

Exercise sessions for the Category Theory course, fall semester 2020-21

Giulio Lo Monaco

Week 1

Exercise Define **Rel** as follows: its objects are sets, and a morphism $A \rightarrow B$ is a relation, that is, a subset $R \subseteq A \times B$. Moreover, define the identity on A to be the diagonal relation $\{(a, a) \in A \times A \mid a \in A\}$, and the composition of $R \subseteq A \times B$ and $S \subseteq B \times C$ as

$$S \circ R = \{(a, c) \in A \times C \mid \exists b \in B \text{ such that } (a, b) \in R \text{ and } (b, c) \in S\}.$$

Prove that **Rel** is a category.

Solution First prove the identity axiom. We need to show that for any $R \subseteq A \times B$, then $\text{id}_B \circ R = R$. The dual is similar. Now

$$\text{id}_B \circ R = \{(a, b) \mid \exists b' \in B \text{ such that } (a, b') \in R \text{ and } (b', b) \in \text{id}_B\}$$

but $(b', b) \in \text{id}_B$ if and only if $b' = b$ so the condition reduces to $\exists b \in B$ such that $(a, b) \in R$. This is precisely R .

Now we show associativity: choose $R \subseteq A \times B$, $S \subseteq B \times C$ and $T \subseteq C \times D$ and compute

$$\begin{aligned} (T \circ S) \circ R &= \{(a, d) \in A \times D \mid \exists b \in B \text{ such that } (a, b) \in R \text{ and } (b, d) \in T \circ S\} \\ &= \{(a, d) \in A \times D \mid \exists b \in B, \exists c \in C \text{ such that } (a, b) \in R, (b, c) \in S \text{ and } (c, d) \in T\} \\ &= \{(a, d) \in A \times D \mid \exists c \in C, \exists b \in B \text{ such that } (a, b) \in R, (b, c) \in S \text{ and } (c, d) \in T\} \\ &= \{(a, d) \in A \times D \mid \exists c \in C \text{ such that } (a, c) \in S \circ R \text{ and } (c, d) \in T\} \\ &= T \circ (S \circ R) \end{aligned}$$

Examples Other commonly used categories:

- **Grp**, objects are groups and morphisms are group homomorphisms;
- **Ab**, objects are abelian groups and morphisms are group homomorphisms;
- **Top**, objects are topological spaces and morphisms are continuous maps;
- **Met**, objects are metric spaces and morphisms are contractions;

- **Haus**, objects are Hausdorff spaces and morphisms are continuous maps;
- **Pos**, objects are partially ordered sets and morphisms are order-preserving maps.

Example In all the examples above, objects are defined as sets with additional structure put on them. Morphisms are functions between the underlying sets which preserve the additional structure. Can we do something similar for categories? We would like to define a “morphism” between categories as a function on objects plus a function on morphisms in a way that the structure of category is preserved.

Definition A functor $F : \mathcal{A} \rightarrow \mathcal{B}$ between categories consists of a function $F_0 : \text{ob}\mathcal{A} \rightarrow \text{ob}\mathcal{B}$ and another function $F_1 : \text{ar}\mathcal{A} \rightarrow \text{ar}\mathcal{B}$ in such a way that for each objects $A, B, C \in \mathcal{A}$ and morphisms $f : A \rightarrow B$ and $g : B \rightarrow C$

- $\text{dom}F_1(f) = F_0(A)$;
- $\text{cod}F_1(f) = F_0(B)$;
- $F_1(g) \circ F_1(f) = F_1(g \circ f)$;
- $F_1(\text{id}_A) = \text{id}_{F_0(A)}$.

Example Let P be a partially ordered set. Then we can define a category \tilde{P} as follows: the objects of \tilde{P} are exactly the points of P . For two $p, q \in P$, the set of morphisms from p to q will be a singleton if $p \leq q$, empty otherwise. In other words

- $\tilde{P}(p, q) = \{(p, q)\}$ whenever $p \leq q$;
- $\tilde{P}(p, q) = \emptyset$ whenever $p \not\leq q$.

If P and Q are posets, then a functor $\tilde{P} \rightarrow \tilde{Q}$ corresponds precisely with an order-preserving map $P \rightarrow Q$.

Example Let M be a monoid. Then we can define a category \mathbf{BM} as follows: $\text{ob}\mathbf{BM} = \{*\}$, that is, there is just one object, so we only need to specify one set of morphisms, and we set $\mathbf{BM}(*, *) = M$. Composition and identity are given by:

- $\forall m, n \in M, n \circ m = mn$
- $\text{id}_* = 1_M$.

This gives indeed the structure of a category. Moreover, if M and N are monoids, then a functor $\mathbf{BM} \rightarrow \mathbf{BN}$ corresponds precisely with a monoid homomorphism $M \rightarrow N$.

Exercise Show that monomorphisms and epimorphisms in **Set** correspond respectively to injections and surjections.

Solution We need to show $\text{Mono} \subseteq \text{Inj}$, $\text{Inj} \subseteq \text{Mono}$, $\text{Epi} \subseteq \text{Surj}$ and $\text{Surj} \subseteq \text{Epi}$.

Take a monomorphism $f : A \rightarrow B$. Consider two points $a, a' \in A$ such that $f(a) = f(a')$. If $*$ is a singleton, there are two maps $g_a, g_{a'} : * \rightarrow A$ selecting a and a' respectively. By hypothesis, $f g_a = f g_{a'}$ which implies $g_a = g_{a'}$ since f is monic. But this means that g_a and $g_{a'}$ select the same point, so that $a = a'$.

Now suppose that f is injective, and choose maps

$$C \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{h} \end{array} A \xrightarrow{f} B.$$

such that $fg = fh$. This means that for every element $c \in C$ we have $fg(c) = fh(c)$. By injectivity of f this implies $g(c) = h(c)$.

Now assume f is epic. Consider the set $B/f(A)$ of equivalence classes of points of B under the relation $b \sim b'$ if and only if either both are in the image of f , or $b = b'$. Let us now take the maps

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{p} \\ \xrightarrow{c} \end{array} B/f(A)$$

where p is the projection on the quotient and c is constant on the equivalence class determined by $f(A)$. Clearly, we have $pf = cf$. Since f is epic, this implies $p = c$ which means that every point of B is in $f(A)$. Thus f is surjective.

Finally, assume that f is surjective, and consider two maps $g, h : B \rightarrow C$ such that $gf = hf$. For each $b \in B$, by surjectivity of f there is some $a \in A$ such that $f(a) = b$, so we can compute

$$g(b) = gf(a) = hf(a) = h(b)$$

which means that $g = h$. We are finished.

Exercise Let \mathbf{Mon} be the category of monoids and monoid homomorphisms. Prove that in \mathbf{Mon} the inclusion $\mathbb{N} \rightarrow \mathbb{Z}$ is an epimorphism.

Solution Consider two morphisms $f, g : \mathbb{Z} \rightarrow M$ such that $f(n) = g(n)$ whenever n is a natural number. It remains to show that the equality holds also for negative integers. Observe that if $h : P \rightarrow Q$ is any monoid homomorphism and $p \in P$ has an inverse $-p$, then $h(-p) = -h(p)$. Using this fact, we compute

$$f(-n) = -f(n) = -g(n) = g(-n)$$

in our case, which completes the proof that $f = g$.

Definition We say that a category \mathcal{C} is balanced if whenever a morphism is both a monomorphism and an epimorphism then it is an isomorphism.

Example We (almost) proved above that the category **Set** is balanced. Indeed, if a morphism is both mono and epi, then it is an injection and a surjection, which means it is a bijection of sets. Isomorphisms in **Set** are precisely bijections.

We have also seen that the inclusion $\mathbb{N} \rightarrow \mathbb{Z}$ is an epimorphism in **Mon**. Moreover, it is a monomorphism, because monomorphisms in **Mon** are precisely injective homomorphisms. So **Mon** is not balanced.

For a similar reason (for instance, take the inclusion $\mathbb{Z} \rightarrow \mathbb{Q}$) the category **CRng** of commutative rings and ring homomorphisms is not balanced.

Example For a poset P , the category $\overline{\tilde{P}}$ is not balanced. Indeed, every morphism in $\overline{\tilde{P}}$ is both monic and epic, but the only isomorphisms are identities.

Example There is an inclusion **Haus** \subset **Top**. We have that a morphism which is epic in **Haus** is not necessarily so in **Top**.

Exercise Prove that epimorphisms in **Haus** are exactly dense maps, and say why this is not the case in **Top**.

Solution Let $f : X \rightarrow Y$ be a dense map of Hausdorff spaces, and let $g, h : Y \rightarrow Z$ be two other continuous maps of Hausdorff spaces such that $gf = hf$. Let us choose any point $y \in Y$. Since f is dense, there is a sequence of points in X such that $\lim_{n \rightarrow +\infty} f(x_n) = y$. Moreover, since g and h are continuous, they preserve limits. So we have

$$g(y) = g(\lim f(x_n)) = \lim gf(x_n) = \lim hf(x_n) = h(\lim f(x_n)) = h(y).$$

Conversely, suppose that f is an epimorphism. If $\overline{f(X)}$ is the closure of $f(X)$ in Y , then we take the disjoint union of two copies of Y and put an equivalence relation on it as follows: two points (x, i) and (y, j) are equivalent if either $(x, i) = (y, j)$ or $x = y \in \overline{f(X)}$. In other words, we are gluing the two copies along the common subspace $\overline{f(X)}$. We call the quotient Q , and observe that it is a Hausdorff space. In particular there are two maps $p_1, p_2 : Y \rightarrow Q$, given by the projection on the quotient of the two distinct copies of Y . Clearly, we have $p_1 f = p_2 f$ so that by assumption we obtain $p_1 = p_2$. Since the two maps are only equal on $\overline{f(X)}$, this means that $Y = \overline{f(X)}$, which is exactly saying that f is dense. In **Top**, the first direction doesn't work because limits are not unique even when they exist in general topological spaces. The second direction actually works, but in addition we may consider the two maps $p, c : Y \rightarrow Y/\overline{f(X)}$ just as we did in the case of sets, which is not possible in **Haus** because that quotient is in general not a Hausdorff space, and this gives strict surjectivity.

Exercise Prove that neither **Top** nor **Haus** are balanced.

Solution In **Top**, consider for instance this map: for a non-trivial set X , call X_d the discrete space and X_i the indiscrete space on X , and take the identity map $X_d \rightarrow X_i$. This is continuous and bijective (therefore both monic and epic), but it is not an isomorphism because its set-theoretical inverse is not continuous.

In **Haus**, consider for instance the inclusion $\{\frac{1}{n}\}_{n \in \mathbb{N}} \rightarrow \{\frac{1}{n}\}_{n \in \mathbb{N}} \cup \{0\}$, with the canonical topology. This is dense, therefore epic, and an inclusion, therefore monic, but not an isomorphism since not surjective.

Exercise What are the initial and terminal objects of **Set**, **Mon**, **Grp**, **Top**, **CRng**, \tilde{P} , **Rel**?

Solution In **Set**, the initial object is \emptyset , because for any other set X the only map $\emptyset \rightarrow X$ is the vacuous map. A terminal object is any singleton $*$, because for any other map X the only map $X \rightarrow *$ is the map sending every element of X to the only point of $*$.

In **Mon**, the one-point monoid $\mathbf{0}$ is both initial and terminal. In particular, it is initial because every monoid homomorphism preserves the unit 0.

In **Grp**, the one-point group $\mathbf{0}$ is both initial and terminal, as above.

In **Top**, the initial and terminal object are as in **Set**, that is, the empty space and the one-point space.

In **CRng**, the terminal object is the one-point ring $\mathbf{0}$, as above. The initial object is \mathbb{Z} . To see this, observe that, if R is any commutative ring, then a ring homomorphism $f : \mathbb{Z} \rightarrow R$ is required to send 0 to 0_R and 1 to 1_R . Moreover, f preserves sums, so that for every $n \in \mathbb{Z}^+$ we have

$$f(n) = f(1 + \dots + 1) = f(1) + \dots + f(1) = 1_R + \dots + 1_R$$

and finally it preserves additive inverses, so that we have a similar formula for negative integers. In other words, if a homomorphism f exists, then it is forced to be of the form above. We conclude by saying that this definition of f gives indeed a ring homomorphism because it clearly preserves sums and, since products in \mathbb{Z} can be expressed in terms of sums, it also preserves products.

In \tilde{P} , an initial and a terminal object correspond respectively to a minimum element and to a maximum element.

In **Rel**, \emptyset is both initial and terminal. To see this, it suffices to observe that for any other set A , the equalities

$$A \times \emptyset = \emptyset = \emptyset \times A$$

which only has the trivial subset.

Definition We define the dual of a category \mathcal{C} as follows: the objects of \mathcal{C}^{op} are exactly those of \mathcal{C} ; for two objects $A, B \in \mathcal{C}$, the set of morphisms $A \rightarrow B$ in \mathcal{C}^{op} is exactly the set of morphisms $B \rightarrow A$ in \mathcal{C} . We write:

- $\text{ob}\mathcal{C}^{op} = \text{ob}\mathcal{C}$;
- $\forall A, B \in \text{ob}\mathcal{C}^{op}, \mathcal{C}^{op}(A, B) = \mathcal{C}(B, A)$.

Week 2

Exercise Show that the following are equivalent for an arrow $f : A \rightarrow B$:

1. f is an isomorphism;
2. f is a monomorphism and a split epimorphism;
3. f is a split monomorphism and an epimorphism;
4. f is a split monomorphism and a split epimorphism.

Solution If f is iso, this trivially implies all the other conditions. Assume that f is mono and split epi. Then there is a map $s : B \rightarrow A$ such that $fs = \text{id}_B$. Therefore, we also have $f s f = f$ which, since f is mono, implies $s f = \text{id}_A$, and f is iso. The case of split mono + epi is analogous.

Solution Show that all sets are projective.

Solution Choose a diagram

$$\begin{array}{ccc} & & A \\ & & \downarrow f \\ X & \xrightarrow{h} & B \end{array}$$

where f is epi. For each $x \in X$, we choose an element $a \in A$ such that $f(a) = h(x)$. This defines a function $g : X \rightarrow A$ such that $fg = h$.

Exercise Show that a retract of a projective object is also projective.

Solution Let X be projective, and Y a retract of X , so that there are maps $r : X \rightarrow Y$ and $s : Y \rightarrow X$ with $rs = \text{id}_Y$. Suppose we have a diagram

$$\begin{array}{ccc} & & A \\ & & \downarrow f \\ X & \xrightarrow{r} & Y \xrightarrow{h} B \end{array}$$

Since X is projective, there exists a lift $g' : X \rightarrow A$ such that $fg' = hr$. Now we have $fg's = hrs = h$, so that $g's$ is the desired lift.

Exercise For an arrow $f : A \rightarrow B$ in a category, show that the arrow $(\text{id}_A, f) : A \rightarrow A \times B$ is a monomorphism.

Solution Consider two arrows $g, h : D \rightarrow A$ such that $(\text{id}_A, f) \circ g = (\text{id}_A, f) \circ h$. By universal property of the product we know that $\pi_A \circ (\text{id}_A, f) = \text{id}_A$, where $\pi_A : A \times B \rightarrow A$ is the projection from the product. In particular, postcomposing with π_A we obtain $g = h$.

Example Given a family of objects $(A_i)_{i \in I}$ in a category \mathcal{C} , take their product $\prod_{i \in I} A_i$, with projections $\pi_j : \prod_{i \in I} A_i \rightarrow A_j$. Given any other object B , we can evaluate the functor $\mathcal{C}(B, -)$ at these maps, obtaining

$$\mathcal{C}(B, \pi_j) : \mathcal{C}(B, \prod_{i \in I} A_i) \rightarrow \mathcal{C}(B, A_j).$$

The universal property of products in **Set** then yields a map

$$\phi : \mathcal{C}(B, \prod_{i \in I} A_i) \rightarrow \prod_{i \in I} \mathcal{C}(B, A_i).$$

The universal property of products in \mathcal{C} says exactly that ϕ is an isomorphism of sets.

Exercise Show that the forgetful functor $U : \mathbf{Mon} \rightarrow \mathbf{Set}$, taking every monoid to its underlying set, is representable. Infer that it preserves products.

Solution For a monoid M , consider the set $\mathbf{Mon}(\mathbb{N}, M)$. We will prove that this is isomorphic to M . To a morphism $f : \mathbb{N} \rightarrow M$ we will associate $f(1) \in M$. Viceversa, for an element $m \in M$, we define a homomorphism by imposing $f(1) = m$. By homomorphism property, necessarily $f(0) = 0_M$ and $f(n) = m + \dots + m$. Since this is always a well-defined homomorphism and it is the only possible choice, we are done. So U is represented by the object \mathbb{N} .

By the example above, this implies that U preserves products.

Definition Given two categories \mathcal{C} and \mathcal{D} and two functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$, a natural transformation $\alpha : F \Rightarrow G$ is an assignment of an arrow $\alpha_C : F(C) \rightarrow G(C)$ in \mathcal{D} for every object $C \in \mathcal{C}$, in such a way that, for every morphism $f : C \rightarrow D$ in \mathcal{C} , the square

$$\begin{array}{ccc} F(C) & \xrightarrow{F(f)} & F(D) \\ \downarrow \alpha_C & & \downarrow \alpha_D \\ G(C) & \xrightarrow{G(f)} & G(D) \end{array}$$

commutes.

Example Many processes give rise to natural transformations:

- Given a functor $F : \mathcal{C} \rightarrow \mathcal{D}$, the identity morphism $F(C) \rightarrow F(C)$ is a natural transformation $\text{id} : F \Rightarrow F$;
- the projection from a group onto its abelianization $G \rightarrow G^{ab}$ is a natural transformation $\text{id}_{\mathbf{Grp}} \Rightarrow (-)^{ab}$;
- the diagonal map $X \rightarrow X \times X$ of sets, groups, spaces and many other structures is a natural transformation from the identity functor to the functor sending X to $X \times X$ and f to $f \times f$;
- the inclusion $X \rightarrow X \coprod \{*\}$ of a set, or topological space, into the set (space) obtained by adjoining one point is a natural transformation $\text{id} \Rightarrow (-) \coprod \{*\}$;
- given a set A and a category \mathcal{C} , we can take an arbitrary functor $F : A \rightarrow \mathcal{C}$ (where A is regarded as a category only containing identity morphisms). Moreover, choosing an object $C \in \mathcal{C}$, there is a constant functor $\delta_C : A \rightarrow \mathcal{C}$. Then a natural transformation $F \Rightarrow \delta_C$ corresponds to the choice of morphisms $F(a) \rightarrow C$, for every $a \in A$;
- later on we will see many other useful examples in the theory of limits.

