It is seen that when $p = q$ and $c = 10$ and both players in the gambling game start with the same capital, the expected duration of the game is 25 rounds. If the total capital is $c = 1000$ and is equally shared by the two players to start with, then the average duration of their game is 250 000 rounds!

Finally we note that when $c = \infty$, the expected times to absorption are

$$\boxed{D_a = \begin{cases} \dfrac{a}{q-p}, & p < q \\ \infty, & p \geq q \end{cases}} \tag{7.24}$$

as will be proved in Exercise 13.

7.8 SMOOTHING THE RANDOM WALK – THE WIENER PROCESS AND BROWNIAN MOTION

In Fig. 7.8a are shown portions of two possible sample paths of a simple unrestricted random walk with steps up or down of equal magnitudes. The illustrations in Fig. 7.8b–f were obtained by successive reductions of Fig. 7.8a. In (a), the 'steps' are discernible, but after several reductions the paths become smooth in appearance. In terms of the position and time scales in (a), the steps in (f) are very small and so is the time between them. The point of this is to illustrate that paths may be discontinuous but appear quite smooth when viewed from a distance.

Consider the time interval $(0, t]$. Subdivide this into subintervals of length Δt so that there are $t/\Delta t$ such subintervals. We now suppose that a particle, initially at $x = 0$, makes a step (in one space dimension) at the times $\Delta t, 2\Delta t, \ldots,$ and that the size of the step is either $+\Delta x$ or $-\Delta x$, the probability being 1/2 that the move is to the left or the right. Thus the position of the particle, $X(t)$, at time t, is a random walk which has executed $t/\Delta t$ steps. Since the position will depend on the choice of Δt and Δx, we write the position as $X(t; \Delta t, \Delta x)$.

