# Homework 3—Global Analysis

Due date:17.11.2020

1. For a topological space $M$ denote by $C^0(M)$ the vector space of continuous real-valued functions $f : M \to \mathbb{R}$. Any continuous map $F : M \to N$ between topological spaces $M$ and $N$ induces a map $F^* : C^0(N) \to C^0(M)$ given by $F^*(f) := f \circ F : M \to \mathbb{R}$.

   (a) Show that $F^*$ is linear.

   (b) If $M$ and $N$ are (smooth) manifolds, show that $F : M \to N$ is smooth $\iff$ $F^*(C^\infty(N)) \subset C^\infty(M)$.

   (c) If $F$ is a homeomorphism between (smooth) manifolds, show that $F$ is a diffeomorphism $\iff$ $F^*$ is an isomorphism.

2. Suppose $M = \mathbb{R}^3$ with standard coordinates $(x, y, z)$. Consider the vector field

$$\xi(x, y, z) = 2\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + 3\frac{\partial}{\partial z}.$$

   How does this vector field look like in terms of the coordinate vector fields associated to the cylindrical coordinates $(r, \phi, z)$, where $x = r\cos\phi$, $y = r\sin\phi$ and $z = z$? Or with respect to the spherical coordinates $(r, \phi, \theta)$, where $x = r\sin\theta\cos\phi$, $y = r\sin\theta\cos\phi$ and $z = r\cos\theta$?

3. Consider $\mathbb{R}^3$ with coordinates $(x, y, z)$ and the vector fields

$$\xi(x, y, z) = (x^2 - 1)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}$$

$$\eta(x, y, z) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2xz^2\frac{\partial}{\partial z}.$$

   Are they tangent to the cylinder $M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\} \subset \mathbb{R}^3$ with radius 1 (i.e. do they restrict to vector fields on $M$)?

4. Suppose $M = \mathbb{R}^2$ with coordinates $(x, y)$. Consider the vector fields $\xi(x, y) = y\frac{\partial}{\partial x}$ and $\eta(x, y) = \frac{x^2}{2}\frac{\partial}{\partial y}$ on $M$. We computed in class their flows and saw that they are complete. Compute $[\xi, \eta]$ and its flow? Is $[\xi, \eta]$ complete?