29})=$$

$$20x+29$$

$$20A=20$$

$$A=1$$

$$9+20B=29$$

$$B=1$$

$$y=Ae^{-2x}+Be^{-x}+(x+1)e^{3x}$$

$$y=x+\tfrac{1}{2}e^{x}$$

$$y''+y=x+e^{x}$$

$$y=Ce^{x}$$

$$y=Ax+B$$

$$Ax+B=x$$

$$A=1,\ B=0$$

$$Ce^{x}+Ce^{x}=e^{x}$$

$$C=\tfrac{1}{2}$$

$$y''+y=\sin x=\frac{e^{ix}-e^{-ix}}{2i}$$