M6140 Topology Exercises - 3rd Week (2020)

1 Separation Axioms

Exercise 1. Show that a disjoint union of T_0 spaces is T_0.

Exercise 2. Prove that a topological space is T_0 iff any two distinct points have distinct closures.

Exercise 3. Show that a topological space X is T_1 iff each singleton $\{x\}$ in X is equal to the intersection of all open neighborhoods of x.

Exercise 4. Prove that a topological space X is T_2 iff each singleton $\{x\}$ in X is equal to the intersection of all closed neighborhoods of x.

Exercise 5. Show that a closed subspace of a T_4 space is T_4.

Exercise 6. Let F_1, F_2, F_3 be a triple of closed sets in a T_4 space X such that $F_1 \cap F_2 \cap F_3 = \emptyset$. Show that there exist open sets U_1, U_2, U_3 in X such that $F_1 \subseteq U_1$, $F_2 \subseteq U_2$, $F_3 \subseteq U_3$ and $U_1 \cap U_2 \cap U_3 = \emptyset$.

Exercise 7. A topological space is called *irreducible* if it cannot be written as a union of two proper closed sets. Furthermore, a topological space is called *sober* if each irreducible closed set in it is a closure of a unique point.

(i) Consider $\mathbb{N}$ with the cofinite topology. Prove that this space is not sober. This gives us an example of a T_1 space that isn't sober.

(ii) Show that every T_2 space is sober.

(iii) Show that every sober space is T_0.

2 Compactness

Exercise 8. Show that a subspace A of a topological space X is compact iff for each collection $\{U_i \mid i \in I\}$ of open sets in X satisfying $A \subseteq \bigcup_{i \in I} U_i$ there exists a finite subcollection $\{U_j \mid j \in J\}$ such that $A \subseteq \bigcup_{j \in J} U_j$.

Exercise 9. Prove that a finite union of compact subspaces of a topological space is a compact subspace.

Exercise 10. Let A, B be disjoint compact subspaces of a Hausdorff topological space X. Show that there exist disjoint open sets U, V in X such that $A \subseteq U$ and $B \subseteq V$.

Exercise 11. A collection of subsets of a topological space is said to have a *finite intersection property* if each finite subcollection has a non-empty intersection. Prove that a topological space X is compact iff each collection of closed sets in X with finite intersection property has a non-empty intersection.

Exercise 12. Show that a continuous map out of a compact space into a T_2 space is closed and proper. (A *proper map* is a continuous map such that a pre-image of a compact subspace is a compact subspace.)

Exercise 13. (i) Suppose that $f\colon X \to Y$ is a continuous map between topological spaces whose image is dense in Y and Y is T_2. Prove that the cardinality of Y is at most $|\mathcal{P}(\mathcal{P}(X))|$.

(ii) Suppose that X is a topological space. By the previous part there exists a set I of isomorphism classes of continous maps $X \to Y$ whose image is dense, and Y is compact and T_2. For each isomorphism class choose a representative $f_i\colon X \to Y_i$. Consider the continous mapping $(f_i)_{i\in I}\colon X \to \prod_{i\in I} Y_i$ and let $\beta(X)$ be the closure of its image. The space $\beta(X)$ is called the *Stone-Čech compactification* of X. Show that $\beta(X)$ is compact and T_2.

(iii) Suppose that $f\colon X \to Y$ is a continuous map between topological spaces such that Y is compact and T_2. Prove that f uniquely factorizes through the mapping $X \to \beta(X)$.