M6140 Topology Exercises - 9th Week (2020)

1 Uniform Spaces

Exercise 1. Suppose that $\mathcal{U}, \mathcal{V}$ are uniformities on X and $\mathcal{U}_1, \mathcal{V}_1$ are their bases, respectively. Prove that $\mathcal{U} \subseteq \mathcal{V}$ if and only if for each $U \in \mathcal{U}_1$ there exists $V \in \mathcal{V}_1$ such that $V \subseteq U$.

Exercise 2. Show that a non-empty set $\mathcal{V} \subseteq X \times X$ is a basis of some uniformity if and only if

(i) $\Delta_X \subseteq U$ for each $U \in \mathcal{V}$,

(ii) for each $U, V \in \mathcal{V}$ there exists $W \in \mathcal{V}$ such that $W \subseteq U \cap V$,

(iii) for each $U \in \mathcal{V}$ there exists $V \in \mathcal{V}$ such that $V^{-1} \subseteq U$, and

(iv) for each $U \in \mathcal{V}$ there exists $V \in \mathcal{V}$ such that $V \circ V \subseteq U$.

Exercise 3. Let $(X, \mathcal{U})$, $(Y, \mathcal{V})$ be uniform spaces with bases $\mathcal{U}_1$, $\mathcal{V}_1$, respectively. Prove that a mapping $f \colon X \to Y$ is uniformly continuous if and only if for each $V \in \mathcal{V}_1$ there exists $U \in \mathcal{U}_1$ such that $U \subseteq (f \times f)^{-1}(V)$.

Definition 1. A non-empty set $\mathcal{V} \subseteq X \times X$ is called a *subbasis* of some uniformity if the set of its finite intersections is a basis of the uniformity.

Exercise 4. Show that a non-empty set $\mathcal{S} \subseteq X \times X$ is a subbasis of some uniformity if and only if

(i) $\Delta_X \subseteq U$ for each $U \in \mathcal{S}$,

(ii) for each $U \in \mathcal{S}$ there exists $V \in \mathcal{S}$ such that $V^{-1} \subseteq U$, and

(iii) for each $U \in \mathcal{S}$ there exists $V \in \mathcal{S}$ such that $V \circ V \subseteq U$.

Exercise 5. Prove that a mapping between metric spaces is uniformly continuous in the sense of metric spaces if and only if it is uniformly continuous in the sense of uniform spaces.

Exercise 6. Let G be a topological group. Show that

$$\Big\{\{(x, y) \in G \times G \mid x \cdot y^{-1} \in V\} \mid V \text{ is a neighborhood of } 1.\Big\}$$

is a basis of some uniformity on G.