M6140 Topology Exercises - 10th Week (2020)

1 Pointless Topology

Definition 1. A *frame* is a complete lattice that satisfies the infinite distributivity law which says that $a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I}(a \wedge b_i)$ for all $a, b_i \in A$. A *homomorphism of frames* is a mapping preserving finite meets and arbitrary joins.

Exercise 1. Show that each topology is a frame.

Exercise 2. Let $f\colon (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ be a continuous mapping. Prove that $\mathcal{T}_2 \to \mathcal{T}_1$ given by $U \mapsto f^{-1}(U)$ is a homomorphism of frames.

Definition 2. A *point* of a frame A is a frame homomorphism $A \to 2$. The set of points of a frame A is denoted by $\mathrm{pt}(A)$.

Exercise 3. How is the notion from the previous definition related to the usual notion of a point in a topological space?

Definition 3. An element p of a frame A is called a *prime element* if $p \neq 1$ and for all $x, y \in A$: $x \wedge y \leq p$ implies $x \leq p$ or $y \leq p$.

Exercise 4. Show that there is a bijective correspondence between points in a frame A and prime elements in a frame A.

Exercise 5. Define a mapping $\Phi\colon A \to \mathcal{P}(\mathrm{pt}(A))$ by

$$\Phi(a) := \{p\colon A \to 2 \mid p(a) = 1\} \cong \{q \text{ prime} \mid a \not\leq q\}.$$

Show that Φ is a homomorphism of frames and conclude that the image of Φ is a topology on $\mathrm{pt}(A)$.

Remark 1. In this way we can relate the theory of frames with the theory of topological spaces and vice versa. In the words of category theory, it can be easily shown that the functor $\mathrm{pt}\colon \mathbf{Frm}^{\mathrm{op}} \to \mathbf{Top}$ is right adjoint to the functor $\Omega\colon \mathbf{Top} \to \mathbf{Frm}^{\mathrm{op}}$ which assigns to each topological space its topology.

Definition 4. A frame A is said to have *enough points* if $\Phi\colon A \to \mathcal{P}(\mathrm{pt}(A))$ is a bijection.

Exercise 6. Prove that each topology has enough points.

Exercise 7. Let A be a frame. Show that the topological space $\mathrm{pt}(A)$ is sober.

Remark 2. Combining the results above allows us to conclude that there's an equivalence of categories between the opposite of the category of frames that have enough points and the category of sober spaces.