We may write

$$X(t; \Delta t, \Delta x) = \sum_{i=1}^{t/\Delta t} Z_i, \tag{7.25}$$

where the Z_i are independent and identically distributed with

$$\Pr[Z_i = +\Delta x] = \Pr[Z_i = -\Delta x] = 1/2, \qquad i = 1, 2, \ldots.$$

For the Z_i we have,

$$E[Z_i] = 0,$$

and

$$\operatorname{Var}[Z_i] = E[Z_i^2] = (\Delta x^2).$$

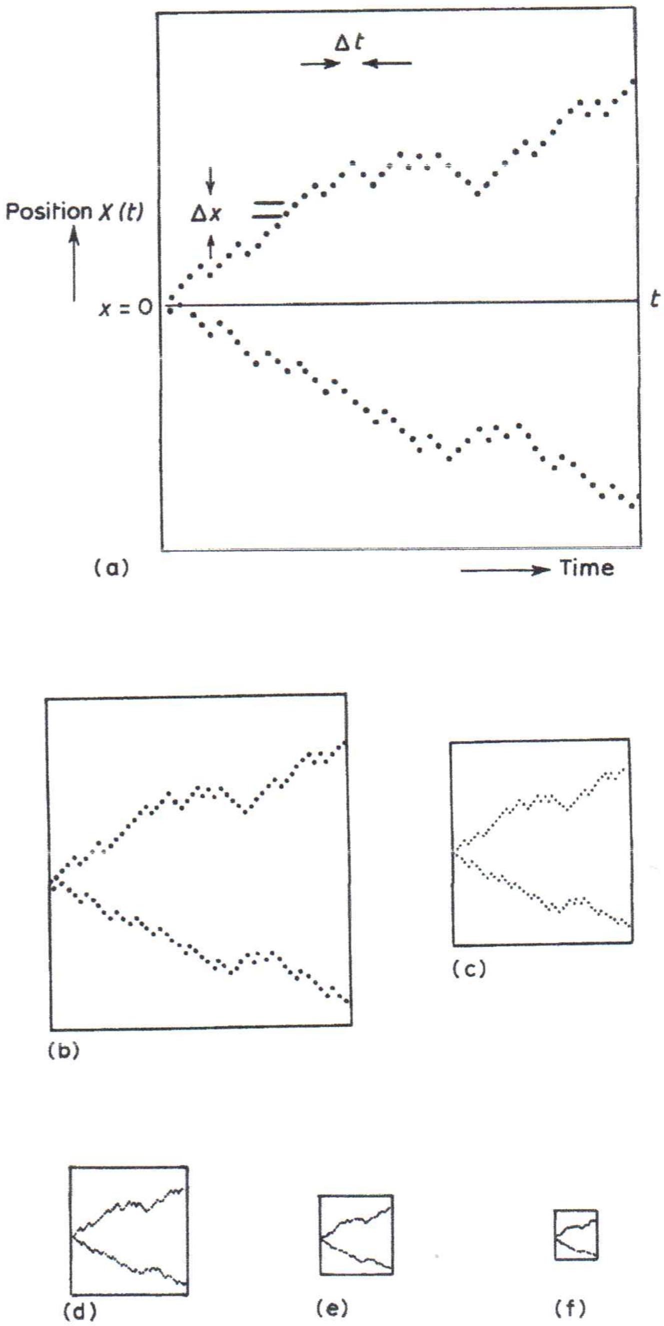

Figure 7.8 In (a) are shown two sample paths of a random walk, (b) to (f) were obtained by successive reductions of (a).

From (7.25) we get

$$E[X(t;\Delta t,\Delta x)]=0,$$

and since the Z_i are independent,

$$\mathrm{Var}[X(t;\Delta t,\Delta x)]=(t/\Delta t)\,\mathrm{Var}[Z_i]=\frac{t(\Delta x)^2}{\Delta t}.$$

Now we let Δt and Δx get smaller so the particle moves by smaller amounts but more often. If we let Δt and Δx approach zero we won't be able to find the limiting variance as this will involve zero divided by zero, unless we prescribe a relationship between Δt and Δx.

A convenient choice is $\Delta x=\sqrt{\Delta t}$ which makes $\mathrm{Var}[X(t;\Delta t,\Delta x)]=t$ for all values of Δt. In the limit $\Delta t\to 0$ the random variable $X(t;\Delta t,\Delta x)$ converges in distribution to a random variable which we denote by $W(t)$. From the central limit theorem (Chapter 6) it is clear that $W(t)$ is normally distributed. Furthermore,

$$E[W(t)]=0$$
$$\mathrm{Var}[W(t)]=t.$$

The collection of random variables $\{W(t),t\geqslant 0\}$, indexed by t, is a continuous process in continuous time called a **Wiener process** or **Brownian motion**, though the latter term also refers to a physical phenomenon (see below).

The Wiener process (named after Norbert Wiener, celebrated mathematician, 1894–1964) is a fascinating mathematical construction which has been much studied by mathematicians. Though it might seem just an abstraction, it has provided useful mathematical approximations to random processes in the real world. One outstanding example is Brownian motion. When a small particle is in a fluid (liquid or gas) it is buffeted around by the molecules of the fluid, usually at an astronomical rate. Each little impact moves the particle a tiny amount. You can see this if you ever watch dust or smoke particles in a stream of sunlight. This phenomenon, the erratic motion of a particle in a fluid, is called Brownian motion after the English botanist Robert Brown who observed the motion of pollen grains in a fluid under a light microscope. In 1905, Albert Einstein obtained a theory of Brownian motion using the same kind of reasoning as we did in going from random walk to Wiener process. The theory was subsequently confirmed by the experimental results of Perrin. For further reading on the Wiener process see, for example, Parzen (1962), and for more advanced aspects, Karlin and Taylor (1975) and Hida (1980).

Random walks have also been employed to represent the voltage in nerve cells (neurons). A step up in the voltage is called **excitation** and a step down is called **inhibition**. Also, there is a critical level (threshold) of excitation of which