Definition A natural isomorphism $\alpha : F \Rightarrow G$ is a natural transformation where each of the components α_C is an isomorphism.

Example The classical example of natural isomorphism is that from a vector space to the double dual. The components $\phi_V : V \rightarrow V^{**}$ assemble to form a natural transformation $\text{id} \Rightarrow (-)^{**}$.

Definition Given two categories \mathcal{C} and \mathcal{D} , we say that two functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ form an equivalence if there are natural isomorphisms $\text{id}_{\mathcal{C}} \Rightarrow GF$ and $\text{id}_{\mathcal{D}} \Rightarrow FG$.

Definition We define a category **CABA**. A complete Boolean algebra is a poset A that has all meets and all joins and such that for every element $a \in A$ there is an element $\neg a \in A$ such that $a \vee \neg a = 1$ and $a \wedge \neg a = 0$. An atom is an element that is minimal in $A \setminus \{0\}$ with respect to the order relation. A complete Boolean algebra is atomic if each element $b \in A$ can be expressed as $b = \bigvee_{i \in I} a_i$, where a_i 's are atoms. A morphism of complete atomic Boolean algebras is a function that preserves order, meets and joins.

Exercise Let B be a complete atomic Boolean algebra. Prove that if a is an atom, then $a \not\leq b \Leftrightarrow a \wedge b = 0$.

Solution Observe that, for general a, b , we have $a \leq b \Leftrightarrow a \wedge b = a$. Now compute

$$a \not\leq b \Leftrightarrow a \wedge b \neq a \Leftrightarrow a \wedge b = 0$$

where the first step is the observation above, the second is true because a is an atom.

Exercise Prove that if B is a complete atomic Boolean algebra, $S \subseteq B$ and $a \in B$ is an atom, then $a \leq \bigvee S$ implies that there is an element $b \in S$ such that $a \leq b$.

Solution Suppose by contradiction that for all $b \in S$ then $a \not\leq b$. By the previous exercise, we have that for all $b \in S$, $a \wedge b = 0$. Then we compute

$$a \wedge \bigvee S = \bigvee_S (a \wedge b) = \bigvee_S 0 = 0$$

which is a contradiction because we know that $a \wedge \bigvee S = a$.

Exercise Prove that if $f : A \rightarrow B$ is a morphism of complete atomic Boolean algebras, and $b \in B$ is an atom, then there is a unique atom $a \in A$ such that $b \leq f(a)$.

Solution Since A is atomic, we have $1_A = \bigvee \{a \in A \mid a \text{ is an atom}\}$. Since f preserves all joins, we also have $1_B = f(1_A) = \bigvee \{f(a) \in B \mid a \in A \text{ is an atom}\}$. Now, if $b \in B$ then $b \in 1_B$. Taking $S = \{f(a) \in B \mid a \in A \text{ is an atom}\}$, then we can use the previous exercise to conclude that there is an atom $a \in A$ such that $b \leq f(a)$. It remains to show that this atom is unique. Suppose that there is another atom a' with the property that $b \leq f(a')$, then

$$b \leq f(a) \wedge f(a') = f(a \wedge a') = f(0) = 0$$

which is impossible. Therefore, there is only one atom with this property.

Exercise Prove that there is an equivalence between the category \mathbf{Set}^{op} of sets and opposite functions and the category \mathbf{CABA} of complete atomic Boolean algebras.

Solution We need two functors $\mathbf{Set}^{op} \rightarrow \mathbf{CABA}$ and $\mathbf{CABA} \rightarrow \mathbf{Set}^{op}$. For each set X , take the power set $\mathcal{P}X$, and for a function $f : X \rightarrow Y$, take the inverse image $f^{-1} : \mathcal{P}Y \rightarrow \mathcal{P}X$. We verify that this is a well-defined functor $\mathcal{P} : \mathbf{Set}^{op} \rightarrow \mathbf{CABA}$. For each set, its power set is a poset via the inclusion relation and, moreover, joins are given by unions and meets are given by intersections. Atoms are precisely singleton. In fact, there are no subsets between \emptyset and a singleton, and if a subset S is not a singleton, we can always take out an element to obtain a subset $S \setminus \{s\}$ which sits between \emptyset and S . Now, every subset $S \subset X$ is given by the union $\bigcup_{x \in X} \{x\}$, so $\mathcal{P}X$ is also atomic. The inverse image maps preserves inclusions, unions and intersections, so it is a morphism in \mathbf{CABA} . Functoriality is readily verified.

For the next step, we need a functor $\mathcal{A}t : \mathbf{CABA} \rightarrow \mathbf{Set}^{op}$. For a complete atomic Boolean algebra B , $\mathcal{A}t B$ will be the set of atoms of B . For a morphism $h : A \rightarrow B$ in \mathbf{CABA} , we define $\mathcal{A}t h : \mathcal{A}t B \rightarrow \mathcal{A}t A$ as follows: given an atom $b \in B$, by the previous exercise there is a unique atom $a \in A$ such that $b \leq h(a)$. Now we set $\mathcal{A}t h(b) = a$. Functoriality follows by the fact that there is always only one possible such atom.

Now we define two natural isomorphisms $\phi : \text{id}_{\mathbf{Set}^{op}} \Rightarrow \mathcal{A}t \mathcal{P}$ and $\psi : \mathcal{P} \mathcal{A}t \Rightarrow \text{id}_{\mathbf{CABA}}$. For a set X , $\mathcal{A}t \mathcal{P}X$ is the set of all singletons of elements of X . Now $\phi_X : X \rightarrow \mathcal{A}t \mathcal{P}X$ is the map which sends a point x to $\{x\}$. This is clearly an isomorphism. For the naturality condition, observe that for a function $f : X \rightarrow Y$, then by definition of $\mathcal{A}t$, the map $\mathcal{A}t \mathcal{P}f : \mathcal{A}t \mathcal{P}X \rightarrow \mathcal{A}t \mathcal{P}Y$ sends a singleton $\{x\}$ to the only singleton $\{y\}$ such that $\{x\} \subseteq f^{-1}(y)$. Therefore it must be $y = f(x)$, and we have $\mathcal{A}t \mathcal{P}f : \{x\} \mapsto \{f(x)\}$. The commutativity of the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow \phi_X & & \downarrow \phi_Y \\ \mathcal{A}t \mathcal{P}X & \xrightarrow{\mathcal{A}t \mathcal{P}f} & \mathcal{A}t \mathcal{P}Y \end{array}$$

now follows trivially.

For a complete atomic Boolean algebra A , $\mathcal{P} \mathcal{A}t A$ is the power set of atoms of A . Now we will define ψ as follows: given a set S of atoms of A , so that $S \in \mathcal{P} \mathcal{A}t A$, we send it to $\bigvee S \in A$. Observe that if $S \subseteq S'$ then $\bigvee S \leq \bigvee S'$, so ψ_A preserves the order relation. Moreover,

$$\begin{aligned} \bigvee \bigcup_{i \in I} S_i &= \bigvee_{i \in I} \bigvee S_i; \\ \bigvee \bigcap_{i \in I} S_i &= \bigwedge_{i \in I} \bigvee S_i, \end{aligned}$$

so it also preserves joins and meets.

Also, ψ_A is surjective because A is atomic. Let us prove that it is injective.

For two sets of atoms $S \neq S'$, there must be at least one atom a such that $a \in S$ and $a \notin S'$. This means in particular that $a \leq \psi_A(S)$. We will see that $a \not\leq \psi_A(S')$. Indeed, if we suppose $a \leq \psi_A(S') = \bigvee S'$, then the previous exercise says that there is $a' \in S'$ such that $a \leq a'$, but this is impossible since a' is an atom. So ψ_A is an isomorphism. Finally, we need to show that it is natural, that is, that the diagram

$$\begin{array}{ccc} \mathcal{P}At A & \xrightarrow{\mathcal{P}At h} & \mathcal{P}At B \\ \downarrow \psi_A & & \downarrow \psi_B \\ A & \xrightarrow{h} & B \end{array}$$

commutes. Observe that if S is a set of atoms of A , then $\mathcal{P}At h(S)$ is the set $T := \{b \in At B \mid \exists a \in S \text{ such that } b \leq h(a)\}$. Now compute

$$\begin{aligned} \psi_B \circ \mathcal{P}At h(S) &= \psi_B(T) = \bigvee T \\ h \circ \mathcal{P}At A(S) &= h(\bigvee S) = \bigvee h(S). \end{aligned}$$

so we need to show that $\bigvee T = \bigvee h(S)$.

For each $b \in T$, then there is $a \in S$ such that $b \leq h(a) \leq \bigvee h(S)$, which implies $\bigvee T \leq \bigvee h(S)$.

Conversely, if $h(a) \in h(S)$, since B is atomic $h(a)$ is a join of atoms. In particular, $h(a) = \bigvee \{b \in At B \mid b \leq h(a)\}$, so it is the join of a subset of T , so also $h(a) \leq \bigvee T$. Again, this implies $\bigvee h(S) \leq \bigvee T$. This concludes the proof.

Week 3

Example We know that the Yoneda lemma states that, for a functor $\mathcal{C} \rightarrow \mathbf{Set}$, where \mathcal{C} is a small category, then there is a natural bijection

$$F(C) \cong \text{Nat}(\mathcal{C}(C, -), F)$$

for each object $C \in \mathcal{C}$. In particular, if $F = \mathcal{C}(D, -)$, this means that there is a bijection

$$\mathcal{C}(D, C) \cong \text{Nat}(\mathcal{C}(C, -), \mathcal{C}(D, -))$$

which is natural in both C and D . What does this mean? There are two functors

$$\mathcal{C}^{op} \times \mathcal{C} \xrightarrow[\text{Nat}(\mathcal{C}(C, -), \mathcal{C}(D, -))]{\mathcal{C}(-, -)} \mathbf{Set}$$

and the bijection above is a natural transformation between these two.

Exercise Show that in a category \mathcal{C} , if two objects C and C' are such that for any other object D we have natural bijections

$$\mathcal{C}(C, D) \cong \mathcal{C}(C', D)$$

then we also have $C \cong C'$.

Solution We know that the natural isomorphism $\phi : \mathcal{C}(C, -) \rightarrow \mathcal{C}(C', -)$ must be represented by a unique morphism $f : C' \rightarrow C$. Moreover, its inverse $\psi : \mathcal{C}(C', -) \rightarrow \mathcal{C}(C, -)$ is uniquely represented by a morphism $g : C \rightarrow C'$. We will show that f and g are inverse to each other.

Consider the map $f \circ g : C \rightarrow C$. This induces a map $\mathcal{C}(C, -) \rightarrow \mathcal{C}(C, -)$ but, since the Yoneda isomorphism is natural, this map must be exactly $\phi \circ \psi$. Since ϕ and ψ are inverse to each other by hypothesis, we have that $\phi \circ \psi = \text{id}_{\mathcal{C}(C, -)}$, so we obtained that the map represented by $f \circ g$ is the identity. But now, $\text{id}_C : C \rightarrow C$ also represents the identity on $\mathcal{C}(C, -)$, and we know that the representing morphism is unique. This implies that $f \circ g = \text{id}_C$.

Analogously, we prove that $g \circ f = \text{id}_{C'}$.

Exercise (Full + faithful = injective up to iso). Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Prove that, if F is full and faithful, then $C \cong C' \in \mathcal{C}$ if and only if $F(C) \cong F(C') \in \mathcal{D}$.

Solution Assume that $C \cong C'$. Let $\mathcal{D}' \subseteq \mathcal{D}$ be the image of F , i.e. the full subcategory of \mathcal{D} spanned by all objects of the form $F(C'')$ for some $C'' \in \mathcal{C}$. Observe that $F(C)$ and $F(C')$ are isomorphic in \mathcal{D} if and only if they are isomorphic in \mathcal{D}' . Now we have by the exercise above that $C \cong C'$ if and only if $\mathcal{C}(C, -) \cong \mathcal{C}(C', -)$ if and only if $\mathcal{D}'(F(C), -) \cong \mathcal{D}'(F(C'), -)$ because F is full and faithful, and finally this is true if and only if $F(C) \cong F(C')$ again by the exercise above.

Example Observe that, if instead of starting off with \mathcal{C} we take \mathcal{C}^{op} instead, we will obtain a dual Yoneda embedding. This means that, for a functor $F : \mathcal{C}^{op} \rightarrow \mathbf{Set}$, there are natural bijections

$$F(C) \cong \text{Nat}(\mathcal{C}(-, C), F)$$

and in particular, when $F = \mathcal{C}(-, D)$, we have natural bijections

$$\mathcal{C}(C, D) \cong \text{Nat}(\mathcal{C}(-, C), \mathcal{C}(-, D))$$

for all $C, D \in \mathcal{C}$.

Exercise Show that in a Cartesian closed category \mathcal{C} every object C defines a functor $\mathcal{C}^{(-)} : \mathcal{C}^{op} \rightarrow \mathcal{C}$.

Solution We first define it on objects, which is easy: just set $B \mapsto C^B$. For a morphism $f : A \rightarrow B$, we need to define a morphism $C^f : C^B \rightarrow C^A$. Such morphisms are in natural bijection with morphisms $A \times C^B \rightarrow C$, so it suffices to give one of these. Let us take

$$A \times C^B \xrightarrow{f \times C^B} B \times C^B \xrightarrow{ev} C.$$

Let's prove that this assignment preserves identities. If $f = \text{id}_B$, the map defined above is precisely ev . We need to prove that the universal property of exponentials associates ev to id_{C^B} . This is clear since the diagram

$$\begin{array}{ccc} C^B \times B & \xrightarrow{ev} & C \\ \text{id}_{C^B} \times \text{id}_B \downarrow & \nearrow ev & \\ C^B \times B & & \end{array}$$

obviously commutes.

Now consider two morphisms $f : A \rightarrow B$ and $g : B \rightarrow D$. We want to show that $C^{gf} = C^f C^g$. Again, it suffices to show that the associated maps $C^D \times A \rightarrow C$ are equal. First observe that there is a square

$$\begin{array}{ccc} C^D \times B & \xrightarrow{C^D \times g} & C^D \times D \\ C^g \times B \downarrow & & \downarrow ev \\ C^B \times B & \xrightarrow{ev} & C \end{array}$$

which commutes by definition of C^g , so we have $ev \circ (C^D \times g) = ev \circ (C^g \times B)$. For the same reason, we have $ev \circ (C^B \times f) = ev \circ (C^f \times A)$. Finally, we clearly have a commutative square

$$\begin{array}{ccc} C^D \times A & \xrightarrow{C^D \times f} & C^D \times B \\ C^g \times A \downarrow & & \downarrow C^g \times B \\ C^B \times A & \xrightarrow{C^B \times f} & C^B \times B. \end{array}$$

We want to use these facts to complete our proof. Now, the map which corresponds to C^{gf} is, by definition, $ev \circ (C^D \times gf)$. Now compute

$$\begin{aligned} ev \circ (C^D \times gf) &= ev \circ (C^D \times g) \circ (C^D \times f) \\ &= ev \circ (C^g \times B) \circ (C^D \times f) \\ &= ev \circ (C^B \circ f) \circ (C^g \times A) \\ &= ev \circ (C^f \times A) \circ (C^g \times A) \\ &= ev \circ (C^f C^g \times A) \end{aligned}$$

Exercise Let \mathcal{C} be a Cartesian closed category. Show that, for objects $A, B, C \in \mathcal{C}$ we have a natural isomorphism

$$A^C \times B^C \cong (A \times B)^C.$$

Solution By the contravariant version of the Yoneda embedding (example above) it suffices to show that the two objects represent naturally isomorphic functors. Now choose an object $D \in \mathcal{C}$ and show that the two sets $\mathcal{C}(D, A^C \times B^C)$ and $\mathcal{C}(D, (A \times B)^C)$ are in natural bijection. In order to do so, compute

$$\begin{aligned}\mathcal{C}(D, A^C \times B^C) &\cong \mathcal{C}(D, A^C) \times \mathcal{C}(D, B^C) \cong \mathcal{C}(D \times C, A) \times \mathcal{C}(D \times C, B) \\ &\cong \mathcal{C}(D \times C, A \times B) \cong \mathcal{C}(D, (A \times B)^C)\end{aligned}$$

and all those isomorphisms are natural, so their composite also is.

Example Remember that the universal property of products can be expressed in the following way. Given two objects X and Y , their products is characterized by the fact that, for every other object Z there are natural isomorphisms

$$\mathcal{C}(Z, X \times Y) \cong \mathcal{C}(Z, X) \times \mathcal{C}(Z, Y).$$

The same line of reasoning leads us to an analogous characterization of coproducts: the coproduct $X + Y$ is characterized by the fact that for every other object Z there are natural isomorphisms

$$\mathcal{C}(X + Y, Z) \cong \mathcal{C}(X, Z) \times \mathcal{C}(Y, Z).$$

Exercise Let \mathcal{C} be a Cartesian closed category. Show that, for objects $A, B, C \in \mathcal{C}$ we have a natural isomorphism

$$A^{(B+C)} \cong A^B \times A^C.$$

Solution As above, we want to prove that there are natural bijections between $\mathcal{C}(D, A^{(B+C)})$ and $\mathcal{C}(D, A^B \times A^C)$. Compute

$$\begin{aligned}\mathcal{C}(D, A^{(B+C)}) &\cong \mathcal{C}(D \times (B+C), A) \cong \mathcal{C}(B+C, A^D) \cong \mathcal{C}(B, A^D) \times \mathcal{C}(C, A^D) \\ &\cong \mathcal{C}(B \times D, A) \times \mathcal{C}(C \times D, A) \cong \mathcal{C}(D, A^B) \times \mathcal{C}(D, A^C) \cong \mathcal{C}(D, A^B \times A^C)\end{aligned}$$

where we are using both universal properties of products and coproducts in their compact form.

