6. Riemannian Geometry

Differential geometry is concerned with the study of manifolds equipped with various types of geometric structure. Important geom. structures on mfds are Riemannian metrics.
The study of Riemannian mfds is an essential part of differ. geometry.

6.1 Basic definitions

Def. 6.1 A Riemannian manifold of dim. n is a (smooth) manifold M equipped with a Riemannian metric, i.e. a $\binom{0}{2}$-tensor $g \in \mathcal{T}^0_2(M)$ s.t. $\underset{\overset{\shortparallel}{g_x}}{g(x)} : T_xM \times T_xM \to \mathbb{R}$ is positive-definite symmetric bilinear form $\forall x \in M$.

Note that, given a Riem. mfd. (M, g), any open subset $U \subseteq M$ inherits a Riem. metric $g|_U$.

<u>Def. 6.2</u> Let (M, g^M) and (N, g^N) be Riem. mfds.

① A diffeomorphism $f: M \longrightarrow N$ is called an isometry,

if $g^M_{(x)}(\xi, \eta) = g^N_{f(x)}(T_x f \xi, T_x f \eta) \quad \forall x \in M, \forall \xi, \eta \in T_x M$.

(i.e. $T_x f : T_x M \to T_{f(x)} N$ is orthogonal w.r.to g^M_x and $g^N_{f(x)}$)

$f^* g^N = g^M$.

② (M, g^M) and (N, g^N) are called isometric, if there exists an isometry between them. Notation: $(M, g^M) \simeq (N, g^N)$.

③ A smooth map $f: M \to N$ is a local isometry at $x \in M$, if $\exists$ an open neighbh. $U \subseteq M$ of x s.t. $f|_U : U \to f(U)$ is a diffeom satisfying ①.

④ (M, g^M) and (N, g^N) are called locally isometric, if for any $x \in M$ $\exists$ an open neighbh. $U \subseteq M$ of x and an isometry $f: U \to f(U)$.

Ex. $M = \mathbb{R}^n$ $\quad g = dx^1 \otimes dx^1 + \dots + dx^n \otimes dx^n$

$(x^1, \dots, x^n)$ coordinates $\quad T_x M = T_x \mathbb{R}^n \simeq \mathbb{R}^n$ $\quad g_x$ is standard inner product on $\mathbb{R}^n$

$= \langle , \rangle$ (under this identification)

standard Euclidean (flat) metric on $\mathbb{R}^n$.

Ex. (Hypersurfaces)

A hypersurface in $\mathbb{R}^{n+1}$ is a submfd. M of $\mathbb{R}^{n+1}$ of dim. n.
If $n = 2$, one speaks also of surfaces in $\mathbb{R}^3$.

Euclidean metric on $\mathbb{R}^{n+1}$ induces a Riemannian metric on M via restriction of $g = \langle , \rangle$ to $T_x M \subseteq T_x \mathbb{R}^{n+1} \simeq \mathbb{R}^{n+1}$.

Ex. $S^n \subseteq \mathbb{R}^{n+1}$ metric on S^n induced by $\langle , \rangle$ on $\mathbb{R}^{n+1}$ is called the standard metric or round metric on S^n.

Ex. If $M \to N$ is an immersion between mfds., then any Riem. metric g^N on N induces a Riem. metric g^M on M given by $g^M := f^* g^N$.

Ex. (M, g^M), (N, g^N) Riem. mfd.

Then $M \times N$ is a Riem. mfds with the product metric

$$g^M \times g^N : \quad T(M \times N) \simeq TM \times TN$$

$$\left(g^M \times g^N\right)_{(x,y)}\left((\xi,\eta),(\xi',\eta')\right) := g^M_x(\xi,\xi') + g^N_y(\eta,\eta')$$

for $\xi, \xi' \in T_x M$, $\eta, \eta' \in T_y N$.

In local coordinates: (M, g) Riem. mf., (U, u) a chart for M

$$g\big|_U = \sum_{i,j} g_{ij}\, du^i \otimes du^j \qquad g_{ij} = g\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right)$$

Here $\{g_{ij}(x)\}_{i,j}$ is a symmetric positive-definite $n \times n$ matrix $\forall x \in M$.

<u>Lemma 6.3</u> Every manifold M admits a Riemannian metric (in fact many).

<u>Proof.</u> Let $\mathcal{A} = \{(U_\alpha, u_\alpha) : \alpha \in I\}$ be an atlas for M and $\{f_\alpha\}_{\alpha \in I}$ a partition of unity subordinate to the cover $\{U_\alpha\}_{\alpha \in I}$ s.t. $\mathrm{supp}(f_\alpha) \subseteq U_\alpha$.

