M6140 Topology Exercises - 4th Week (2020)

1 Connectedness

Exercise 1. Is the Sierpinski space connected?

Exercise 2. Show that a non-empty topological space X is connected iff each continuous mapping $\chi\colon X \to \{0,1\}$ is constant, where the codomain has discrete topology.

Exercise 3. Is the union of two connected topological subspaces necessarily connected? What about the intersection of two connected topological subspaces?

Exercise 4. Prove that the closure of a connected subspace is connected.

Exercise 5. Show that discrete spaces are totally disconnected.

Exercise 6. Let $\sim$ be an equivalence on a topological space X. Suppose that $X/{\sim}$ is connected and that each equivalence class is connected. Prove that X is connected.

Exercise 7. Show that a locally constant map whose domain is connected and whose codomain is T_1 is necessarily constant.

Exercise 8. Let x and y be points of a topological space X. A *chain from x to y* in a cover $\mathcal{U}$ of X is a finite sequence $U_1, \dots, U_n \in \mathcal{U}$ such that the intersection of each pair of consecutive U_i's is non-empty and $x \in U_1$, $y \in U_n$. Show that a topological space is connected if and only if for each open cover $\mathcal{U}$ there exists a chain in $\mathcal{U}$ between each pair of points of X.

Exercise 9. Let $f\colon X \to Y$ be a continuous mapping, where X is a connected topological space and Y is a totally ordered set with the order topology[1]. Suppose that $x, y \in X$ and $r \in Y$ is such that $f(x) < r < f(y)$, then there exists $z \in X$ such that $f(z) = r$. This is the *intermediate value theorem.*

[1] The *order topology* is given by the base of all the "intervals" (x,y), $(-\infty, y)$, $(x, +\infty)$, where $x, y \in Y$