Homework 5—Global Analysis

Due date: 1.15.2020

1. Suppose $p : E \to M$ and $q : F \to M$ are vector bundles over M. Show that their direct sum $E \oplus F := \sqcup_{x \in M} E_x \oplus F_x \to M$ and their tensor product $E \otimes F := \sqcup_{x \in M} E_x \otimes F_x \to M$ are again vector bundles over M.

2. Suppose $E \subset TM$ is a smooth distribution of rank k on a manifold M of dimension n and denote by $\Omega(M)$ the vector space of differential forms on M.

   (a) Show that locally around any point $x \in M$ there exists (local) 1-forms $\omega^1, ..., \omega^{n-k}$ such that for any (local) vector field ξ one has: ξ is a (local) section of $E \iff \omega_i(\xi) = 0$ for all $i = 1, ..., n-k$.

   (b) Show that E is involutive $\iff$ whenever $\omega^1, ..., \omega^{n-k}$ are local 1-forms as in (a) then there exists local 1-forms $\mu^{i,j}$ for $i, j = 1, ..., n-k$ such that

   $$d\omega^i = \sum_{j=1}^{n-k} \mu^{i,j} \wedge \omega^j.$$

   (c) Show

   $$\Omega_E(M) := \{\omega \in \Omega(M) : \omega|_E = 0\} \subset \Omega(M)$$

   is an ideal of the algebra $(\Omega(M), \wedge)$. Here, $\omega|_E = 0$ for a ℓ-form ω means that $\omega(\xi_1, ..., \xi_\ell) = 0$ for any sections $\xi_1, ...\xi_\ell$ of E.

   (d) An ideal $\mathcal{J}$ of $(\Omega(M), \wedge)$ is called differential ideal, if $d(\mathcal{J}) \subset \mathcal{J}$. Show that $\Omega_E(M)$ is a differential ideal $\iff$ E is involutive.

3. Suppose M is a manifold and $D_i : \Omega^k(M) \to \Omega^{k+r_i}(M)$ for $i = 1, 2$ a graded derivation of degree r_i of $(\Omega(M), \wedge)$.

   (a) Show that

   $$[D_1, D_2] := D_1 \circ D_2 - (-1)^{r_1 r_2} D_2 \circ D_1$$

   is a graded derivation of degree $r_1 + r_2$.

   (b) Suppose D is a graded derivation of $(\Omega(M), \wedge)$. Let $\omega \in \Omega^k(M)$ be a differential form and $U \subset M$ an open subset. Show that $\omega|_U = 0$ implies $D(\omega)|_U = 0$.

   Hint: Think about writing 0 as $f\omega$ for some smooth function f and use the defining properties of a graded derivation.

(c) Suppose D and $\tilde{D}$ are two graded derivations such that $D(f) = \tilde{D}(f)$ and $D(df) = \tilde{D}(df)$ for all $f \in C^\infty(M, \mathbb{R})$. Show that $D = \tilde{D}$.

4. Suppose M is a manifold and $\xi, \eta \in \Gamma(TM)$ vector fields.

(a) Show that the insertion operator $i_\xi : \Omega^k(M) \to \Omega^{k-1}(M)$ is a graded derivation of degree -1 of $(\Omega(M), \wedge)$.

(b) Recall from class that $[d, d] = 0$. Verify (the remaining) graded-commutator relations between $d, \mathcal{L}_\xi, i_\eta$:

(i) $[d, \mathcal{L}_\xi] = 0$.

(ii) $[d, i_\xi] = d \circ i_\xi + i_\xi \circ d = \mathcal{L}_\xi$.

(iii) $[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi,\eta]}$.

(iv) $[\mathcal{L}_\xi, i_\eta] = i_{[\xi,\eta]}$.

(v) $[i_\xi, i_\eta] = 0$.

Hint: Use (c) from 2.