Definition Let B be a Boolean algebra. Then a filter U is a proper subset of B such that

1. $1 \in U$;
2. $x, y \in U$ implies $x \wedge y \in U$;
3. $x \leq y$ and $x \in U$ implies $y \in U$.

Moreover, a filter is called an ultrafilter if it is maximal, that is, there are no filters U' of B such that $U \subset U'$ or, equivalently, for each $x \in B$, precisely one between x and $\neg x$ is in U .

Exercise Show that, for any Boolean algebra, B , the set of ultrafilters on B is isomorphic to the set of morphisms of Boolean algebras $B \rightarrow \mathbf{2}$, so that there is an isomorphism $\text{Ult}(B) \cong \mathbf{BA}(B, \mathbf{2})$.

Solution We already know that, if we regard B as a set, there is a bijection $\mathcal{P}(B) \cong \mathbf{Set}(B, \mathbf{2})$, given by sending a subset $A \subseteq B$ to the map χ_A that sends elements of A to 1 and all other elements to 0. We need to show that this map is a morphism of Boolean algebras if and only if A is an ultrafilter. The preservation of the order relation is equivalent to condition 3 of a filter.

For the preservation of meets, observe that when $\chi_A(x) = 0$ (or $\chi_A(y) = 1$) it just follows by $x \wedge y \leq x$ plus condition 3; when $\chi_A(x) = \chi_A(y) = 1$, it is exactly condition 2.

Similarly, for the preservation of joins, when $\chi_A(x) = 1$ (or $\chi_A(y) = 1$) it follows by $x \vee y \geq x$ plus condition 3. When $\chi_A(x) = \chi_A(y) = 0$, we consider two cases: if $x \leq y$, then $x \vee y = y$ and it becomes trivial; if $x \not\leq y$ and $y \not\leq x$, then $\chi_A(x \vee y) = 1$ would imply $x \vee y \in A$, so that $A \subsetneq A \cup \{z \in B \mid x \leq z\} \subsetneq B$ which contradicts maximality.

Conversely, assuming that χ_A is a morphism of Boolean algebras, A is a proper set because $\chi_A(0_B) = 0 \neq 1$. Condition 1 is trivial, condition 2 and 3 were already discussed. For maximality, suppose by contradiction that A is not maximal, so there is either an element $x \in B$ such that $x, \neg x \in A$ or such that $x, \neg x \notin A$. In the former case, we obtain $0 = x \wedge \neg x \in A$ which is impossible. In the latter, we obtain $0 = 0 \vee 0 = \chi_A(x) \vee \chi_A(\neg x) = \chi_A(x \vee \neg x) = \chi_A(1) = 1$, which also is a contradiction.

Week 4

Example Take two functions of sets, $f : A \rightarrow X$ and $g : B \rightarrow X$. We want to give a concrete description of their pullback. We claim that their pullback is the subset of the product $S_{f=g} \subseteq A \times B$ containing all the pairs (a, b) such that $f(a) = g(b)$, and the two projections are the restrictions of the two canonical projections from the product $p_1 : A \times B \rightarrow A$ and $p_2 : A \times B \rightarrow B$. To see that, take two other functions $h : D \rightarrow A$ and $k : D \rightarrow B$ such that $fh = gk$. We want to find a factorization $t : D \rightarrow S_{f=g}$, and see that it is unique.

For an element $d \in D$, define $t(d) = (h(d), k(d)) \in A \times B$. First, we need to see that this is well-defined, that is, that this function lands in the subset $S_{f=g}$. Observe that $fh(d) = gk(d)$ by hypothesis, so this is clearly satisfied. Moreover, $p_1 t(d) = p_1(h(d), k(d)) = h(d)$, so that $p_1 t = h$, and analogously we obtain $p_2 t = k$, so this map is indeed a factorization. To see that it is unique, observe that its definition is forced on the two components respectively by the requirements that $p_1 t = h$ and $p_2 t = k$.

Example Now we want to understand equalizers and coequalizers in \mathbf{Set} . Take two functions $f, g : A \rightarrow B$ between sets. We claim that their equalizer is the subset $E \subseteq A$ of all elements $a \in A$ such that $f(a) = g(a)$. To see that, choose another function $h : D \rightarrow A$ such that $fh = gh$, and prove that it factors uniquely through the inclusion $E \subseteq A$.

We want to define a function t as in the diagram

$$\begin{array}{ccc}
 D & & \\
 \downarrow t & \searrow h & \\
 E & \hookrightarrow & A \xrightarrow[f]{g} B.
 \end{array}$$

For an element $d \in D$, let us set $t(d) = h(d)$. Since $fh = gh$ by hypothesis, $h(d)$ is indeed in the subset E , so the map is well-defined. Moreover, the commutativity of the left triangle forces t to have this form.

Exercise Show that the pullback square

$$\begin{array}{ccc}
 A \times_X B & \xrightarrow{p_1} & A \\
 \downarrow p_2 & & \downarrow f \\
 B & \xrightarrow{g} & X
 \end{array}$$

is also the product of f and g when regarded as objects of \mathcal{C}/X .

Solution Take an object $h : C \rightarrow X \in \mathcal{C}/X$ with maps $q_1 : C \rightarrow A, q_2 : C \rightarrow B$ such that $f q_1 = h = g q_2$, so that they are morphisms in \mathcal{C}/X . This condition exactly means that the outer square in the diagram

$$\begin{array}{ccccc}
 C & & & & \\
 \downarrow q_1 & \searrow r & & & \\
 A \times_X B & \xrightarrow{p_1} & A & & \\
 \downarrow p_2 & & \downarrow f & & \\
 B & \xrightarrow{g} & X & & \\
 \uparrow q_2 & & & & \\
 C & & & &
 \end{array}$$

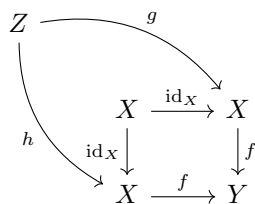
commutes. By the universal property of pullbacks, we thus obtain a unique factorization $r : C \rightarrow A \times_X B$, which is precisely the factorization we were looking for, and it is a morphism in \mathcal{C}/X .

Exercise Let \mathcal{C} be a category with pullbacks. Show that a morphism $f : X \rightarrow Y$ is monic if and only if the square

$$\begin{array}{ccc}
 X & \xrightarrow{\text{id}_X} & X \\
 \text{id}_X \downarrow & & \downarrow f \\
 X & \xrightarrow{f} & Y
 \end{array}$$

is a pullback diagram.

Solution Suppose that the diagram above is a pullback. We take two morphisms $g, h : Z \rightarrow X$ such that $fg = fh$. This means that the outer square of the diagram



commutes. The universal property provides a factorization $Z \rightarrow X$ for the two bent arrows, which means that $g = h$.

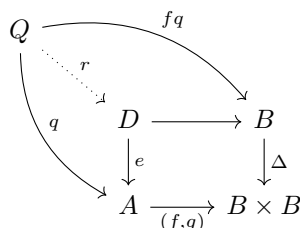
Conversely, suppose that f is monic, we want to prove that the square above is a pullback. Let us take a commutative outer square as the one above. Since f is monic, the condition $fg = fh$ which is exactly the commutativity of the square implies that $g = h$, so we choose this as a factorization $Z \rightarrow X$. It is unique by the identity property of id_X .

Exercise Let \mathcal{C} be a category with products and pullbacks. Consider the pullback square

$$\begin{array}{ccc} E & \longrightarrow & B \\ \downarrow e & & \downarrow \Delta \\ A & \xrightarrow{(f,g)} & B \times B \end{array}$$

and show that the map $e : E \rightarrow A$ is an equalizer of f and g .

Solution Let us take an arrow $q : Q \rightarrow A$ such that $f q = g q$. Firstly, we want to show that $(f, g)q = \Delta f q$. By the universal property of the product, it suffices to show it after postcomposing with the two projections $p_1, p_2 : B \times B \rightarrow B$. Now we have $p_1(f, g)q = f q = p_1 \Delta f q$ and also $p_2(f, g)q = g q = f q = p_2 \Delta f q$. This means that the outer square in the diagram



commutes. By universal property of the pullback, there is a dotted arrow r making everything commute, so that in particular $q = e r$. This factorization is unique because it is also a factorization for the pullback.

Example In **Grp**, pullbacks are computed as in **Set**. We already saw a proof in **Set**, what remains to show is that the subset $S_{f=g} \subseteq A \times B$ as in the previous example is a subgroup, and that the factorization $t : D \rightarrow S_{f=g}$ we defined is a group homomorphism. For the first, observe that 1 is preserved by f and g , so that $1 \in S_{f=g}$. Moreover, choose $(a, b), (a', b') \in S_{f=g}$ and take their product in $A \times B$, that is, (aa', bb') . Now we have that $f(aa') =$

$f(a)f(a') = g(b)g(b') = g(bb')$, so the product lies in $S_{f=g}$.
 Finally, for two elements $d, d' \in D$, we need to show that $t(dd') = t(d)t(d')$.
 For that, let us compute

$$\begin{aligned} t(dd') &= (h(dd'), k(dd')) \\ &= (h(d)h(d'), k(d)k(d')) \\ &= (h(d), k(d))(h(d'), k(d')) \\ &= t(d)t(d') \end{aligned}$$

where we used that h and k are group homomorphisms.

Example In **Top**, pullbacks are also computed in the same manner. The topology on the pullback is the subspace topology of the product topology.

Example In **CRng** and **Mon**, the same thing happens.

Example We now want to look at the dual notion of coequalizer. What is a coequalizer in **Set**? We claim that, given two maps $f, g : A \rightarrow B$, their coequalizer is the quotient of B obtained by the following equivalence relation: $b \sim_{=} b'$ whenever there is an element $a \in A$ such that $f(a) = b$ and $g(a) = b'$. This is in general not reflexive, symmetric, or transitive. We simply take \sim to be the smallest equivalence relation generated by $\sim_{=}$. More explicitly, two elements b and b' are \sim -equivalent if there is a finite sequence of elements $(b_i)_{i=1, \dots, n}$ such that $b = b_1$, $b' = b_n$ and $\forall i = 1, \dots, n-1$ we have either $b_i \sim_{=} b_{i+1}$ or $b_{i+1} \sim_{=} b_i$. Consider a function $h : B \rightarrow D$ such that $hf = hg$. We want to show that this factors uniquely through the projection $p : B \rightarrow B/\sim$, as in the diagram

$$\begin{array}{ccc} A & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & B & \xrightarrow{p} & B/\sim \\ & & & \searrow h & \vdots k \\ & & & & D. \end{array}$$

For an element $[b] \in B/\sim$, define $k([b]) = h(b)$. We first need to see that this is well defined, that is, if $b \sim b'$ then $h(b) = h(b')$. We know that there is a sequence of elements b_i 's as described above. By induction, it suffices to restrict our attention to b_i and b_{i+1} , for $i = 1, \dots, n-1$. In other words, we may assume that $b \sim_{=} b'$. This means that there is an element $a \in A$ such that $f(a) = b$ and $g(a) = b'$. Since $hf = hg$ by hypothesis, we now that $h(b) = hf(a) = hg(a) = h(b')$, so the map k is well-defined. Moreover, it clearly satisfies $kp = h$.

To see that it is unique, observe that the commutativity of the right triangle forces the form we just gave.

Example In a similar way, we can construct pushouts in **Set**. Consider two maps $f : A \rightarrow B$ and $g : A \rightarrow C$. We claim that the pushout is computed as the quotient on the disjoint union $B \amalg C$ given by the following equivalence

relation: if two elements belong to the same component of the disjoint union, we do nothing; if $b \in B$ and $c \in C$, we say that $a \sim_{=} c$ whenever there is an element $a \in A$ such that $f(a) = b$ and $g(a) = c$. As in the case of coequalizers, this relation is in general far from being an equivalence relation, but we can complete it to an equivalence relation \sim on $B \amalg C$. This gives a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow \\ C & \longrightarrow & (B \amalg C)/\sim \end{array}$$

where the map $B \rightarrow (B \amalg C)/\sim$ is given as the composition of the inclusion in the disjoint union $B \hookrightarrow B \amalg C$ and the projection to the quotient $B \amalg C \rightarrow (B \amalg C)/\sim$, and similarly for the map $C \rightarrow (B \amalg C)/\sim$. The proof that this is a pushout square is analogous to that for coequalizers.

Remark As in previous examples, pullbacks and equalizers in categories of algebras are computed on the underlying sets. This is generally not the case for pushouts or coequalizers.

However, this is the case in other categories such as **Top**, where both pushouts and coequalizers are computed on the underlying sets, which are then endowed with the quotient topology.

Week 5

Exercise Given a functor $F : \mathbf{J} \rightarrow \mathcal{C}$, we denote by $\text{Cone}_{j \in \mathbf{J}}(A, Fj)$ the set of all cones on F with vertex A .

Give a description of limits in **Set** in terms of cones. By the theorem of existence of limits given products and equalizers, we can obtain another explicit description. Verify that the two are isomorphic.

Solution Given a functor $F : \mathbf{J} \rightarrow \mathbf{Set}$, we can compute

$$\lim_{j \in \mathbf{J}} Fj \cong \mathbf{Set}(*, \lim_{j \in \mathbf{J}} Fj) \cong \text{Cone}_{j \in \mathbf{J}}(*, Fj)$$

where the first isomorphism is a peculiarity of **Set**, and the second is simply the universal property of limits. Now, let us spell out in detail what a cone on F with vertex $*$ is.

For each $j \in \mathbf{J}$, we are given a map $* \rightarrow Fj$, which corresponds to an element $x_j \in Fj$. The cone condition says that, for a morphism $f : j \rightarrow j'$ in \mathbf{J} , we need to have $x_{j'} = Ff(x_j)$. This is all we need. A cone is then a tuple of elements $(x_j \in Fj)_{j \in \mathbf{J}}$ satisfying the condition above. So the set of cones on F with vertex $*$ is the subset of $\prod_{j \in \mathbf{J}} Fj$ determined by that property, which is exactly the description given by the existence theorem.

Example We know that, for an object $C \in \mathcal{C}$ in a category, the representable functor $\mathcal{C}(C, -) : \mathcal{C} \rightarrow \mathbf{Set}$ preserves limits. This follows directly by the universal property of limits. In particular, we can obtain the following equivalent

definition of limits.

Given a diagram $(C_i)_i$ in \mathcal{C} and a cone $(C \rightarrow C_i)_i$, these induce maps of the form

$$\mathcal{C}(D, C) \rightarrow \mathcal{C}(D, C_i)$$

for every other object $D \in \mathcal{C}$. This is a cone in **Set** with vertex $\mathcal{C}(D, C)$. By universal property of limits in **Set**, this cone induces a map

$$\phi_D : \mathcal{C}(D, C) \rightarrow \lim_i \mathcal{C}(D, C_i).$$

Now, C is a limit of the diagram $(C_i)_i$ if and only if for every object $D \in \mathcal{C}$ the natural map ϕ_D is an isomorphism of sets. This implies that there is a natural isomorphism $\mathcal{C}(D, \lim_i C_i) \cong \lim_i \mathcal{C}(D, C_i)$.

Example There is a completely analogous discussion for colimits. Suppose we have a diagram $(C_i)_i$ and a cocone $(C_i \rightarrow C)_i$. For every object $D \in \mathcal{C}$, the maps to the vertex of this cocone induce maps

$$\mathcal{C}(C, D) \rightarrow \mathcal{C}(C_i, D)$$

which form a cone (not a cocone!) in **Set** with vertex $\mathcal{C}(C, D)$. Therefore, again by the universal property of limits (not colimits!) in **Set**, we have a canonical map

$$\psi_D : \mathcal{C}(C, D) \rightarrow \lim_i \mathcal{C}(C_i, D).$$

Then the object C is a colimit for the diagram if and only if for every object $D \in \mathcal{C}$ the map ψ_D is an isomorphism of sets. In other words, there are natural isomorphisms $\mathcal{C}(\text{colim}_i C_i, D) \cong \lim_i \mathcal{C}(C_i, D)$.

Example Unfortunately, there is no simple way to describe colimits in **Set** as in the exercise above. Therefore, we have to rely on the existence theorem and deduce what they look like from what we know about coproducts and coequalizers.

In short, given a functor $F : \mathbf{J} \rightarrow \mathbf{Set}$, the colimit of F may be thus computed. First take the coproduct $\coprod_{j \in \mathbf{J}} Fj$. By the existence theorem and the description of coequalizers, we know that the colimit has to be a quotient of this.

