M mfd. equipped with an affine connection ∇

$\rightsquigarrow$ notion of parallel transport of tangent vectors along curves: $c : [t_0, t_1] \to M$

$$Pt_{t_0}^{t_1}(c) : T_{c(t_0)}M \longrightarrow T_{c(t_1)}M \quad \text{linear isomorphism.}$$

<u>Remark</u>

(1) There exist abstract notion of parallel transport along curves of M and it can be shown that it is equivalent to an affine conn. on M.

(2) Given an affine connection ∇, then it induces a linear connection on any tensor bundle:

$$\nabla : \mathfrak{X}(M) \times \mathcal{T}^p_q(M) \longrightarrow \mathcal{T}^p_q(M)$$

$$\nabla_\xi t$$

(bilinear over $\mathbb{R}$, tensorial in ξ (recall (a) of Def. 6.18)

+ (b) of Def. 6.18. $\nabla_\xi f t = f \nabla_\xi t + (\xi . f) t$.)

∇ on trivial bundle $M \times \mathbb{R} \to M$ is just d

and you require $\nabla t \otimes s = (\nabla t) \otimes s + t \otimes \nabla s$.

③ In the case of $M = \mathbb{R}^n$ with the standard connection, the parallel transport is independent of c and it is given by translation. ($T_x\mathbb{R}^n \simeq \mathbb{R}^n \simeq T_y\mathbb{R}^n$ $x, y \in \mathbb{R}^n$).

On a mfd M with affine we can talk about the acceleration of a smooth curve in:

Note that on a mfd M

$$c''(t) = \lim_{h \to 0} \frac{c'(t+h) - c'(t)}{h} \qquad (*)$$

makes no sense, since $c'(t+h) \in T_{c(t+h)}M$ and $c'(t) \in T_{c(t)}M$

lie in different vector spaces, so the minus in (*) makes no sense.

However, if M is equipped with an affine connection ∇ (equiv. a notion of parallel transp.), then we can define the acceleration of c w.r.to ∇ by

$$\nabla_{c'} c' = \lim_{h \to 0} \frac{P_{t}^{t+h}(c)\left(\underline{c'(t+h)}\right) - c'(t)}{h}.$$

<u>Def. 6.22</u>. (M, ∇) mfd. with affine connection. A C^∞-curve $c : I \to M$ ($I \subseteq \mathbb{R}$ open interval) is called a geodesic of ∇, if

$$(\nabla_{c'} c')(t) = 0 \quad \forall t \in I.$$

Remark.

- $M = \mathbb{R}^n$ $\quad c'' = c' \cdot c' = \nabla_{c'} c'$ for ∇ standard connection on $\mathbb{R}^n$,

  $\rightsquigarrow$ geodesics are affine lines.

- mfd. with affine connections generalize affine geometry to the setting of mfds.

In local coordinates the geodesic equation has the following form: $c: I \to M$ with values in a chart (U, u) and $c^i = u^i \circ c$ then Lemma 6.12 implies

that $$\underbrace{\frac{d^2 c^k}{dt} + \sum_{i,j} \Gamma^k_{ij} \frac{dc^i}{dt} \frac{dc^j}{dt} = 0}_{} \quad \forall k = 1, \dots, n$$

1) geodesic equation.

Theory of ODEs implies:

<u>Thm. 6.23</u> (M,∇) mfd. with affine connection.

① Given $x \in M$, $\xi_x \in T_xM$, then $\exists!$ maximal interval $I \subseteq \mathbb{R}$ with $0 \in I$ and a unique (maximal) geodesic $c^{\xi_x} : I \to M$ s.t. $c(0) = x$ $c'(0) = \xi_x$

② Given $x \in M$, $\exists$ an open neighbh. U of 0 in T_xM s.t. for each $\xi_x \in U$ the interval I in ① contains $[0,1]$ and

$$\exp_x : U \to M \quad \text{is smooth.}$$

$$\exp_x(\xi_x) = c^{\xi_x}(1)$$

It is called the exponential map at x of ∇.

③ $\exp_x$ satisfies $\exp_x(0) = x$ and $T_0 \exp : T_x M \to T_x M$ is the identity on $T_x M$. $(T_0 T_x M \simeq T_x M)$.

Hence, $\exp_x$ is a local diffeom. from a open neigh. U of 0 in $T_x M$ to an open neighb. V of x in M.

④ $\exists$ an open neighb. $\tilde{U}$ of the zero section of $p : TM \to M$ s.t. for any $\xi \in \tilde{U} \subseteq TM$, $\exp_{p(\xi)}(\xi) \in M$ is defined.

Moreover, choosing $\tilde{U}$ small enough, $(p, \exp) : \tilde{U} \to M \times M$ is a diffeom. onto an open neighbhd. of the diagonal in $M \times M$.