On U_α we can define a Riem. metric by $\sum_{i=1}^{n} du_\alpha^i \otimes du_\alpha^i =: g_\alpha$.

and $g := \sum_{\alpha \in I} f_\alpha g_\alpha$ defines a Riemannian metric on M.

Remark. Analogue of Lemma 6.3 is not true for pseudo-Riem. metrics of a given signature.

For example $\nexists$ Lorentzian metric on S^{2n}.

Suppose M is an oriented manifold equipped with a Riem metric. (an oriented Riem. mfd). Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in I\}$ be an oriented atlas for M. For $(U_\alpha, \varphi_\alpha) \in \mathcal{A}$

$$\mathrm{vol}(g)\big|_{U_\alpha} := \sqrt{\det(g^\alpha_{ij})}\, du^1_\alpha \wedge \dots \wedge du^n_\alpha$$

where $g^\alpha_{ij} = g\left(\frac{\partial}{\partial u^i_\alpha}, \frac{\partial}{\partial u^j_\alpha}\right)$. Since $(g^\alpha_{ij})(x)$ is a positive-def. symmetric matrix $\forall x \in U_\alpha$, $\det(g^\alpha_{ij}(x)) > 0$ $\forall x \in U_\alpha$.

and $\mathrm{vol}(g)\big|_{U_\alpha}$ a well-def. nowhere vanishing n-form on U_α.

Since for different $(U_\alpha, u_\alpha), (U_\beta, u_\beta) \in \mathcal{U}$, $\mathrm{vol}(g)\big|_{U_\alpha}$ and $\mathrm{vol}(g)\big|_{U_\beta}$ coincide on $U_\alpha \cap U_\beta$, $\mathrm{vol}(g)$ defines a nowhere vanishing n-form on all of M.

<u>Def. 6.4</u> Suppose (M,g) is an oriented Riem. mfd. Then the nowhere vanishing n-form $\mathrm{vol}(g) \in \Omega^n(M)$ is called the volume form of (M,g).

- On an oriented Riem. mfd of dim. n we can identify functions with n-forms:

$$
\begin{aligned}
C^\infty(M,\mathbb{R}) &\longrightarrow \Omega^n(M) \\
f &\longmapsto f\,\mathrm{vol}(g).
\end{aligned}
$$

In particular, we can integrate fcts $f \in C^\infty(M,\mathbb{R})$ with compact support on oriented Riem. mfd.: $\int_M f = \int_M f\,\mathrm{vol}(g)$.

<u>Ex.</u> $M = \mathbb{R}^n$ with its standard Euclid. metric g and standard orientation. Then $\mathrm{vol}(g) = dx^1 \wedge \dots \wedge dx^n$.

Integral of $f \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ is usual integral $\int_{\mathbb{R}^n} f \, dx^1 \dots dx^n$

$$= \int_{\mathbb{R}^n} f \, dx^1 \wedge \dots \wedge dx^n.$$

For any Riemannian mf. (M, g):

$$\hat{g}_x = g_x^\# : T_xM \to T_x^*M \quad \text{is an isomorphism } \forall x \in M.$$
$$\xi \longmapsto g_x(\xi, -)$$

Hence, $g^\# : TM \to T^*M$ is a vector bundle isomorphism and induces isomorphism $g^\# : \Gamma(TM) \xrightarrow{\sim} \Omega^1(M)$
$$\xi \longmapsto g(\xi, -).$$

In particular, for any $f \in C^\infty(M,\mathbb{R})$ $\exists!$ $\mathrm{grad}(f) \in \Gamma(TM)$ s.t. $g^\#(\mathrm{grad}(f)) = df$, which is called the gradient of f w.r.t. g.

<u>Remark</u> More generally, on a Riemannian mfd. contravariant tensors can be identified with covariant tensors.

- On an oriented Riem. mf (M,g) we can use $\mathrm{vol}(g)$ to identify vector fields with $(n-1)$-forms:

$$\begin{aligned} \Gamma(TM) &\longrightarrow \Omega^{n-1}(M) \\ \xi &\longmapsto i_\xi \mathrm{vol}(g) \end{aligned}$$

Divergence of ξ: $\mathrm{Div}: \Gamma(TM) \to C^\infty(M,\mathbb{R})$

$$\mathrm{Div}(\xi)\,\mathrm{vol}(g) = d\left(i_\xi \mathrm{vol}(g)\right)$$

Ex $M = \mathbb{R}^n$ equipped with standard orient. and metric

- $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ $\quad \mathrm{grad}(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}$

$\xi = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i}$ $\quad \mathrm{div}(\xi) = \sum_{i=1}^{n} \frac{\partial \xi^i}{\partial x^i}$