Define an equivalence relation on $\coprod_{j \in \mathbf{J}} Fj$ as follows: for $x \in Fj$ and $y \in Fj'$, we say that $x \sim y$ if there exists a $k \in \mathbf{J}$ and two morphisms

$$\begin{array}{ccc} & & j \\ & \nearrow f & \\ k & & \\ & \searrow f' & \\ & & j' \end{array}$$

with an element $z \in Fk$ such that $Ff(z) = x$ and $Fg(z) = y$. This relation is reflexive (because we allow identities) and symmetric (because we are not considering j and j' in any particular order), but it is in general not transitive. We define \sim as the transitive closure of this relation. Clearly, it gives an equivalence relation. A bit more visually, $x, y \in \coprod_{j \in \mathbf{J}} Fj$ are equivalent if and only if there is a zig-zag of morphisms in \mathbf{J} that move x to y through evaluating functions or choosing preimages. Taking the quotient of this equivalence relation, we get the desired description

$$\operatorname{colim}_{j \in \mathbf{J}} Fj \cong (\coprod_{j \in \mathbf{J}} Fj) / \sim.$$

Exercise Let \mathcal{C} be a category admitting limits of shape \mathbf{J} , and let $\operatorname{Fun}(\mathbf{J}, \mathcal{C})$ be the category of functors $\mathbf{J} \rightarrow \mathcal{C}$, having functors as objects and natural transformations as morphisms. Show that there is a functor

$$\lim_{\mathbf{J}} : \operatorname{Fun}(\mathbf{J}, \mathcal{C}) \rightarrow \mathcal{C}.$$

Solution For a functor $F : \mathbf{J} \rightarrow \mathcal{C}$, we simply choose a limit $\lim_{\mathbf{J}} F$. Given a natural transformation $\alpha : F \rightarrow G$, we want to obtain a morphism $\lim F \rightarrow \lim G$. Note that there is a cone $(t_j : \lim F \rightarrow Fj)_{j \in \mathbf{J}}$. We can compose this objectwise with α , so to obtain maps $(\alpha_j \circ t_j : \lim F \rightarrow G_j)_{j \in \mathbf{J}}$. We want to check that this is also a cone. Given a morphism $j \rightarrow j'$ in \mathbf{J} , we have a diagram

$$\begin{array}{ccc} & \lim F & \\ t_j \swarrow & & \searrow t_{j'} \\ Fj & \xrightarrow{\quad} & Fj' \\ \alpha_j \downarrow & & \downarrow \alpha_{j'} \\ Gj & \xrightarrow{\quad} & Gj'. \end{array}$$

Here, the upper triangle commutes because (t_j) is a cone, the lower square commutes because α is natural. Therefore, $(\alpha_j \circ t_j)$ is a cone over G . By universal property of limits, this induces a unique map, which we will call $\lim \alpha : \lim F \rightarrow \lim G$, such that $s_j \circ \lim \alpha = \alpha_j \circ t_j$, where $(s_j : \lim G \rightarrow Gj)$ is a limiting cone for G . We need to check that this is functorial.

Given the identity natural transformation $\operatorname{id} : F \rightarrow F$, the composed cone as in the diagram above is still (t_j) , so the identity $\operatorname{id} : \lim F \rightarrow \lim F$ will be a possible factorization for it. Since this factorization is unique, the identity is the only possible choice.

Now, given two natural transformations $\alpha : F \rightarrow G$ and $\beta : G \rightarrow H$, we obtain as above the induced factorizations $\lim \alpha : \lim F \rightarrow \lim G$ and $\lim \beta : \lim G \rightarrow \lim H$. By uniqueness of factorizations, it will suffice to show that $\lim \beta \circ \lim \alpha$ is a factorization for the cone $(\beta_j \circ \alpha_j \circ t_j)$ over H with vertex $\lim F$. If we use the same notation as above, plus $(r_j : \lim H \rightarrow Hj)$ for the limiting cone of H , we compute, for each $j \in \mathbf{J}$,

$$r_j \circ \lim \beta \circ \lim \alpha = \beta_j \circ s_j \circ \lim \alpha = \beta_j \circ \alpha_j \circ t_j$$

which is exactly what we need.

Example This is the key example of density of a functor. Let \mathcal{C} be any category, and let $\mathbf{y} : \mathcal{C} \rightarrow \mathbf{Set}^{\mathcal{C}^{op}}$ the Yoneda embedding, so that $\mathbf{y}(C) = \mathcal{C}(-, C)$. Here, embedding means full and faithful (and we know that this functor is such). Then \mathbf{y} is a dense functor. In particular, for every functor $F \in \mathbf{Set}^{\mathcal{C}^{op}}$, we have a slice category $\mathbf{Set}_{/F}^{\mathcal{C}^{op}}$, so we can take the subcategory spanned by all representable functors over F , and we call it $\mathcal{C}_{/F}$. This comes with a natural projection $\mathcal{C}_{/F} \rightarrow \mathcal{C}$. We claim that F is the colimit of the composite functor

$$\mathcal{C}_{/F} \rightarrow \mathcal{C} \hookrightarrow \mathbf{Set}^{\mathcal{C}^{op}}.$$

We will not spell out all the details of the proof here, but the strategy is the following: given a functor $G \in \mathbf{Set}^{\mathcal{C}^{op}}$ and a cocone on $\mathcal{C}_{/F}$ with vertex G , we need to show that it factors uniquely through F .

We will give each component $\theta_C : F(C) \rightarrow G(C)$ of the factorization. Given an element $x \in F(C)$, by the Yoneda lemma it corresponds to a natural transformation $\alpha_x : \mathcal{C}(-, C) \rightarrow F$, which is then an object of $\mathcal{C}_{/F}$. Therefore, since we have a cocone on $\mathcal{C}_{/F}$ with vertex G there is a corresponding natural transformation $t_{\alpha_x} : \mathcal{C}(-, C) \rightarrow G$. Using the Yoneda lemma again, this yields an element $z \in G(C)$. Now we set $\theta_C(x) := z$.

Moreover, θ is natural, which is a consequence of the fact that the Yoneda isomorphisms are all natural. That it is the desired factorization is directly a consequence of its very definition and of the fact that the Yoneda lemma applied to G gives a bijection

$$G(C) \cong \text{Nat}(\mathcal{C}(-, C), G).$$

Finally, it is uniquely determined for the same reason (assuming that there are two different ones, we would obtain that there is at least one $C \in \mathcal{C}$ that falsifies Yoneda).

Week 6

Exercise Show that limits in a functor category are computed objectwise. This means that, assuming that \mathcal{C} has all limits of shape \mathbf{J} , and choosing a diagram $F : \mathbf{J} \rightarrow \text{Fun}(\mathcal{D}, \mathcal{C})$, the value of the limit of F on an object $D \in \mathcal{D}$ can be calculated as

$$(\lim_{j \in \mathbf{J}} Fj)(D) = \lim_{j \in \mathbf{J}} (Fj(D)).$$

Solution Note that the functor $F : \mathbf{J} \rightarrow \text{Fun}(\mathcal{D}, \mathcal{C})$ corresponds to a functor $\mathbf{J} \times \mathcal{D} \rightarrow \mathcal{C}$, which in turn corresponds to a functor $\tilde{F} : \mathcal{D} \rightarrow \text{Fun}(\mathbf{J}, \mathcal{C})$ sending an object $D \in \mathcal{D}$ to the functor $j \mapsto Fj(D)$. In a previous exercise, we have seen that the computation of limits can be assembled in a functor $\lim : \text{Fun}(\mathbf{J}, \mathcal{C}) \rightarrow \mathcal{C}$. Now, the functor on the right-hand side of the statement is precisely the composite

$$\lim \circ \tilde{F} : \mathcal{D} \rightarrow \text{Fun}(\mathbf{J}, \mathcal{C}) \rightarrow \mathcal{C}.$$

We want to show that this functor has the universal property of limits with respect to the diagram F .

First, we observe that for an object $D \in \mathcal{D}$ we have a cone $p_j^D : \lim_{j \in \mathbf{J}} (Fj(D)) \rightarrow Fj(D)$. The maps forming this cone are natural, because given a morphism $k : D \rightarrow D'$ in \mathcal{D} , the square

$$\begin{array}{ccc} \lim_{j \in \mathbf{J}} (Fj(D)) & \xrightarrow{p_j^D} & Fj(D) \\ \lim Fj(k) \downarrow & & \downarrow Fj(k) \\ \lim_{j \in \mathbf{J}} (Fj(D')) & \xrightarrow{p_j^{D'}} & Fj(D') \end{array}$$

commutes. This is true by definition of the left vertical map. So we have a cone $(p_j : \lim_{j \in \mathbf{J}} Fj \rightarrow Fj)$ in $\text{Fun}(\mathcal{D}, \mathcal{C})$. Let us show it has the universal property.

Choose another cone $(\alpha_j : G \rightarrow Fj)_{j \in \mathbf{J}}$ in $\text{Fun}(\mathcal{D}, \mathcal{C})$. For every object $D \in \mathcal{D}$, this yields a cone $(\alpha_j^D : G(D) \rightarrow Fj(D))_{j \in \mathbf{J}}$ in \mathcal{C} . So by the universal property of limits in \mathcal{C} , we obtain a unique natural morphism $\theta^D : G(D) \rightarrow \lim_{j \in \mathbf{J}} (Fj(D))$ such that $p_j^D \circ \theta^D = \alpha_j^D$. It only remains to show that the morphisms $(\theta^D)_{D \in \mathcal{D}}$ form a natural transformation.

Given a morphism $k : D \rightarrow D'$, consider the diagram

$$\begin{array}{ccccc} G(D) & \xrightarrow{\theta^D} & \lim_{j \in \mathbf{J}} (Fj(D)) & & \\ G(k) \downarrow & & \downarrow \lim Fj(k) & & \\ G(D') & \xrightarrow{\theta^{D'}} & \lim_{j \in \mathbf{J}} (Fj(D')) & \xrightarrow{p_j^{D'}} & Fj(D'). \end{array}$$

We need to show that $\lim Fj(k) \circ \theta^D$ and $\theta^{D'} \circ G(k)$ are equal. By the universal property of $\lim_{j \in \mathbf{J}} (Fj(D'))$, it suffices to show that they coincide after composing with all the projections $p_j^{D'}$. Now compute

$$\begin{aligned} p_j^{D'} \circ \lim Fj(k) \circ \theta^D &= Fj(k) \circ p_j^D \circ \theta^D \\ &= Fj(k) \circ \alpha_j^D \\ &= \alpha_j^{D'} \circ G(k) \\ &= p_j^{D'} \circ \theta^{D'} \circ G(k) \end{aligned}$$

where we are using the fact that the θ^D components are factorizations for the α^D components, plus the definition of $\lim Fj(k)$ and the naturality of α .

In conclusion, we have obtained a natural transformation $\theta : G \rightarrow \lim_{j \in \mathbf{J}} (Fj(-))$ such that $p_j \circ \theta = \alpha_j$ for every j . This factorization is unique because if it weren't, there would be two different factorizations for at least one component D , which contradicts the universal property of limits in \mathcal{C} .

Remark Exactly by the same reason, colimits are computed objectwise in functor categories. So if we have a diagram $F : \mathbf{J} \rightarrow \text{Fun}(\mathcal{D}, \mathcal{C})$, the functor $\text{colim}_{j \in \mathbf{J}} Fj : \mathcal{D} \rightarrow \mathcal{C}$ sends an object $D \in \mathcal{D}$ to $\text{colim}_{j \in \mathbf{J}} (Fj(D))$, and the action on morphisms is determined by the universal property of colimits.

Remark Another way to express the fact that limits and colimits are computed objectwise is saying that, for every object $D \in \mathcal{D}$, the evaluation functor

$$\text{ev}_D : \text{Fun}(\mathcal{D}, \mathcal{C}) \rightarrow \mathcal{C}$$

preserves limits and colimits.

Exercise Let \mathcal{C} be a category. Prove that the Yoneda embedding $\mathbf{y} : \mathcal{C} \rightarrow \mathbf{Set}^{\mathcal{C}^{op}}$ preserves all limits that exist in \mathcal{C} .

Solution Let us choose a diagram $F : \mathbf{J} \rightarrow \mathcal{C}$, and assume that it has a limit. We want to show that $\mathbf{y}(\lim F) \cong \lim \mathbf{y}F$. The left-hand side is the functor $\mathcal{C}(-, \lim F)$, the right-hand side is the limit of the functor $\mathbf{J} \rightarrow \mathbf{Set}^{\mathcal{C}^{op}}$ sending an object $j \in \mathbf{J}$ to $\mathcal{C}(-, Fj)$. It suffices to show that these two are naturally isomorphic, that is, we need to show that, once evaluated in $C \in \mathcal{C}$, they yield a natural isomorphism. Now the statement follows immediately by observing that, by the universal property of limits, we have natural isomorphisms

$$\mathcal{C}(C, \lim_{j \in \mathbf{J}} Fj) \cong \lim_{j \in \mathbf{J}} \mathcal{C}(C, Fj).$$

Remark The Yoneda embedding does not preserve colimits, even in the trivial case $\mathbf{y} : * \rightarrow \mathbf{Set}^* \cong \mathbf{Set}$.

Furthermore, the Yoneda embedding $\mathbf{y} : \mathcal{C}^{op} \rightarrow \mathbf{Set}^{\mathcal{C}}$ does not preserve colimits either. Indeed, if we have a diagram $(C_i)_{i \in \mathbf{I}}$ in \mathcal{C} , then we know that $\mathcal{C}(\text{colim } C_i, D) \cong \lim \mathcal{C}(C_i, D)$ for any other object $D \in \mathcal{C}$. Since colimits in \mathcal{C} correspond with limits in \mathcal{C}^{op} , the conclusion is that the dual Yoneda embedding preserves limits, just as the Yoneda embedding. Another way to see it is that this functor is simply a special case of the Yoneda embedding which, as we already know, preserves limits but not colimits.

This might seem strange at first, but the following should clarify things. Observe that, since the Yoneda embedding of \mathcal{C}^{op} preserves limits, this means exactly that the embedding $\mathbf{y}^{op} : \mathcal{C} \rightarrow (\mathbf{Set}^{\mathcal{C}})^{op}$ preserves colimits. This polarization between limits and colimits is closely connected to the fact that $\mathbf{Set}^{\mathcal{C}^{op}}$ is the free cocompletion, and $(\mathbf{Set}^{\mathcal{C}})^{op}$ is the free completion of \mathcal{C} .

Definition Recall that a subobject of an object C in a category is an equivalence class of monomorphisms with codomain C , where two such $f : A \hookrightarrow C$ and $g : B \hookrightarrow C$ are equivalent if there exists a commutative triangle

$$\begin{array}{ccc} A & \xrightarrow{q} & B \\ & \searrow f & \swarrow g \\ & & C \end{array}$$

where q is an isomorphism.

We say that a category with a final object has a subobject classifier if there is an object Ω with a monomorphism $t : * \rightarrow \Omega$ such that for every subobject $A \rightarrow C$ there is a unique map $C \rightarrow \Omega$ such that the square

$$\begin{array}{ccc} A & \longrightarrow & * \\ \downarrow & & \downarrow t \\ C & \longrightarrow & \Omega \end{array}$$

is a pullback.

Remark In other words, this means that the operation of pulling back gives a bijection $\mathcal{C}(C, \Omega) \cong \text{Sub}(C)$. Moreover, since monomorphisms and isomorphisms are both stable under pullback, there is a functor $\text{Sub} : \mathcal{C}^{op} \rightarrow \mathbf{Set}$. The present statement says that if \mathcal{C} has a subobject classifier this is a representable functor.

Exercise Assume that there is a monomorphism $t : Q \rightarrow \Omega$ which satisfies the universal property of the subobject classifier. Prove that Q is terminal.

Solution Given an object C , we need to show that there exists a unique morphism $C \rightarrow Q$. For the existence, observe that the identity $\text{id}_C : C \rightarrow C$ is a monomorphism, so there is a pullback square

$$\begin{array}{ccc} C & \xrightarrow{f} & Q \\ \text{id}_C \downarrow & & \downarrow t \\ C & \longrightarrow & \Omega. \end{array}$$

This gives a morphism $f : C \rightarrow Q$. Moreover, the bottom horizontal morphism is tf . Suppose there are two morphisms $f, g : C \rightarrow Q$. We have a diagram

$$\begin{array}{ccccc} C & \xrightarrow{g} & Q & \xrightarrow{\text{id}_Q} & Q \\ \text{id}_C \downarrow & & \downarrow \text{id}_Q & & \downarrow t \\ C & \xrightarrow{g} & Q & \xrightarrow{t} & \Omega \end{array}$$

where the right square is a pullback because t is monic, and the left square is a pullback by direct check. Thus the composite tg classify the identity $C \rightarrow C$, but by universal property of the subobject classifier there can only be one morphism that does that, so we must have $tf = tg$. Finally, since t is a monomorphism, we get $f = g$.

Exercise Given a small category \mathcal{C} , find the subobject classifier in the category $\mathbf{Set}^{\mathcal{C}}$.

Solution If Ω exists, it is a functor $\mathcal{C} \rightarrow \mathbf{Set}$ which satisfies the universal property $\text{Nat}(F, \Omega) \cong \text{Sub}(F)$ for every other functor $F : \mathcal{C} \rightarrow \mathbf{Set}$, where the isomorphism is given by pulling back a (yet to be found) natural transformation $* \rightarrow \Omega$. If this is true, then by the Yoneda lemma we can evaluate the functor Ω at an object C as

$$\Omega(C) \cong \text{Nat}(\mathbf{y}(C), \Omega) \cong \text{Sub}(\mathbf{y}(C)).$$

We use this to define Ω as sending C to the set $\text{Sub}(\mathbf{y}(C))$, and a morphism $f : C \rightarrow C'$ to the pullback map $f^* : \text{Sub}(\mathbf{y}(C')) \rightarrow \text{Sub}(\mathbf{y}(C))$. The natural transformation $t : * \rightarrow \Omega$ has components $t_C : * \rightarrow \text{Sub}(\mathbf{y}(C))$, defined as the map selecting the maximal subobject $\text{id} : \mathbf{y}(C) \rightarrow \mathbf{y}(C)$. The universal property holds by construction.

Week 7

Definition Recall that given two categories \mathcal{C} and \mathcal{D} , we define $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ to be adjoint, and write $F \dashv G$, if for every object $C \in \mathcal{C}$ and $D \in \mathcal{D}$ there are bijections

$$\phi_{C,D} : \mathcal{D}(F(C), D) \cong \mathcal{C}(C, G(D))$$

that are natural in both C and D . This means that for a morphism $h : C' \rightarrow C$ the square

$$\begin{array}{ccc} \mathcal{D}(F(C), D) & \xrightarrow{\phi_{C,D}} & \mathcal{C}(C, G(D)) \\ \downarrow F(f)^* & & \downarrow f^* \\ \mathcal{D}(F(C'), D) & \xrightarrow{\phi_{C',D}} & \mathcal{C}(C', G(D)) \end{array}$$

commutes. Analogously, for a morphism $k : D \rightarrow D'$, the square

$$\begin{array}{ccc} \mathcal{D}(F(C), D) & \xrightarrow{\phi_{C,D}} & \mathcal{C}(C, G(D)) \\ \downarrow k_* & & \downarrow G(k)_* \\ \mathcal{D}(F(C), D') & \xrightarrow{\phi_{C,D'}} & \mathcal{C}(C, G(D')) \end{array}$$

commutes.