Proof.

(1) Theory of ODEs (cf. also Prop. 6.20).

(2) Follows from the fact that solutions to ODEs depend smoothly on the initial data and the fact that for a geodesic $c : I \to M$, $0 \in I$, $\tilde{c}(t) := c(st)$ for any $s \in \mathbb{R}$ is again a geodesic with $\tilde{c}(0) = c(0)$ and $\tilde{c}'(0) = s c'(0)$.

(3) Since constant curve $c(t) = x$ $\forall t$ is the unique geodesic with $c(0) = x$ and $c'(0) = 0$, $\exp_x(0) = x$.

For $\xi \in U$ and small t, $t \mapsto \exp_x(t\xi_x)$ is defined and equals c^{ξ_x}.

$$\exp_x(t\xi_x) = c^{t\xi_x}(1) \qquad c^{t\xi_x}(1) = c^{\xi_x}(t)$$

$$\Rightarrow \underline{T_0 \exp_x \xi_x} = \frac{d}{dt}\Big|_{t=0} \underline{\exp_x(t\xi_x)} = \underline{\xi_x}$$

$$\Rightarrow T_0 \exp_x = \mathrm{Id}_{T_x M}.$$

(4) Existence of $\tilde{U}$ follows from smooth depend. of solutions of ODEs on initial conditions and (3). $\square$

Remark. 3 shows that $\exp_x : \underset{\subseteq T_x M}{U} \to V$ is a diffeo.

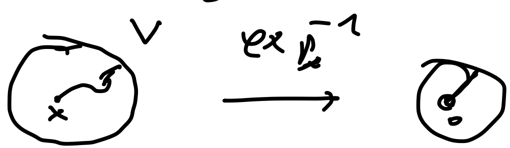
onto neigh. V of x in M $\Longrightarrow$ $\exp_x^{-1} : V \to U \subseteq T_x M \cong \mathbb{R}^n$

defines distinguished coordinates on M with values in $T_x M \cong \mathbb{R}^n$,

called normal coordinates.

Geodesics in V emanating from x correspond in these coordinates

to straight lines through 0 in $T_x M \cong \mathbb{R}^n$:



Def. 6.24 (M,∇) mfd. with of the connection.
(M,∇) is called complete, if $\forall x\in M$, $\exp_x$ is defined $\forall \xi_x \in T_x M$.

Prop. 6.25 (M,∇) mfd. with affine conn., $\xi,\eta,\zeta\in\Gamma(TM)$.

① $R(\xi,\eta)(\zeta) := \nabla_\xi\nabla_\eta\zeta - \nabla_\eta\nabla_\xi\zeta - \nabla_{[\xi,\eta]}\zeta$

defines a $\binom{1}{3}$-tensor, called the curvature of ∇.

$(R\in\Gamma(\Lambda^2T^*M\otimes T^*M\otimes TM) = \Gamma(\Lambda^2T^*M\otimes \mathrm{End}(TM)))$

∇ is called flat, if $R_x = 0$ $\forall x\in M$.

② $T(\xi,\eta) = \nabla_\xi \eta - \nabla_\eta \xi - [\xi,\eta]$ is a $\binom{1}{2}$-tensor, called the torsion of ∇. $\left(T \in \Gamma(\Lambda^2 T^*M \otimes TM)\right)$.

∇ is called torsion-free, if $T_x = 0 \;\; \forall x \in M$

(is equiv. to $\Gamma^k_{ij} = \Gamma^k_{ji} \;\; \forall i,j$).

<u>Proof.</u> ① Exercise.

② $T(-,-)$ $\mathbb{R}$-bilinear, since $\nabla_- -$ and $[-,-]$ are. $f \in C^\infty(M,\mathbb{R})$

$$T(f\xi,\eta) = \nabla_{f\xi}\eta - \nabla_\eta f\xi - [f\xi,\eta] = f\nabla_\xi\eta - f\nabla_\eta\xi \;\cancel{+(\eta(f))\xi}$$

$$- \cancel{(\eta \cdot f)\xi} - f[\xi,\eta]$$

$$= f\,T(\xi,\eta)$$

and by skew-symmetry also $T(\xi, f\eta) = -T(f\eta, \xi)$

$$= -fT(\eta, \xi)$$

$$= fT(\xi, \eta).$$

6.3.2. The Levi-Civita connection of a Riem. mfd.

Fundamental Theorem in Riem. geometry:

Thm. 6.26 Suppose (M, g) is a Riemannian mfd. Then there exists a unique torsion-free affine connection ∇ s.t.

$$(*) \qquad \xi \cdot g(\eta, \zeta) = g(\nabla_\xi \eta, \zeta) + g(\eta, \nabla_\xi \zeta),$$

(⊛ can be also written as $\underline{\nabla g = 0}$, because connection on S^2T^*M induced by ∇ is given by $(\nabla_\xi g)(\eta,\zeta) = \xi\cdot g(\eta,\zeta) - g(\nabla_\xi\eta,\zeta) - g(\eta,\nabla_\xi\zeta)$).

