Homework 1—Global Analysis

Due date:18.10.2020

1. Consider the **cylinder** in $\mathbb{R}^3$ given by the equation

   $$M := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = R^2\},$$

   where $R > 0$. Show that M is a 2-dimensional submanifold in $\mathbb{R}^3$. Moreover, give formula for local parametrizations and local trivializations, and a description of M as a local graph.

2. Consider a **double cone** given by rotating a line through 0 of slope α around the z-axis in $\mathbb{R}^3$. It is given by the equation

   $$z^2 = (\tan \alpha)^2(x^2 + y^2).$$

   At which points is the double cone a smooth submanifold of $\mathbb{R}^3$? Around the points where it is give a formula for local parametrizations and trivializations, and a description of it as a local graph.

3. Denote by $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ the nm-dimensional vector space of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. Consider the subset $\mathrm{Hom}_r(\mathbb{R}^n, \mathbb{R}^m)$ of linear maps in $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ of rank r. Show that $\mathrm{Hom}_r(\mathbb{R}^n, \mathbb{R}^m)$ is a submanifold of dimension of $r(n + m - r)$ in $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$.

   Hint: Let $T_0 \in \mathrm{Hom}_r(\mathbb{R}^n, \mathbb{R}^m)$ be a linear map of rank r and decompose $\mathbb{R}^n$ and $\mathbb{R}^m$ as follows

   $$\mathbb{R}^n = E \oplus E^\perp \quad \text{and} \quad \mathbb{R}^m = F \oplus F^\perp, \tag{0.1}$$

   where F equals the image of T_0 and $E^\perp$ the kernel of T_0, and $(\cdot)^\perp$ denotes the orthogonal complement. Note that $\dim E = \dim F = r$. With respect to (0.1) any $T \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ can be viewed as a matrix

   $$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

   where $A \in \mathrm{Hom}(E, F)$, $B \in \mathrm{Hom}(E^\perp, F)$, $C \in \mathrm{Hom}(E, F^\perp)$ and $D \in \mathrm{Hom}(E^\perp, F^\perp)$. Show that the set of matrices T with A invertible defines an open neighbourhood of T_0 and characterize the elements in this neighbourhood that have rank r (equivalently, the ones that have an $(n - r)$-dimensional kernel).