There is a more compact way of expressing naturality of these isomorphisms. Consider the two functors $\mathcal{C}^{op} \times \mathcal{D} \rightarrow \mathbf{Set}$ given as the composites

$$\begin{array}{ccc} & \mathcal{D}^{op} \times \mathcal{D} & \\ F^{op} \times \text{id} \nearrow & & \searrow \mathcal{D}(-,-) \\ \mathcal{C}^{op} \times \mathcal{D} & & \mathbf{Set} \\ \text{id} \times G \searrow & & \nearrow \mathcal{C}(-,-) \\ & \mathcal{C}^{op} \times \mathcal{C} & \end{array}$$

What we are saying is now that $\phi_{C,D}$ is a natural isomorphism between these two functors.

Exercise Show that whenever a category has exponentials, then for an object $C \in \mathcal{C}$ there is an adjoint pair $(-) \times C \dashv (-)^C$.

Solution By definition of exponential, we know that there is a map $\text{ev}_A : C \times A^C \rightarrow A$ such that for every $f : C \times B \rightarrow A$ there is a unique $\tilde{f} : B \rightarrow A^C$ fitting in a commutative diagram

$$\begin{array}{ccc} C \times B & \xrightarrow{f} & A \\ C \times \tilde{f} \downarrow & \nearrow \text{ev}_A & \\ C \times A^C & & \end{array}$$

In particular, we may define $\phi_{B,A}$ to send f to \tilde{f} . This is a bijection by universal property. We want to prove that it is natural. Choose a morphism $h : B' \rightarrow B$ and a morphism $k : A \rightarrow A'$. We need to show that the square

$$\begin{array}{ccc} \mathcal{C}(C \times B, A) & \xrightarrow{\phi_{B,A}} & \mathcal{C}(B, A^C) \\ \mathcal{C}(C \times h, k) \downarrow & & \downarrow \mathcal{C}(h, k^C) \\ \mathcal{C}(C \times B', A') & \xrightarrow{\phi_{B',A'}} & \mathcal{C}(B', A'^C) \end{array}$$

is commutative. Choosing a map $f : C \times B \rightarrow A$, the right-then-down composite takes it to $k^C \circ \tilde{f} \circ h$, the down-then-right composite takes it to the map associated to $k \circ f \circ (C \times h)$ by the universal property of exponentials. Since this map is unique, it suffices to check that $k^C \circ \tilde{f} \circ h$ fits in the appropriate diagram. This diagram is

$$\begin{array}{ccc} C \times B' & \xrightarrow{k \circ f \circ (C \times h)} & A' \\ C \times (k^C \circ \tilde{f} \circ h) \downarrow & \nearrow \text{ev}_{A'} & \\ C \times A'^C & & \end{array}$$

This can be decomposed into the following, a bit more complicated diagram

$$\begin{array}{ccccc} & & C \times B' & & \\ & & \downarrow C \times h & \searrow k \circ f \circ (C \times h) & \\ & & C \times B & & \\ & & \downarrow C \times \tilde{f} & \searrow f & \\ & & C \times A^C & \nearrow \text{ev}_A & A \xrightarrow{k} A' \\ & & \downarrow C \times k^C & \nearrow \text{ev}_{A'} & \\ & & C \times A'^C & & \end{array}$$

Here the upper square commutes evidently, the middle triangle by universal property of exponentials and the lower square by definition of k^C .

Example As a consequence of the theorem above, we have many interesting examples where the functor taking products with a fixed object has a right adjoint: **Set**, **Mon**, **Grp**, **Ab**, **Haus**, **CRng**, just to name a few, are special cases of it.

Example The product-exponential pattern is very common, and many adjunction that arise naturally are variants of this form. Another very common pattern is constituted by the free-forgetful adjunctions. We give one example, the others are more or less similar to this.

Given a set X , we say that FX is the free group generated by X if there is a map $\eta_X : X \rightarrow FX$ such that, for any other group G and every function $X \rightarrow G$ of underlying sets, there exists a unique group homomorphism $FX \rightarrow G$ fitting in the commutative triangle

$$\begin{array}{ccc} X & \longrightarrow & G \\ \eta_X \downarrow & \nearrow & \\ FX & & \end{array}$$

Denoting by $U : \mathbf{Grp} \rightarrow \mathbf{Set}$ the forgetful functor associating to every group its underlying set, observe that the map $X \rightarrow G$ above should more precisely written as $X \rightarrow UG$. The universal property of free groups then associates to every such map $X \rightarrow UG$ a unique group homomorphism $FX \rightarrow G$, thus giving a bijection

$$\mathbf{Grp}(FX, G) \cong \mathbf{Set}(X, UG)$$

which is natural in both variables. This is indeed an adjunction

$$F : \mathbf{Set} \rightleftarrows \mathbf{Mon} : U.$$

Example The evaluation map $\text{ev}_A : C \times A^C \rightarrow A$ and the injection $\eta_X : X \rightarrow FX$ play dual roles in their respective settings. Indeed, if we denote a general left adjoint functor by L and a right adjoint by R , the former is of the form $\varepsilon_Q : LRQ \rightarrow Q$ and the latter is of the form $\eta_Q : Q \rightarrow RLQ$ for an object Q . Moreover, they are natural in Q , so they are in fact natural transformations of the forms $\varepsilon : LR \Rightarrow \text{id}$ and $\eta : \text{id} \Rightarrow RL$ respectively. These are precisely the counit and the unit of the corresponding adjunctions.

Exercise What are the counits of the product-exponential adjunction in **Set**? And what about the counit of the free-forgetful adjunction between **Set** and **Mon**?

Solution There is a natural bijection

$$\mathbf{Set}(A \times C, A \times C) \cong \mathbf{Set}(A, (A \times C)^C) \cong \mathbf{Set}(A, A^C \times C^C)$$

and we know that the unit corresponds to the identity in the left-hand side. An explicit calculation yields that the identity is sent to the map sending an element $a \in A$ to the pair (c_a, id_C) , where the first component is the constant map in a , and the second is simply the identity $C \rightarrow C$. Similarly, the counit of $F \dashv U$ corresponds to the identity via the bijection

$$\mathbf{Set}(UG, UG) \cong \mathbf{Mon}(FUG, G).$$

Now, elements of the group FUG are finite sequences (g_1, \dots, g_n) , where the g_i 's are elements of G . The counit map $FUG \rightarrow G$ sends such a sequence to the product $g_1 \cdots g_n$.

Exercise Show that, if \mathcal{C} is a category having all \mathbf{J} -indexed limits, there is an adjunction

$$\delta : \mathcal{C} \rightleftarrows \mathcal{C}^{\mathbf{J}} : \lim$$

where δ is the functor sending C to the functor constant at C .

Solution We need to show that there are natural bijections of the form

$$\mathcal{C}^{\mathbf{J}}(\delta_C, F) \cong \mathcal{C}(C, \lim F).$$

But now the left-hand side is precisely $\text{Cone}(C, F)$, which is naturally isomorphic to $\mathcal{C}(C, \lim F)$ by universal property of limits.

Example By a dual argument, we know that there are natural bijections

$$\mathcal{C}(\text{colim } F, C) \cong \mathcal{C}^{\mathbf{J}}(F, \delta_C)$$

so that we have an adjunction

$$\text{colim} : \mathcal{C}^{\mathbf{J}} \rightleftarrows \mathcal{C} : \delta$$

Exercise Show that a left adjoint preserves colimits and that a right adjoint preserves limits.

Solution We will show that a right adjoint preserves limits, the other argument is dual. Let $(D_i)_{i \in I}$ be a diagram in the category \mathcal{D} , and let there be an adjunction

$$L : \mathcal{C} \rightleftarrows \mathcal{D} : R$$

we want to show that $R(\lim_{i \in I} D_i)$ is a limit of the diagram $(RD_i)_{i \in I}$. Let us compute, for an object $C \in \mathcal{C}$,

$$\begin{aligned} \mathcal{C}(C, R(\lim_{i \in I} D_i)) &\cong \mathcal{D}(LC, \lim_{i \in I} D_i) \\ &\cong \lim_{i \in I} \mathcal{D}(LC, D_i) \\ &\cong \lim_{i \in I} \mathcal{C}(C, RD_i) \\ &\cong \mathcal{C}(C, \lim_{i \in I} RD_i) \end{aligned}$$

where all the isomorphisms are natural by various universal properties. Now the result follows by a characterization of limits.

Week 8

Exercise Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a functor. Show that F has a right adjoint if and only if for every object $B \in \mathcal{B}$ the functor $\mathcal{B}(F-, B) : \mathcal{A}^{op} \rightarrow \mathbf{Set}$ is representable.

Solution Clearly, if F has a right adjoint G , then

$$\mathcal{B}(FA, B) \cong \mathcal{A}(A, GB)$$

naturally in both A and B . This means that the functor above is represented by GB .

Conversely, suppose that every functor of the form $\mathcal{B}(F-, B)$ is represented by some object \bar{B} . This means that there are natural bijections

$$\phi_{A,B} : \mathcal{B}(FA, B) \cong \mathcal{A}(A, \bar{B})$$

that are natural in A . Now choose such an object $\bar{B} \in \mathcal{A}$ and such a bijection for every object $B \in \mathcal{B}$ and define a functor $G : \mathcal{B} \rightarrow \mathcal{A}$ as follows: on objects, the action is given by $GB := \bar{B}$. For a morphism $f : B \rightarrow B'$, consider the composite

$$\mathcal{A}(A, GB) \xrightarrow{\phi_{A,B}} \mathcal{B}(FA, B) \xrightarrow{f^*} \mathcal{B}(FA, B') \xrightarrow{\phi_{A,B'}^{-1}} \mathcal{A}(A, GB').$$

By Yoneda lemma, this corresponds to an arrow $GB \rightarrow GB'$, which we call Gf .

Now, the square

$$\begin{array}{ccc} \mathcal{B}(FA, B) & \xrightarrow{\phi_{A,B}} & \mathcal{A}(A, GB) \\ \downarrow f^* & & \downarrow \\ \mathcal{B}(FA, B') & \xrightarrow{\phi_{A,B'}} & \mathcal{A}(A, GB') \end{array}$$

commutes by the very definition of the right vertical map, giving naturality in B as well.

It only remains to show that G is a functor. If we pick the identity, the composite $\phi_{A,B'}^{-1} \circ \text{id}_B^* \circ \phi_{A,B} : \mathcal{B}(FA, B) \rightarrow \mathcal{B}(FA, B)$ as above is the identity, therefore it corresponds to the identity via the Yoneda isomorphism. If $f : B \rightarrow B'$ and $g : B' \rightarrow B''$ are two composable maps, then we have

$$\begin{aligned} \phi_{A,B''}^{-1} \circ g^* \circ \phi_{A,B'} \circ \phi_{A,B'}^{-1} \circ f^* \circ \phi_{A,B} &= \phi_{A,B''}^{-1} \circ g^* \circ f^* \circ \phi_{A,B} \\ &= \phi_{A,B''} \circ (gf)^* \circ \phi_{A,B}. \end{aligned}$$

Since the left-hand side corresponds to $Gg \circ Gf$ and the right-hand side to $G(gf)$ via the Yoneda isomorphism, this concludes the proof.

Remark We also have the dual statement, that is, given a functor $G : \mathcal{B} \rightarrow \mathcal{A}$, then it has a left adjoint if and only if for every object $A \in \mathcal{A}$ the functor $\mathcal{A}(A, G-) : \mathcal{B} \rightarrow \mathbf{Set}$ is representable. The proof is analogous, and uses the covariant Yoneda lemma instead of the contravariant one.

Exercise Let Ω be a subobject classifier in an elementary topos, and let $\phi : \Omega \rightarrow \Omega$ be a monomorphism. Prove that $\phi \circ \phi = \text{id}_\Omega$. In particular, ϕ is an isomorphism.

Solution First, define the two pullback squares

$$\begin{array}{ccccc} V & \longrightarrow & U & \longrightarrow & 1 \\ \downarrow & & \downarrow k & & \downarrow t \\ 1 & \xrightarrow{t} & \Omega & \xrightarrow{\phi} & \Omega. \end{array}$$

The fact that the outer square is a pullback can be expressed as $\phi t = \chi_V$. Moreover, there is a double pullback

$$\begin{array}{ccccc} V & \longrightarrow & V & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow t \\ U & \longrightarrow & 1 & \xrightarrow{\chi_V} & \Omega \end{array}$$

where the right part is a pullback by definition, and the left part is easily checked to verify the universal property, using that the map $U \rightarrow 1$ is monic. By uniqueness of the classifying map, this means that $\chi_V i_U = k$. Using this and the fact that k is monic, we see that there is a double pullback

$$\begin{array}{ccccc} U & \longrightarrow & T & \longrightarrow & U \\ \downarrow & & \downarrow & & \downarrow k \\ U & \longrightarrow & 1 & \xrightarrow{\chi_V} & \Omega \end{array}$$

so that T is a subobject of 1 such that $U \leq T$ and $T \leq U$, which implies $T = U$. As a consequence, we know that the left square (and the right one as well) is a pullback in the diagram

$$\begin{array}{ccccc} U & \longrightarrow & U & \longrightarrow & 1 \\ \downarrow & & \downarrow k & & \downarrow t \\ 1 & \xrightarrow{\chi_V} & \Omega & \xrightarrow{\phi} & \Omega. \end{array}$$

Again by uniqueness of the classifying map, this means that $\phi \chi_V = \chi_U$. The key facts that we proved and need to keep in mind for the following step are:

- $\phi t = \chi_V$;
- $\chi_V i_U = k$;

$$- \phi\chi_V = \chi_U.$$

Now, in the big diagram

$$\begin{array}{ccccccc}
 V & \longrightarrow & V & \longrightarrow & 1 & \longrightarrow & 1 \\
 \downarrow & & \downarrow & & \downarrow t & & \downarrow t \\
 U & \longrightarrow & U & \xrightarrow{k} & \Omega & \longrightarrow & \Omega \\
 \downarrow & & \downarrow & & \downarrow \phi & & \downarrow \phi \\
 U & \longrightarrow & 1 & \xrightarrow{t} & \Omega & \longrightarrow & \Omega \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \phi \\
 U & \longrightarrow & 1 & \xrightarrow{t} & \Omega & \xrightarrow{\phi} & \Omega
 \end{array}$$

every square is a pullback. By the three formulas above, the outer square can be rewritten as

$$\begin{array}{ccc}
 V & \longrightarrow & 1 \\
 \downarrow & & \downarrow \chi_U \\
 U & \xrightarrow{k} & \Omega.
 \end{array}$$

This fact combined with the first double square in the proof implies that $\phi\chi_U = \phi t$. Since ϕ is monic, this implies $\chi_U = t$. Now we have

$$\phi^2 t = \phi\chi_V = \chi_U = t.$$

To conclude, compute the following pullback

$$\begin{array}{ccccc}
 1 & & & & \\
 \downarrow t & \searrow & & & \\
 W & \longrightarrow & 1 & & \\
 \downarrow & & \downarrow t & & \\
 \Omega & \xrightarrow{\phi^2} & \Omega & &
 \end{array}$$

Since $\phi^2 t = t$, the outer square commutes so there is a factorization. Thus, W is a subobject of 1 that has a section, which means that it is in fact isomorphic to 1 , so that the square

$$\begin{array}{ccc}
 1 & \longrightarrow & 1 \\
 \downarrow t & & \downarrow t \\
 \Omega & \xrightarrow{\phi^2} & \Omega
 \end{array}$$

is a pullback. Using the universal property of Ω one last time yields that $\phi^2 = \text{id}_\Omega$ as desired.

Exercise Show that in an elementary topos the subobject classifier is injective.

Solution Remember that Ω being injective means that for every monomorphism $A \hookrightarrow B$ and every morphism $A \rightarrow \Omega$, there is a lift in the diagram

$$\begin{array}{ccc} A & \longrightarrow & \Omega \\ \downarrow & \nearrow & \\ B & & \end{array}$$

Compute the pullback diagram

$$\begin{array}{ccc} M & \longrightarrow & 1 \\ \downarrow & & \downarrow t \\ A & \longrightarrow & \Omega \end{array}$$

So that in particular we have a composite monomorphism $M \hookrightarrow A \hookrightarrow B$. In turn, this is classified by a morphism $B \rightarrow M$. We want to show that this is the lift we are looking for.

By universal property of Ω , it suffices to show that the given morphism $A \rightarrow \Omega$ and the composite $A \hookrightarrow B \rightarrow \Omega$ classify the same subobject of B . To see this, observe the following diagram

$$\begin{array}{ccccc} M & \xrightarrow{\text{id}} & M & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow t \\ A & \xrightarrow{\text{id}} & A & & \\ \downarrow \text{id} & & \downarrow & & \\ A & \hookrightarrow & B & \longrightarrow & \Omega. \end{array}$$

The right square is a pullback by definition, the left lower square by a characterization of monomorphisms and the left upper square evidently. Therefore, the bigger square is a pullback, so the composite $A \hookrightarrow B \rightarrow \Omega$ is a classifying map for the subobject $M \hookrightarrow A$, but this is classified by the morphism $A \rightarrow \Omega$ by definition. This concludes the proof.

Week 9

Definition Recall that a functor $G : \mathcal{D} \rightarrow \mathcal{C}$ is said to satisfy the solution set condition if, for every object $C \in \mathcal{C}$ there is a set of objects $(D_i)_{i \in I}$ in \mathcal{D} such that every map $C \rightarrow GD$ can be factored as $C \rightarrow GD_i \rightarrow GD$ for some $i \in I$ and some map $D_i \rightarrow D$ in \mathcal{C} .

Theorem Also recall the following theorem, of the utmost importance in category theory.