<u>Proof.</u> Assume such a connection exists.

$$\Longrightarrow \quad 0 = \xi\cdot g(\eta,\zeta) - g(\nabla_\xi\eta,\zeta) - g(\eta,\nabla_\xi\zeta)$$

$$0 = \eta\cdot g(\zeta,\xi) - g(\nabla_\eta\zeta,\xi) - g(\zeta,\nabla_\eta\xi)$$

$$0 = \zeta\cdot g(\xi,\eta) - g(\nabla_\zeta\xi,\eta) - g(\xi,\nabla_\zeta\eta).$$

Adding the first two and substracting the last identity and using torsion-freeness to replace terms $-\nabla_\xi \zeta + \nabla_\zeta \xi$ by $-[\xi,\zeta]$, $-\nabla_\eta \zeta + \nabla_\zeta \eta$ by $-[\eta,\zeta]$ and $-\nabla_\xi \eta - \nabla_\eta \xi$ by $-2\nabla_\xi \eta + [\xi,\eta]$.

implies Koszul formula

$$g(\nabla_\xi \eta, \zeta) = \underbrace{\frac{1}{2}\Big(\xi \cdot g(\eta,\zeta) + \eta \cdot g(\zeta,\xi) - \zeta \cdot g(\xi,\eta) + g([\xi,\eta],\zeta) - g([\xi,\zeta],\eta) - g([\eta,\zeta],\xi) \Big)}_{=:\, \varphi(\xi,\eta)(\zeta)} \qquad (**)$$

RHS just involves g and $[\,,\,]$ and so by non-deg. of g implies that ∇ is unique if it exists.

$$\left(g(\tilde{\nabla}_\xi \eta - \nabla_\xi \eta, e) = 0 \ \forall e \Rightarrow \tilde{\nabla}_\xi \eta = \nabla_\xi \eta \right).$$

(**) can be used to prove existence:

For $\xi, \eta \in \Gamma(TM)$, $\varphi(\xi, \eta)$ defines a 1-form on M (Check this!).

Define $\nabla_\xi \eta$ as the unique vf. s.t. $g(\nabla_\xi \eta, -) = \varphi(\xi, \eta)$

One verifies directly that this defines an affine torsion-free connection on M satisfying $\nabla g = 0$.

Check the remaining claims ;

$\square$.

Def. 6.27 (M, g) Riem. mfd. w

① The affine connection of Thm. 6.26 is called the Levi-Civita connection of (M, g). Γ_{ij}^{k} are called the Christoffel symbols of (M, g).

② The Riemannian curvature (or Riemann tensor) of (M, g) is the curvature of its Levi-Civita connection.

③ Geodesics of (M, g) are the geodesics of the Levi-Civita connection.

Prop. 6.28 Suppose (M, g) is a Riem. mf with Levi-Civita connection ∇ and $c : I \to M$ a C^∞-curve.

(1) If ξ, η are vector fields along c, then

$$\frac{d}{dt} g(\xi, \eta) = g(\nabla_{c'} \xi, \eta) + g(\xi, \nabla_{c'} \eta)$$

In particular, ~~that~~ if c is a geodesic, then $\frac{d}{dt} g(c'(t), c'(t)) = 0 \; \forall t$, hence $\|c'(t)\|_g = \sqrt{g(c'(t), c'(t))}$ is constant in t.

(2) If $[t_0, t_1] \subseteq I$, then the isomorphism $Pt_{t_0}^{t_1}(c) : T_{c(t_0)}M \to T_{c(t_1)}M$ (determ. by ∇) is orthogonal w.r.t. to g.

(3) Suppose (U, u) is a chart, ~~and~~ Then Christoffel symbols of ∇ w.r.t. to (U, u) are given by:

$$\underline{\Gamma_{ij}^{k}} = \frac{1}{2} \sum_{l} \left(\frac{\partial g_{jl}}{\partial u^i} + \frac{\partial g_{il}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^l} \right) g^{kl}$$

where (g_{ij}) are the coefficients of g w.r.t. to (U, u) and (g^{ij}) the inverse matrix.

Proof.

1. Follows from (*) in Thm. 6.26.

2. Follows from 1, since $g(\xi, \eta)$ is constant for vector fields ξ, η parallel along c.

3. Insert $\xi = \frac{\partial}{\partial u^i}$, $\eta = \frac{\partial}{\partial u^j}$ and $\zeta = \frac{\partial}{\partial u^l}$ in the Koszul for the Levi-Civ. conn. (in proof of Thm. 6.26).