Let $G : \mathcal{D} \rightarrow \mathcal{C}$ be a functor, where \mathcal{D} is complete. Then G has a left adjoint if and only if the two following conditions are satisfied:

1. G preserves all small limits;
2. G satisfies the solution set condition.

Example The forgetful functor $U : \mathbf{CHaus} \rightarrow \mathbf{Set}$, assigning to every compact Hausdorff space its underlying set of points, has a left adjoint.

To see this, we need to verify the two conditions of the above theorem. By a direct check of the universal property, we see that the product topology and the subspace topology provide products and equalizers, which means that U preserves limits.

We need the solution set condition. Fixing a set S , we need to find a suitable set of compact Hausdorff space. We will get there through the following reasoning: given a function $f : S \rightarrow UX$ (that we would like to factor through the set we are looking for), we construct a map $\overline{fS} \rightarrow \mathcal{PPS}$ by sending every point $x \in \overline{fS}$ to the set

$$Lx = \{D \subseteq S \mid x \in \overline{fD}\}.$$

Using that X is compact Hausdorff, for two distinct points $x, x' \in X$ we can always construct two open sets $U \ni x$ and $U' \ni x'$ in such a way that $\overline{U} \cap \overline{U'} = \emptyset$. In particular, $f^{-1}U \in Lx$ but $f^{-1}U \notin Lx'$. This implies that the function $L : X \rightarrow \mathcal{PPS}$ is injective. Now, observe that f factors as

$$S \rightarrow \overline{fS} \subseteq UX$$

and we have just proven that \overline{fS} can be regarded as a subset of \mathcal{PPS} with a suitable topology.

This construction motivates the following choice: once we fix a set S , we take the set of all subsets of \mathcal{PPS} with all possible compact Hausdorff topologies on them. The above discussion proves that this is indeed a solution set.

Therefore, the adjoint functor theorem yields a left adjoint to U . More explicitly, this left adjoint assigns to each set S the Stone-Ćech compactification of the discrete topology on S .

Exercise Show that the forgetful functor $U : \mathbf{Grp} \rightarrow \mathbf{Set}$ has a left adjoint using the adjoint functor theorem.

Solution We already know that there is a natural isomorphism $U \cong \mathbf{Grp}(\mathbb{Z}, -)$, so that it preserves limits.

We need to show that U satisfies the solution set condition. Choose a set X . There exists a cardinal λ such that X has less than λ elements. Now we take the set of all groups that admit a generating set smaller than λ . This is indeed a set, because up to isomorphism these groups are in bijection with quotients of groups of the form \mathbb{Z}^i , with $i < \lambda$.

Given a group G and an arbitrary map $X \rightarrow UG$, observe that, since $|X| < \lambda$, its image in G is smaller than λ . Therefore, the subgroup $G_X \subseteq G$ generated by the image certainly belongs to the set of groups that we described above. Now the image inclusion provides the factorization

$$X \rightarrow UG_X \hookrightarrow UG$$

that we were looking for.

Counterexample Consider the forgetful functor $U : \mathbf{CBool} \rightarrow \mathbf{Set}$. The category \mathbf{CBool} of complete Boolean algebras is complete, and the forgetful functor preserves small limits. However, it has been (non-trivially) proven that for every set S there are arbitrarily large complete Boolean algebras generated by S , which falsifies the solution set condition. As a consequence, there is no "free complete Boolean algebra" functor.

Remark In the case of groups, we used essentially that every set is bounded by some cardinal and that every group can be constructed out of its subgroups with a similar bound on the respective generating sets. This motivates the following definitions.

Definition Let λ be a regular cardinal. We say that a small category \mathbf{J} is λ -filtered if every diagram in \mathbf{J} which is smaller than λ admits a cocone. For an intuition, think of a set X . The poset of all subsets of X of cardinality $< \lambda$ is λ -filtered, since the union of λ -small subsets is itself λ -small.

Definition Let $C \in \mathcal{C}$ be an object of a category. Let us fix a regular cardinal λ . We say that C is λ -compact if the functor $\mathcal{C}(C, -)$ preserves λ -filtered colimits. In other words, if \mathbf{J} is a λ -filtered category and $(D_i)_{i \in \mathbf{J}}$ is a diagram in \mathcal{C} , the natural map

$$\operatorname{colim}_{\mathbf{J}} \mathcal{C}(C, D_i) \rightarrow \mathcal{C}(C, \operatorname{colim}_{\mathbf{J}} D_i)$$

is an isomorphism.

Remark The above definition can be expressed more explicitly by saying that every map $C \rightarrow \operatorname{colim}_{\mathbf{J}} D_i$ factors through one of the colimit inclusions $c_i : D_i \rightarrow \operatorname{colim}_{\mathbf{J}} D_i$. Moreover, this factorization is essentially unique in the sense that, if there is a set of factorizations $(C \rightarrow D_i)_{i \in I}$, where $|I| < \lambda$, then there is an object D_t , with $t \in \mathbf{J}$, and a cocone $(D_i \rightarrow D_t)_{i \in I}$ such that all the composite maps

$$(C \rightarrow D_i \rightarrow D_t)_{i \in I}$$

are equal. In other words, a set of less than λ factorizations can always be amalgamated to a single object.

Example In \mathbf{Set} , λ -compact objects are precisely sets of cardinality smaller than λ . In \mathbf{Grp} (and categories of algebras generally), λ -compact objects are precisely groups that admit a λ -small generating set.

\mathbf{Top} is ill-behaved in this respect, in that its λ -compact objects are precisely discrete spaces of cardinality smaller than λ , so it does not capture topological notions of compactness. In particular, compact spaces are not necessarily finitely compact in the present sense.

Definition We say that a category \mathcal{K} is locally λ -presentable if

- it is cocomplete,
- the subcategory $\mathcal{K}^\lambda \subseteq \mathcal{K}$ of λ -compact objects is essentially small and

- every object $K \in \mathcal{K}$ is a λ -filtered colimit of λ -compact objects.

We say that \mathcal{K} is locally presentable if it is locally λ -presentable for some λ .

Example Typically, categories arising in algebra are presentable, whereas categories arising in topology are not. In particular, **Set**, **Grp**, **Mon**, **R-Mod**, **Pos** and all categories of the form **Set** ^{\mathcal{C}} with \mathcal{C} small are locally presentable. The categories, **Top**, **Haus**, **CHaus**, **CBool**, **CABA** are not locally presentable.

Tools These facts are very handy but highly non-trivial, so we are just stating them for future usage.

1. In a locally presentable category, for every μ the subcategory of μ -compact objects is essentially small.
2. In a locally presentable category, every object is μ -compact for some μ . As a consequence, a locally presentable category \mathcal{K} can be written as a union of small categories $\bigcup_{\mu} \mathcal{K}^{\mu}$.
3. If a category is locally presentable, there are arbitrarily large cardinals μ such that it is locally μ -presentable.

Moreover, observe that for $\lambda < \mu$ every λ -compact object is also μ -compact.

Theorem (Adjoint functor theorem, presentable version).

Let $G : \mathcal{D} \rightarrow \mathcal{C}$ be a functor between locally presentable categories. Then G has a left adjoint if and only if it preserves limits and λ -filtered colimits for some λ .

Proof We only need to verify the solution set condition. Fix an object $C \in \mathcal{C}$. It is μ -compact for some μ , and we may assume that $\mu \geq \lambda$. Then take the set of μ -compact objects of \mathcal{D} . Now, given a morphism $C \rightarrow GD$, express D as a μ -filtered colimit of μ -compact objects, $D \cong \text{colim } D_i$. Since $\mu \geq \lambda$, we know that G preserves μ -filtered colimits, so that we also have $GD \cong \text{colim } GD_i$. Since C is μ -compact, there is a factorization

$$C \rightarrow GD_i \rightarrow GD$$

as we wanted.

For the converse, we only need to show that G preserves λ -filtered colimits. Let F be a left adjoint to G . Let λ be such that \mathcal{C} is locally λ -presentable and let μ be such that the image of every λ -compact object along F is μ -compact, and take a μ -filtered diagram (D_i) . By the Yoneda lemma, it suffices to show that there are natural isomorphisms $\mathcal{C}(C, \text{colim } GD_i) \cong \mathcal{C}(C, G \text{colim } D_i)$ for every $C \in \mathcal{C}$. Express C as a colimit of λ -compact objects C_j 's. Remembering that F preserves all colimits, we may now compute

$$\begin{aligned}
\mathcal{C}(C, G \operatorname{colim}_i D_i) &\cong \mathcal{D}(FC, \operatorname{colim}_i D_i) \\
&\cong \mathcal{D}(\operatorname{colim}_j FC_j, \operatorname{colim}_i D_i) \\
&\cong \lim_j \mathcal{D}(FC_j, \operatorname{colim}_i D_i) \\
&\cong \lim_j \operatorname{colim}_i \mathcal{D}(FC_j, D_i) \\
&\cong \lim_j \operatorname{colim}_i \mathcal{C}(C_j, GD_i) \\
&\cong \lim_j \mathcal{C}(C_j, \operatorname{colim}_i GD_i) \\
&\cong \mathcal{C}(\operatorname{colim}_j C_j, \operatorname{colim}_i GD_i) \\
&\cong \mathcal{C}(C, \operatorname{colim}_i GD_i)
\end{aligned}$$

which completes the proof.

Week 10

Remark Observe that a λ -small colimit of λ -compact objects is λ -compact. To see this, let $C = \operatorname{colim}_i C_i$ be a λ -small colimit, where each of the C_i 's is λ -compact, and let $D = \operatorname{colim}_j D_j$ be any λ -filtered colimit. Using that λ -filtered colimit commute with λ -small limits in **Set**, we compute

$$\begin{aligned}
\mathcal{C}(C, \operatorname{colim}_j D_j) &\cong \lim_i \mathcal{C}(C_i, \operatorname{colim}_j D_j) \cong \lim_i \operatorname{colim}_j \mathcal{C}(C_i, D_j) \cong \\
&\operatorname{colim}_j \lim_i \mathcal{C}(C_i, D_j) \cong \operatorname{colim}_j \mathcal{C}(C, D_j)
\end{aligned}$$

which proves the claim.

Exercise Show that for a small category \mathcal{C} the category $\mathbf{Set}^{\mathcal{C}^{op}}$ is locally presentable.

Solution It is clear that $\mathbf{Set}^{\mathcal{C}^{op}}$ has all small colimits. Moreover, every functor $F : \mathcal{C}^{op} \rightarrow \mathbf{Set}$ is a colimit of representables indexed by the diagram \mathcal{C}/F . Let us take the diagram $\mathcal{D} \supseteq \mathcal{C}/F$ of all finite colimits of representables over F . This is small, filtered, and a check of the universal property shows that its colimit must be the same as that of \mathcal{C}/F . We only need to show that objects of \mathcal{D} are compact. By the lemma above, we are reduced to showing that representable functors are compact. In fact, we can show the stronger result that they are *absolutely compact*, that is, the functors $\mathbf{Set}^{\mathcal{C}^{op}}(\mathcal{C}(-, C), -)$ (in the second variable) preserve *all* colimits. To see this, choose a colimit $F = \operatorname{colim} F_i$ in $\mathbf{Set}^{\mathcal{C}^{op}}$ and compute

$$\mathbf{Set}^{\mathcal{C}^{op}}(\mathcal{C}(-, C), F) \cong F(C) \cong \operatorname{colim} F_i(C) \cong \operatorname{colim} \mathbf{Set}^{\mathcal{C}^{op}}(\mathcal{C}(-, C), F_i)$$

which concludes the proof.

Exercise Prove that, if \mathcal{K} is locally presentable, there is a small category \mathcal{C} such that \mathcal{K} is a reflective full subcategory of $\mathbf{Set}^{\mathcal{C}^{op}}$.

Solution We know that there is λ such that the full subcategory \mathcal{K}^λ is small. Take $\mathcal{C} = \mathcal{K}^\lambda$. First, we want to show that there is a full embedding $E : \mathcal{C} \hookrightarrow \mathbf{Set}^{\mathcal{K}^{\lambda op}}$. We define it to be the restriction of the Yoneda embedding, so that its action on objects is

$$K \mapsto \mathcal{K}(-, K)$$

where the functor $\mathcal{K}(-, K)$ is only intended to take λ -compact objects as inputs. We want to show that $\mathcal{K}(K, H) \cong \mathbf{Set}^{\mathcal{K}^{\lambda op}}(\mathcal{K}(-, K), \mathcal{K}(-, H))$. Suppose $f \neq g : K \rightarrow H$. Let us express K as a λ -filtered colimit of λ -compact objects $(K_j)_{j \in J}$. By the universal property of colimits, there is at least one j such that the two composites

$$K_j \rightarrow K \rightrightarrows H$$

are different, which means that the induced natural transformations $\mathcal{K}(-, f)$ and $\mathcal{K}(-, g)$ are different as well, and E is faithful. Now, take a natural transformation $\mathcal{K}(-, K) \rightarrow \mathcal{K}(-, H)$, where the input is restricted to λ -compact objects. Taking the same diagram J as above, the components

$$\mathcal{K}(K_j, K) \rightarrow \mathcal{K}(K_j, H)$$

gives rise, by taking limits on the opposite diagram J^{op} , to a map

$$\lim_j \mathcal{K}(K_j, K) \rightarrow \lim_j \mathcal{K}(K_j, H)$$

which is isomorphic to

$$\mathcal{K}(K, K) \rightarrow \mathcal{K}(K, H)$$

so that we get a morphism $K \rightarrow H$ as image of id_K , which represents our natural transformation.

Now we need to show that E has a left adjoint. By the adjoint functor theorem in its presentable version, it suffices to show that E preserves λ -filtered colimits. So if $H = \text{colim}_i H_i$ is a λ -filtered colimit in \mathcal{C} , what we need to show is that $EL(K) \cong \text{colim}_i EH_i(K)$ naturally for every λ -compact object K . Compute

$$EH(K) \cong \mathcal{K}(K, H) \cong \text{colim}_i \mathcal{K}(K, H_i) \cong \text{colim}_i EH_i(K)$$

so we are done.

For additional clarity, let us write a left adjoint explicitly. The idea is that, since every functor $\mathcal{K}^{\lambda op} \rightarrow \mathbf{Set}$ is a λ -filtered colimit of objects of \mathcal{K}^λ , this functor concretely calculates this colimit inside \mathcal{C} .

In detail, given a functor $F : \mathcal{K}^{\lambda op} \rightarrow \mathbf{Set}$, this is a colimit of the diagram $\mathcal{K}_{/F}^\lambda \rightarrow \mathcal{K}^\lambda \hookrightarrow \mathbf{Set}^{\mathcal{K}^{\lambda op}}$. We define LF to be a colimit of the diagram $\mathcal{K}_{/F}^\lambda \rightarrow \mathcal{K}^\lambda \hookrightarrow \mathcal{C}$. This is clearly functorial, it remains to show that it is in

fact a left adjoint to E . Let $H \in \mathcal{K}$, and express it as a λ -filtered colimit of λ -compact objects $L = \operatorname{colim}_i H_i$. Now compute

$$\begin{aligned}
\mathcal{K}(LF, H) &\cong \mathcal{K}(\operatorname{colim} K_j, H) \\
&\cong \lim \mathcal{K}(K_j, H) \\
&\cong \lim \operatorname{colim}_i \mathcal{K}^\lambda(K_j, H_i) \\
&\cong \lim \operatorname{colim}_i \mathbf{Set}^{\mathcal{K}^{\lambda op}}(\mathcal{K}(-, K_j), \mathcal{K}(-, H_i)) \\
&\cong \lim \mathbf{Set}^{\mathcal{K}^{\lambda op}}(\mathcal{K}(-, K_j), \operatorname{colim}_i \mathcal{K}(-, H_i)) \\
&\cong \lim \mathbf{Set}^{\mathcal{K}^{\lambda op}}(\mathcal{K}(-, K_j), EH) \\
&\cong \mathbf{Set}^{\mathcal{K}^{\lambda op}}(\operatorname{colim} \mathcal{K}(-, K_j), EH) \\
&\cong \mathbf{Set}^{\mathcal{K}^{\lambda op}}(F, EH).
\end{aligned}$$

Corollary Every locally presentable category is complete.

Proof Let \mathcal{K} be locally presentable and let $E : \mathcal{K} \hookrightarrow \mathbf{Set}^{cop}$ be an embedding as above, with L being its left adjoint. Observe that, since E is fully faithful, we have $LE \cong \operatorname{id}_{\mathcal{K}}$.

Given a small diagram $p : \mathbf{J} \rightarrow \mathcal{K}$, compose it with E . Since \mathbf{Set}^{cop} is already known to be complete, the diagram Ep has a limit. Take a limiting cone

$$(\lim EK_j \rightarrow EK_j)_{j \in \mathbf{J}}$$

and plug it through the left adjoint L . This yields a cone in \mathcal{K} of the form

$$(L \lim EK_j \rightarrow LEK_j \cong K_j)_{j \in \mathbf{J}}.$$

We want to show that this is a limiting cone. Given another cone

$$(Q \rightarrow K_j)_{j \in \mathbf{J}}$$

using that $Q \cong LEQ$, this corresponds precisely to a cone

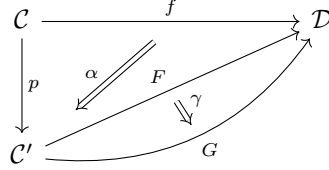
$$(EQ \rightarrow LK_j)_{j \in \mathbf{J}}$$

in \mathbf{Set}^{cop} which, by universal property, yields a unique factorization $\lim EK_j \rightarrow EQ$. By adjunction, this corresponds uniquely to a morphism $L \lim EK_j \rightarrow LEQ \cong Q$, which is the unique factorization we were looking for.

Remark The proof that we have seen of the adjoint functor theorem in its presentable version actually uses that locally presentable categories are complete. Therefore, if we don't know this fact from other sources, we can't use AFT in the above theorem, and the explicit construction of L becomes necessary.

Remark The above corollary does not mean that L preserves limits. Rather, it preserves all limits of objects that are in the image of E .

Definition Given two functors $f : \mathcal{C} \rightarrow \mathcal{D}$ and $p : \mathcal{C} \rightarrow \mathcal{C}'$, a left Kan extension of f along p is a pair (F, α) , where $F : \mathcal{C}' \rightarrow \mathcal{D}$ and $\alpha : f \Rightarrow Fp$ such that for any other such pair (G, β) there is a unique natural transformation $\gamma : F \Rightarrow G$ such that $\beta = \gamma p \circ \alpha$.



Remark Given a functor $G \in \mathcal{D}^{\mathcal{C}'}$ with a natural transformation $\beta \in \mathcal{D}^{\mathcal{C}}(f, Gp)$, the universal property of left Kan extensions gives a unique natural transformation $\gamma \in \mathcal{D}^{\mathcal{C}'}(F, G)$. This means that, if we conveniently denote F by $\text{Lan}_p f$, the assignment $f \mapsto F$ (which is functorial by naturality of the construction) gives rise to a natural bijection

$$\mathcal{D}^{\mathcal{C}'}(\text{Lan}_p f, G) \cong \mathcal{D}^{\mathcal{C}}(f, p^* G)$$

thus, if all left Kan extensions of functors $\mathcal{C} \rightarrow \mathcal{D}$ along p exist, they give rise to an adjunction

$$\text{Lan}_p : \mathcal{D}^{\mathcal{C}} \quad \perp \quad \mathcal{D}^{\mathcal{C}'} : p^*$$

The natural transformation $\alpha : f \rightarrow Fp = \text{Lan}_p f \circ p = p^* \text{Lan}_p f$ plays the role of the unit of this adjunction.

Definition Dually, there is a definition for right Kan extension. Given f and p as above, we say that a right Kan extension of f along p is a pair (F, α) , where $\alpha : Fp \Rightarrow f$ such that for any other pair (G, β) , where $G : \mathcal{C}' \rightarrow \mathcal{D}$ and $\beta : Gp \Rightarrow f$ there is a unique natural transformation $\gamma : G \Rightarrow F$ such that $\beta = \alpha \circ \gamma p$.

In this case, provided all the relevant right Kan extensions exist, there is an adjunction

$$p^* : \mathcal{D}^{\mathcal{C}'} \quad \perp \quad \mathcal{D}^{\mathcal{C}} : \text{Ran}_p$$

Remark Kan extensions do not always exist, but of course when they do they are unique up to natural isomorphism. However, there are some circumstances in which the existence of Kan extensions is granted.

The most usual is in the assumption that \mathcal{C} is small and \mathcal{D} is cocomplete. In this case, a left Kan extension $\text{Lan}_p f$ always exists, and it can be computed by the following formula for an object $C \in \mathcal{C}'$: first take the pullback square

$$\begin{array}{ccc}
\mathcal{C}/C & \longrightarrow & \mathcal{C}'/C \\
\downarrow \pi_C & & \downarrow \\
\mathcal{C} & \xrightarrow{p} & \mathcal{C}'
\end{array}$$

and then define $\text{Lan}_p f(C) := \text{colim } f\pi_C$. In other words, we take the colimit of all objects of \mathcal{C} that live over C via the functor p .

The natural map α is defined objectwise as the inclusion in the colimit.

Exercise Show that, if \mathcal{C} is small, \mathcal{D} is cocomplete and moreover p is a full embedding, then α is an isomorphism.

Solution In the case above, we are enabled to use the colimit formula. Moreover, if $\mathcal{C} \subseteq \mathcal{C}'$ is a full subcategory, then for an object $C \in \mathcal{C}$ the category \mathcal{C}/C is an honest slice category, so it has a terminal object. Therefore, the colimit of $f\pi_C$ must be the image of its terminal object, which is $f(C)$.

Remark Another way to phrase this is saying that if p is a full embedding then the diagram

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{f} & \mathcal{D} \\
p \downarrow & \nearrow & \\
\mathcal{C}' & & \text{Lan}_p f
\end{array}$$

commutes (up to isomorphism).

Week 11

The (co)limit formula We want to show that, in a diagram

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{f} & \mathcal{D} \\
\downarrow p & & \\
\mathcal{C}' & &
\end{array}$$

where \mathcal{C} is small and \mathcal{D} is cocomplete, the mentioned colimit formula actually gives a left Kan extension of f along p .

For an object $X \in \mathcal{C}'$, we denote by $\mathcal{C}/_X$ the category whose objects are pairs $(M, s : p(M) \rightarrow X)$ with M being an object of \mathcal{C} , and where a morphism from (M, s) to (M', s') is a morphism $M \rightarrow M'$ in \mathcal{C} such that the triangle

$$\begin{array}{ccc}
p(M) & \longrightarrow & p(M') \\
& \searrow s & \swarrow s' \\
& & X
\end{array}$$

commutes. This category has an obvious forgetful functor $\pi_X : \mathcal{C}_{/X} \rightarrow \mathcal{C}$. Then define $F(X) := \operatorname{colim} f\pi_X$. A morphism $X \rightarrow Y$ in \mathcal{C}' induces a functor $\mathcal{C}_{/X} \rightarrow \mathcal{C}_{/Y}$ by postcomposition, and the triangle

$$\begin{array}{ccc} \mathcal{C}_{/X} & \xrightarrow{\quad} & \mathcal{C}_{/Y} \\ & \searrow \pi_X & \swarrow \pi_Y \\ & \mathcal{C} & \end{array}$$

obviously commutes. This means that $\operatorname{colim} f\pi_Y$ is the vertex of a cocone on the diagram $f\pi_X$, therefore there is a unique factorization $\operatorname{colim} f\pi_X \rightarrow \operatorname{colim} f\pi_Y$. All this defines a functor $F : \mathcal{C}' \rightarrow \mathcal{D}$. Moreover, for an object $A \in \mathcal{C}$, the indexing diagram of $Fp(A)$ is given by objects $M \in \mathcal{C}$ with a morphism $p(M) \rightarrow p(A)$, so that A itself is such an object. Therefore, there is a colimit inclusion $\alpha_A : f(A) \rightarrow \operatorname{colim} f\pi_{p(A)}$, which is natural in A and therefore defines $\alpha : f \Rightarrow Fp$. We want to prove that this has the universal property of Kan extensions.

Assume that we are given another functor $G : \mathcal{C}' \rightarrow \mathcal{D}$ with a natural transformation $\beta : f \Rightarrow Gp$. Given an object $(M, s) \in \mathcal{C}_{/X}$, we have a composite morphism

$$f(M) \xrightarrow{\beta_M} Gp(M) \xrightarrow{Gs} G(X)$$

and, since β is natural and Gs is compatible with morphisms in $\mathcal{C}_{/X}$ by definition, this defines a cocone on $f\pi_X$ with vertex $G(X)$. The universal property of colimits now yields a unique factorization morphism $\gamma_X : \operatorname{colim} f\pi_X \rightarrow G(X)$, which is readily verified to be natural in X by the same universal property. Moreover, for an object $A \in \mathcal{C}$, the triangle

$$\begin{array}{ccc} & f(A) & \\ & \swarrow \alpha_A & \searrow \beta_A \\ \operatorname{colim} f\pi_{p(A)} & \xrightarrow{\gamma_{p(A)}} & Gp(A) \end{array}$$

commutes. Incidentally, this also proves that γ is uniquely determined in the image of p . It remains to show that it is uniquely determined on all \mathcal{C}' .

For this, take another natural transformation $\gamma' : F \Rightarrow G$ with the property that $\gamma'p \circ \alpha = \beta$. We want to show that for each $X \in \mathcal{C}'$ it must be $\gamma'_X = \gamma_X$. Consider such an object X and its corresponding component $\gamma'_X : F(X) \rightarrow G(X)$. If $\mathcal{C}_{/X}$ is empty, then $F(X) = \operatorname{colim} \emptyset$ which is an initial object, so there is no choice for γ'_X . Otherwise, for every $(M, s) \in \mathcal{C}_{/X}$ we have a commutative diagram

$$\begin{array}{ccccc}
& & Fp(M) & \xrightarrow{\gamma'_{p(M)}} & Gp(M) \\
& \nearrow^{\alpha_M} & \downarrow F(s) & & \downarrow G(s) \\
f(M) & & & & \\
& \searrow & F(X) & \xrightarrow{\gamma'_X} & G(X)
\end{array}$$

which exhibits γ'_X as the unique factorization of the cocone

$$(Gs \circ \gamma'_{p(M)} \circ \alpha_M)_{(M,s) \in \mathcal{C}/X} = (Gs \circ \beta_M)_{(M,s) \in \mathcal{C}/X}.$$

But this was precisely γ_X by definition, so we are done.

Remark Dually, if we assume that \mathcal{D} is complete instead of cocomplete, we get a limit formula for right Kan extensions, so for an object $X \in \mathcal{C}'$ we will have

$$\text{Ran}_p f(X) = \lim f \pi_X$$

where π_X is now the projection $\mathcal{C}_{X/} \rightarrow \mathcal{C}$.

Corollary As we have already shown, this implies that, with the additional hypothesis that p is fully faithful, then $Fp(A) \cong f(A)$ for every $A \in \mathcal{C}$.

Example Let \mathcal{K} be a locally λ -presentable category, and let \mathcal{K}^λ be its full subcategory of λ -compact objects. Then the diagram

$$\begin{array}{ccc}
\mathcal{K}^\lambda & \xleftarrow{i} & \mathcal{K} \\
\mathbf{y} \downarrow & & \\
\mathbf{Set}^{\mathcal{K}^{\lambda op}} & &
\end{array}$$

gives rise to an adjunction $\text{Lan}_{\mathbf{y}} i \dashv \text{Lan}_i \mathbf{y}$.

To see this, note that the colimit formula applies in both cases, so we have an explicit way of computing the two Kan extensions. In the first case, given $F \in \mathbf{Set}^{\mathcal{K}^{\lambda op}}$, we have

$$\text{Lan}_{\mathbf{y}} i(F) \cong \text{colim } i \pi_F = \text{colim}_{K_j \in \mathcal{K}_{/F}^\lambda} K_j$$

which is precisely the functor L from last week.

The computation for the other functor, given an object $K \in \mathcal{K}$, is

$$\text{Lan}_i \mathbf{y}(K) \cong \text{colim } \mathbf{y} \pi_K = \text{colim}_{K_j \in \mathcal{K}_{/K}^\lambda} \mathcal{K}(-, K_j)$$

but since this functor only takes values in λ -compact objects, and the category $\mathcal{K}_{/K}^\lambda$ is λ -filtered, this is isomorphic to the functor represented by $\text{colim}_{K_j \in \mathcal{K}_{/K}^\lambda} K_j$, which is precisely the E of last week. We conclude by remembering that we already know that $L \dashv E$ is an adjunction.

Remark The same proof applies to a more general result. Given a functor $p : \mathcal{C} \rightarrow \mathcal{D}$, where \mathcal{C} is small and \mathcal{D} is cocomplete, then there is an adjunction

$$\text{Lan}_{\mathbf{y}} p : \mathbf{Set}^{\mathcal{C}^{op}} \perp \quad \mathcal{D} : \text{Lan}_p \mathbf{y}$$

Example Special cases of the above whose role is of the greatest importance are the adjunctions $h : s\mathbf{Set} \rightleftharpoons \mathbf{Cat} : \mathbf{N}$ and $|-| : s\mathbf{Set} \rightleftharpoons \mathbf{Top} : \mathbf{Sing}$.

Example If \mathcal{K} is locally λ -presentable and $i : \mathcal{K}^\lambda \hookrightarrow \mathcal{K}$ is the inclusion, then we have $\text{Lan}_i i \cong \text{id}_{\mathcal{K}}$, which can be seen directly by applying the colimit formula. This suggests an alternative, equivalent definition of density of a subcategory. We say that a subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ is dense if the left Kan extension of the inclusion along itself is isomorphic to the identity on \mathcal{C} .

Example Consider a functor $f : \mathcal{C} \rightarrow \mathcal{D}$. A left Kan extension of f along the terminal map $\mathcal{C} \rightarrow \bullet$ is a functor $\bullet \rightarrow \mathcal{D}$ which selects a colimit of f . In fact, this can be taken as an equivalent definition of colimit. Dually, right Kan extensions of f classify limits of f .

Example A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ has a left adjoint if and only if $\text{Ran}_F \text{id}_{\mathcal{C}}$ exists, in which case the Kan extension is itself a left adjoint. Similarly, F has a right adjoint if and only if $\text{Lan}_F \text{id}_{\mathcal{C}}$ exists. Note the mismatch between the direction of the extensions and the resulting adjoint functors. This is essentially due to the direction of the corresponding units and counits. Indeed, assuming for instance that $\text{Ran}_F \text{id}_{\mathcal{C}}$ exists, then the natural map $\alpha : \text{Ran}_F \text{id}_{\mathcal{C}} \circ F \rightarrow \text{id}_{\mathcal{C}}$ has the direction of a counit, so the right Kan extension is a left adjoint.

Example This technology can be used to prove the universal property of the free cocompletion in a slick way. Since the Yoneda embedding is fully faithful, we know that $\mathbf{y}^* \text{Lan}_{\mathbf{y}} \cong \text{id}$. Now take a small category \mathcal{C} and a cocomplete category \mathcal{D} , and observe that, for two functors $f, g : \mathcal{C} \rightarrow \mathcal{D}$, we have

$$\mathcal{D}^{\mathcal{P}(\mathcal{C})}(\text{Lan}_{\mathbf{y}} f, \text{Lan}_{\mathbf{y}} g) \cong \mathcal{D}^{\mathcal{C}}(f, \mathbf{y}^* \text{Lan}_{\mathbf{y}} g) \cong \mathcal{D}^{\mathcal{C}}(f, g)$$

which means that the functor $\text{Lan}_{\mathbf{y}} : \text{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Fun}(\mathcal{P}(\mathcal{C}), \mathcal{D})$ is fully faithful, so that there is an equivalence of categories with the image

$$\text{Fun}(\mathcal{C}, \mathcal{D}) \simeq \text{Fun}^{\text{Lan}}(\mathcal{P}(\mathcal{C}), \mathcal{D}).$$

It only remains to show that a functor $F : \mathcal{P}(\mathcal{C}) \rightarrow \mathcal{D}$ is a left Kan extension if and only if it preserves colimits. One direction is true because such left Kan extensions are always left adjoint, as seen above, therefore they preserve colimits. For the other direction, observe that, defining $f = F\mathbf{y}$, the condition that F preserves colimits forces it to be defined via the colimit formula (because every presheaf is a colimit of $\mathcal{C}_{/F}$), so it is automatically a left Kan extension of f .

Week 12

Lambek's lemma For an endofunctor $T : \mathcal{C} \rightarrow \mathcal{C}$, if $i : TI \rightarrow I$ is an initial T -algebra, then i is an isomorphism.

Proof Consider the diagram

$$\begin{array}{ccccc} TI & \longrightarrow & T^2I & \longrightarrow & TI \\ \downarrow i & & \downarrow & & \downarrow \\ I & \longrightarrow & TI & \longrightarrow & I \end{array}$$

where both composites in the right square are $i \circ Ti$, and the horizontal morphisms in the left triangle are uniquely given by initiality of i among T -algebras.

In particular, we have an arrow $s : I \rightarrow TI$ and, by uniqueness of morphisms from i in $T\text{-Alg}$, the lower horizontal composite must be the identity, so $is = \text{id}_I$. Moreover, the upper horizontal composite says that $Ti \circ Ts = \text{id}_{TI}$, and the left square says that $Ti \circ Ts = si$, so that we may conclude $si = \text{id}_{TI}$, and s is a two-side inverse of i .

Exercise Check that, given an adjunction $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$, the composite functor RL indeed gives a monad on \mathcal{C} .

Solution We call $T := RL$. Now, the multiplication $\mu : T^2 \rightarrow T$ and the unit $\eta : \text{id} \rightarrow T$ of the monad will be defined respectively as

$$\begin{aligned} R\varepsilon_L &: RLRL \rightarrow RL \\ \eta &: \text{id} \rightarrow RL. \end{aligned}$$

We need to show the associativity and the unitality axioms. The first amounts to showing that the diagram

$$\begin{array}{ccc} RLRLRLC & \xrightarrow{R\varepsilon_{LRLC}} & RLRLC \\ \downarrow RL R\varepsilon_{LC} & & \downarrow R\varepsilon_{LC} \\ RLRLC & \xrightarrow{R\varepsilon_{LC}} & RLC \end{array}$$

commutes for every object $C \in \mathcal{C}$. By an application of R , it clearly suffices to show that the diagram

$$\begin{array}{ccc} LRLRLC & \xrightarrow{\varepsilon_{LRLC}} & LRLC \\ \downarrow LR\varepsilon_{LC} & & \downarrow \varepsilon_{LC} \\ LRLC & \xrightarrow{\varepsilon_{LC}} & LC \end{array}$$

commutes. It will be enough to show that both composites correspond to the same morphism $RLRLC \rightarrow RLC$ via the adjunction $L \dashv R$. Using the notation $\tilde{\sim}$ for adjunct morphisms and remembering that adjunctions are natural in both variables, we obtain that the right-then-down composite corresponds to

$$R\varepsilon_{LC} \circ \tilde{\varepsilon}_{LRLC} = R\varepsilon_{LC}$$

since ε_{LRLC} corresponds to id_{RLRLC} by definition. The down-then-right composite corresponds to

$$\tilde{\varepsilon}_{LC} \circ R\varepsilon_{LC} = R\varepsilon_{LC}$$

since, similarly, ε_{LC} corresponds to id_{RLC} . This concludes the verification of associativity.

As for unitality, we will need an alternative characterization of adjunctions. With the data of two functors and two natural transformations with the same domains and codomains as the unit and counit, we have an adjunction if and only if the two triangles of natural transformations

$$\begin{array}{ccc} & LRL & \\ L\eta \nearrow & & \searrow \varepsilon_L \\ L & \xrightarrow{\text{id}_L} & L \end{array} \qquad \begin{array}{ccc} & RLR & \\ \eta_R \nearrow & & \searrow R\varepsilon \\ R & \xrightarrow{\text{id}_R} & R \end{array}$$

commute. They are often called the triangular identities. Now back to the case at issue, we need to check that the two triangles

$$\begin{array}{ccccc} RLC & \xrightarrow{RL\eta_C} & RLRLC & \xleftarrow{\eta_{RLC}} & RLC \\ & \searrow & \downarrow R\varepsilon_{LC} & & \swarrow \\ & & RLC & & \\ \text{id}_{RLC} \nearrow & & & & \searrow \text{id}_{RLC} \end{array}$$

commute for every object $C \in \mathcal{C}$. The left one is simply the former triangular identity applied to the object C and then plugged into the functor R . The right one is the latter triangular identity applied to the object LC .

Definition Recall the definition of the category of algebras and its universal property. Given a monad (T, μ, η) on a category \mathcal{C} , a T -algebra is an object $C \in \mathcal{C}$ with a morphism $h : TC \rightarrow C$ such that the diagrams

$$\begin{array}{ccc} T^2C & \xrightarrow{T h} & TC \\ \mu_C \downarrow & & \downarrow h \\ TC & \xrightarrow{h} & C \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\mu_C} & TC \\ & \searrow & \downarrow h \\ & & C \\ \text{id}_C \nearrow & & \end{array}$$

commute. A homomorphism of T -algebras is a morphism in \mathcal{C} that is compatible with the algebra structures in the obvious way. This defines a category, called the category of T -algebras and usually denoted by \mathcal{C}^T . There is a free-forgetful adjunction $F : \mathcal{C} \rightleftarrows \mathcal{C}^T : U$.

Theorem The category of T -algebras is universal among all adjunctions that give rise to the monad T , in the following sense. Given an adjunction $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ that induces the monad T , then there is a unique functor $K : \mathcal{D} \rightarrow \mathcal{C}^T$ such that $KL = F$ and $UK = R$. In other words, in the diagram

$$\begin{array}{ccc}
\mathcal{D} & \xrightarrow{K} & \mathcal{C}^T \\
\swarrow R & & \nearrow F \\
& & \mathcal{C} \\
\searrow L & & \swarrow U
\end{array}$$

the two triangles comprising respectively both left adjoints and both right adjoints commute.

Remark This means that the category of T -algebras is terminal among adjunctions inducing T . We will now see that there is also an initial such category.

Definition Given a monad as above, we define its Kleisli category \mathcal{C}_T as follows. The objects are the same as those of \mathcal{C} , but conventionally denoted as $C_T, D_T \dots$, and a morphism $f_T : C_T \rightarrow D_T$ is precisely a morphism $f : C \rightarrow TD$ in \mathcal{C} . The composite of $f_T : C_T \rightarrow D_T$ and $g_T : D_T \rightarrow E_T$ is defined as

$$\mu_E \circ Tg \circ f : C \rightarrow TD \rightarrow T^2E \rightarrow TE.$$

Exercise Check that the Kleisli category is indeed a category.

Solution We first check that composition is associative. Take morphisms

$$C_T \xrightarrow{f_T} D_T \xrightarrow{g_T} E_T \xrightarrow{h_T} F_T.$$

Unwinding the definitions, we see that $h_T \circ (g_T \circ f_T)$ is given by

$$C \xrightarrow{f} TD \xrightarrow{Tg} T^2E \xrightarrow{\mu_E} TE \xrightarrow{Th} T^2F \xrightarrow{\mu_F} TF$$

while $(h_T \circ g_T) \circ f_T$ is given by

$$C \xrightarrow{f} TD \xrightarrow{Tg} T^2E \xrightarrow{T^2h} T^3F \xrightarrow{T\mu_F} T^2F \xrightarrow{\mu_F} TF.$$

It suffices to show that they are already equal when leaving the first two components out.

For this, observe that there is a diagram

$$\begin{array}{ccccc}
T^2E & \xrightarrow{T^2h} & T^3F & \xrightarrow{T\mu_F} & T^2F \\
\downarrow \mu_E & & \downarrow \mu_{TF} & & \downarrow \mu_F \\
TE & \xrightarrow{Th} & T^2F & \xrightarrow{\mu_F} & TF
\end{array}$$

where both squares commute by naturality of μ . The outer square gives precisely the equation that we want.

As for unitality, let us define the identity on C_T as the morphism $\eta_C : C \rightarrow TC$. For one side, need to verify that the composite

$$C \xrightarrow{f} TD \xrightarrow{T\eta_D} T^2D \xrightarrow{\mu_D} TD$$

equals f , which follows directly by one of the unitality axioms of T . For the other side, we need to verify that the composite

$$C \xrightarrow{\eta_C} TC \xrightarrow{Tf} T^2D \xrightarrow{\mu_D} TD$$

equals f , which follows by observing that $Tf \circ \eta_C = \eta_{TC} \circ f$ by naturality of η , and then applying the other unitality axiom of T .

Exercise Show that there is an adjunction $F_T : \mathcal{C} \rightleftarrows \mathcal{C}_T : G_T$ where

$$\begin{aligned} F_T \text{ sends } f : C \rightarrow D \text{ to } \eta_D \circ f : C \rightarrow TD; \\ G_T \text{ sends } f_T : C_T \rightarrow D_T \text{ to } \mu_D \circ Tf : TC \rightarrow TD. \end{aligned}$$

Solution We will only show the bijection between homsets, leaving naturality as a routine computation. Observe that, when considering only their actions on objects, F_T is the identity and G_T corresponds with T . Now compute

$$\mathcal{C}_T(F_T C, D) \cong \mathcal{C}_T(C, D) \cong \mathcal{C}(C, TD) \cong \mathcal{C}(C, G_T D).$$

Exercise Show that, whenever there is an adjunction $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ whose induced monad on \mathcal{C} is T , then there is a unique comparison functor $H : \mathcal{C}_T \rightarrow \mathcal{D}$ such that in the diagram

$$\begin{array}{ccc} \mathcal{C}_T & \xrightarrow{H} & \mathcal{D} \\ \swarrow F_T & \begin{array}{c} \nearrow G_T \\ \searrow L \end{array} & \nearrow R \\ & \mathcal{C} & \end{array}$$

the two triangles comprising respectively both left adjoints and both right adjoints commute.

Solution Since, F_T is the identity on objects, the condition $H F_T = L$ forces the assignment $H : C_T \mapsto LC$. Since $RL = T$, we also have $RH(C) = RL(C) = T(C) = G_T(C_T)$, so the other triangle commutes on objects. Now suppose that H has been defined, and look at the following diagram

$$\begin{array}{ccc} \mathcal{C}_T(C_T, D_T) & \xrightarrow{H} & \mathcal{D}(LC, LD) \\ & \searrow G_T & \downarrow R \\ & & \mathcal{C}(RLC, RLD) \xrightarrow{\eta_C^*} \mathcal{C}(C, RLD) \end{array}$$

which commutes by remembering that $RH = G_T$. Note that the adjunctions $F_T \dashv G_T$ and $L \dashv R$ have the same unit η , so by general properties of adjunctions we have that the composite $\eta_C^* \circ G_T$ transposes morphisms with respect to the former, while the composite $\eta_C^* \circ R$ transposes morphisms with respect to the latter. In other words, the functor H must preserve transpositions. This forces the action of H on morphisms as follows. A morphism $f_T : F_T C = C_T \rightarrow D_T$ is the transpose of $f : C \rightarrow TD = G_T D$, therefore H must send it to the transpose of f with respect to the other adjunction, that is, to the composite

$$LC \xrightarrow{Lf} LRLD \xrightarrow{\varepsilon_{LD}} LD.$$

Another routine argument will show that this is indeed functorial, and we already saw that $RH = G_T$. We need to show that $HF_T = L$ also on morphisms. But the left-hand side, when computed on a morphism $f : C \rightarrow D$, gives

$$LC \xrightarrow{Lf} LD \xrightarrow{L\eta_D} LRLD \xrightarrow{\varepsilon_{LD}} LD$$

so the result follows by one of the triangular identities.

Week 13

Exercise Let **StrMon** be the category of strict monoidal categories (the associators and unitors are identities) and strict monoidal functors. There is a forgetful functor $U : \mathbf{StrMon} \rightarrow \mathbf{Cat}$. Prove that it has a left adjoint.

Solution One can use the adjoint functor theorem with a direct check, taking all strict monoidal categories of the same size as one given category as its solution set.

More explicit, one can define the free strict monoidal category $SM(\mathcal{C})$ on a category \mathcal{C} as follows: its objects are finite ordered sequences of objects of \mathcal{C} . A morphism from (C_0, \dots, C_n) to (D_0, \dots, D_m) is an order-preserving map $\alpha : \mathbf{n} \rightarrow \mathbf{m}$ with morphisms $C_i \rightarrow D_{\alpha(i)}$, for each $i = 0, \dots, n$. The tensor product is given by concatenation. The reflection $\mathcal{C} \rightarrow SM(\mathcal{C})$ sends every morphism $f : C \rightarrow D$ to $f : (C) \rightarrow (D)$.

Since strict monoidal functors must strictly preserve tensor products, a functor $\mathcal{C} \rightarrow \mathcal{M}$ where \mathcal{M} is strictly monoidal determines a unique assignment $SM(\mathcal{C}) \rightarrow \mathcal{M}$ on objects. Moreover, since morphisms in $SM(\mathcal{C})$ are determined by morphisms in \mathcal{C} , a whole functor $SM(\mathcal{C}) \rightarrow \mathcal{M}$ is thus already determined, and it is unique with the property that the diagram

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{M} \\ \downarrow & \nearrow & \\ SM(\mathcal{C}) & & \end{array}$$

commutes.

Example Similarly, we can define a free weak monoidal category functor. This time, an object in $WM(\mathcal{C})$ will be a finite planar binary tree T (where every branch splits into at most two subbranches and there is a total ordering on the set of leaves) with a map $\mathbf{leaf}(T) \rightarrow \mathbf{ob}(\mathcal{C})$. Morphisms are generated by the ones described above, with $\alpha : \mathbf{leaf}(T) \rightarrow \mathbf{leaf}(S)$, plus unique morphisms (which are necessarily isomorphisms) whenever there is a commutative triangle

$$\begin{array}{ccc}
\mathbf{leaf}(T) & \xrightarrow{\cong} & \mathbf{leaf}(S) \\
& \searrow & \swarrow \\
& \mathit{ob}(\mathcal{C}) &
\end{array}$$

with the horizontal arrow being an order-preserving map. The proof that this is the correct definition is similar to the one above but slightly more complicated.

Example In a yet similar way, it is possible to define a free weak symmetric monoidal category functor. Keeping in mind the construction above, we modify it by instead freely adding to it unique morphisms (necessarily isomorphisms) whenever there are diagrams

$$\begin{array}{ccc}
\mathbf{leaf}(T) & \xrightarrow{\cong} & \mathbf{leaf}(S) \\
& \searrow & \swarrow \\
& \mathit{ob}(\mathcal{C}) &
\end{array}$$

where the horizontal map is no longer required to preserve the order relation.

Example A key example of closed symmetric monoidal category is the category **CGHaus** of compactly generated Hausdorff spaces. A topological space X is compactly generated when a subspace $A \subseteq X$ is closed precisely if its intersection $A \cap K$ with every compact subspace $K \subseteq X$ is compact. The category of compactly generated Hausdorff spaces corrects the ill behavior of **Top** in this respects: under the product-exponential adjunction of underlying sets

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, Z^Y)$$

continuous maps do not in general transpose to continuous maps in either direction. Therefore, **Top** is not Cartesian closed.

A lengthy proof shows that, in addition to being complete, cocomplete and coreflective in **Haus**, the category **CGHaus** is Cartesian closed. The exponential Z^Y is given as follows: the underlying set is **Top**(Y, Z), and a subbase for the topology is given by all sets $N(K, U)$, where $K \subseteq Y$ is compact, $U \subseteq Z$ is open and $N(K, U)$ is the set of all maps $f : Y \rightarrow Z$ such that $f(K) \subseteq U$.

Weak Yoneda lemma Recall that the underlying category \mathcal{C}_0 of \mathcal{C} is given by taking the same objects and $\mathcal{V}(1, \mathcal{C}(C, D))$ as morphisms between C and D . Moreover, there is a functor $(-)_0 = \mathcal{V}(1, -) : \mathcal{V} \rightarrow \mathbf{Set}$.

Let $F : \mathcal{C} \rightarrow \mathcal{V}$ be a \mathcal{V} -functor. Then for every object $C \in \mathcal{C}$ there is a bijection

$$(FC)_0 \cong \mathcal{V} - \mathrm{Nat}(\mathcal{C}(C, -), F).$$

Remark Although this bijection is natural in C , we have no way yet to say that it is natural in F , because we haven't defined enriched functor categories. What we lack for the moment is a \mathcal{V} -object of natural transformations, in that we only have a way to define a set of such. Moreover, this version of the enriched Yoneda lemma only gives a bijection of sets, so it still relies on the structure of ordinary underlying categories and thus does not employ the full power of the enrichment. In what follows, we will fix these two problems in one shot, by defining \mathcal{V} -objects of natural transformations and by giving a stronger version of the enriched Yoneda lemma that solely stands on the enriched structure of our objects.

Definition Let \mathcal{C} and \mathcal{D} be categories, and let $S, T : \mathcal{C}^{op} \times \mathcal{C} \rightarrow \mathcal{D}$ be two functors. A dinatural transformation $S \rightarrow T$ is an assignment of a morphism $\alpha_C : S(C, C) \rightarrow T(C, C)$ for every object $C \in \mathcal{C}$ in such a way that for every morphism $f : C \rightarrow C'$ there is a commutative diagram

$$\begin{array}{ccccc}
 & & S(C, C) & \xrightarrow{\alpha_C} & T(C, C) & & \\
 & \nearrow^{S(f,1)} & & & & \searrow_{T(1,f)} & \\
 S(C', C) & & & & & & T(C, C') \\
 & \searrow_{S(1,f)} & & & & \nearrow_{T(f,1)} & \\
 & & S(C', C') & \xrightarrow{\alpha_{C'}} & T(C', C') & &
 \end{array}$$

Example If the two functors ignore either the first or the second variable, then we are reduced to a natural transformation of covariant and contravariant functors respectively.

Example Suppose that S is the constant functor at $D \in \mathcal{D}$. Then the dinaturality diagrams above reduces to a family of maps $(\alpha_C : D \rightarrow T(C, C))_{C \in \mathcal{C}}$ such that the diagrams

$$\begin{array}{ccc}
 D & \xrightarrow{\alpha_C} & T(C, C) \\
 \alpha_{C'} \downarrow & & \downarrow T(1,f) \\
 T(C', C') & \xrightarrow{T(f,1)} & T(C, C')
 \end{array}$$

are commutative. This is sort of a two-variable version of a cone, and it is known as a wedge on T with vertex D .

Definition Let $T : \mathcal{C}^{op} \times \mathcal{C} \rightarrow \mathcal{D}$ be a functor. We say that an end of T is a wedge on T which is final among such wedges. We write it as

$$\int_{C \in \mathcal{C}} T(C, C).$$

It does not always exist, but it is of course unique whenever it does.

Remark We might be tempted to write an end of a functor as a limit of some sort. In fact, this is always possible in the following way. Define the twisted category $\text{Tw}(\mathcal{C})$ of \mathcal{C} as the category having for objects the set of symbols C^\clubsuit for every object $C \in \mathcal{C}$ and f^\clubsuit for every morphism $f \in \mathcal{C}$. The only morphisms in $\text{Tw}(\mathcal{C})$ are the identities plus morphisms

$$C^\clubsuit \rightarrow f^\clubsuit \leftarrow D^\clubsuit$$

for every morphism $f : C \rightarrow D$.

Now let us define a functor $T^\clubsuit : \text{Tw}(\mathcal{C}) \rightarrow \mathcal{D}$ as

$$\begin{array}{ccccc} C^\clubsuit & \xrightarrow{\quad} & f^\clubsuit & \xleftarrow{\quad} & D^\clubsuit \\ \downarrow & & \downarrow & & \downarrow \\ T(C, C) & \xrightarrow{T(1, f)} & T(C, D) & \xleftarrow{T(f, 1)} & T(D, D). \end{array}$$

A direct verification will yield that $\int_{C \in \mathcal{C}} T(C, C) \cong \lim T^\clubsuit$.

Example We use the description above in a very specific case. Consider the hom-functor $\mathcal{D}(-, -) : \mathcal{D}^{op} \times \mathcal{D} \rightarrow \mathbf{Set}$ and, furthermore, consider two functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$. Precomposing the functor $F^{op} \times G$ to the hom-functor gives $\mathcal{D}(F-, G-) : \mathcal{C}^{op} \times \mathcal{C} \rightarrow \mathbf{Set}$. Remembering the construction above and the explicit construction of limits in \mathbf{Set} , we obtain that there is an isomorphism

$$\int_{C \in \mathcal{C}} \mathcal{D}(FC, GC) \cong \text{Nat}(F, G).$$

This motivates the following definition in the case of enriched categories.

Definition Let \mathcal{V} be a closed symmetric monoidal category, and let \mathcal{C} and \mathcal{D} be \mathcal{V} -categories. Then we can define the \mathcal{V} -enriched functor category $\text{Fun}(\mathcal{C}, \mathcal{D})$ in the following way.

An object is a \mathcal{V} -functor $F : \mathcal{C} \rightarrow \mathcal{D}$. Given two such \mathcal{V} -functors F and G , the \mathcal{V} -object of natural transformations from F to G is given as the end of the composite functor

$$\mathcal{C}^{op} \times \mathcal{C} \xrightarrow{F^{op} \times G} \mathcal{D}^{op} \times \mathcal{D} \xrightarrow{\mathcal{D}(-, -)} \mathcal{V}.$$

In other words, we define

$$\text{Nat}(F, G) := \int_{C \in \mathcal{C}} \mathcal{D}(FC, GC)$$

and composition is given by universal property of ends.

Strong Yoneda lemma Let $F : \mathcal{C} \rightarrow \mathcal{V}$ be a \mathcal{V} -functor. Then for every $C \in \mathcal{C}$ there is a natural isomorphism

$$FC \cong \text{Nat}(\mathcal{C}(C, -), F)$$

which is natural in both C and F .