$n = 3$ $\quad M = \mathbb{R}^3$

$$\begin{array}{ccccccc}
C^\infty(\mathbb{R}^3, \mathbb{R}) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3) \\
\| & & \wr\| & & \wr\| & & \wr\| \\
C^\infty(\mathbb{R}^3, \mathbb{R}) & \xrightarrow{\mathrm{grad}} & \Gamma(T\mathbb{R}^3) & \xrightarrow{\mathrm{Curl}} & \Gamma(T\mathbb{R}^3) & \xrightarrow{\mathrm{Div}} & C^\infty(\mathbb{R}^3, \mathbb{R})
\end{array}$$

$d^2 = 0 \implies \mathrm{Curl} \circ \mathrm{grad} = 0$ and $\mathrm{Div} \circ \mathrm{Curl} = 0$

where $\operatorname{curl}\left(\sum_{i=1}^{3} \zeta^i \frac{\partial}{\partial x^i}\right) = \left(\frac{\partial \zeta^3}{\partial x^2} - \frac{\partial \zeta^2}{\partial x^3}\right)\frac{\partial}{\partial x^1} + \left(\frac{\partial \zeta^1}{\partial x^3} - \frac{\partial \zeta^3}{\partial x^1}\right)\frac{\partial}{\partial x^2}$

$$+ \left(\frac{\partial \zeta^2}{\partial x^1} - \frac{\partial \zeta^1}{\partial x^2}\right)\frac{\partial}{\partial x^3}.$$

$\leadsto$ Recover from Stokes Thm. formulas of Green, Gauss and Stokes.

6.2 Hypersurfaces

$M \subseteq \mathbb{R}^{n+1}$ hypersurface equipped with the Riem. metric g induced by the Euclidean metric on $\mathbb{R}^{n+1}$.

Def. 6.5 $(M, g) \subseteq (\mathbb{R}^{n+1}, g) = (\mathbb{R}^{n+1}, \langle , \rangle)$ hypersurface.

A local unit normal vector field for M is a local vector field ν of $\mathbb{R}^{n+1}$ defined on open subset $\tilde{U} \subseteq \mathbb{R}^{n+1}$ s.t. $\forall x \in U := \tilde{U} \cap M$ one has:

- $g_x(\nu(x), \nu(x)) = 1$
- $g_x(\nu(x), \xi) = 0 \quad \forall \xi \in T_xM \subseteq T_x\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$ (i.e. $\nu(x) \in T_xM^\perp$)

Note that $(T_xM)^\perp \subseteq T_x\mathbb{R}^{n+1}$ is 1-dimensional.

Hence, there exist exactly two unit normal vectors in $T_x\mathbb{R}^{n+1}$.

Lemma 6.6

① Locally around any point $x \in M$ $\exists$ a (local) unit normal vector field.

② M is orientable $\iff$ $\exists$ a globally defined unit normal vector field (i.e. $\nu : M \to T\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$).

Proof. ② exercise.

① Fix $x \in M$ and let $\varphi : \tilde{U} \to V$ be local trivialization, where $\tilde{U}, V \subseteq \mathbb{R}^{n+1}$ open subsets, $x \in \tilde{U}$ and $\varphi(\tilde{U} \cap M) = V \cap W$ for an n-dim. subspace $W \subseteq \mathbb{R}^{n+1}$.

Choose basis $\{w_1, \dots, w_n\}$ of W and let $0 \neq w_{n+1} \in W^\perp \subseteq \mathbb{R}^{n+1}$.

Set $\xi_j(y) := T_{\varphi(y)} \varphi^{-1} w_j$ for $y \in \tilde{U}$.

Then $\{\xi_1, \dots, \xi_{n+1}\}$ is a local frame of $T\mathbb{R}^{n+1}$ defined on $\tilde{U}$ s.t. $\{\xi_1(y), \dots, \xi_n(y)\}$ is a basis of T_yM $\forall y \in U = \tilde{U} \cap M$.

Apply Gram-Schmidt orthonormalization procedure to $\{\xi_1(y), \dots, \xi_{n+1}(y)\}$

Result leads to a local orthonormal frame $\{\eta_1, \dots, \eta_n\}$ on $\tilde{U}$ for TM and $\nu := \eta_{n+1}\big|_{\tilde{U} \cap M}$ satisfies the claim.

Suppose $(M,g) \subseteq (\mathbb{R}^{n+1}, g)$ hypersurface and let ν be a local unit normal vector field, which we can view as a map

$$\nu : U \longrightarrow S^n \subseteq \mathbb{R}^{n+1} \qquad U \subseteq M \text{ open}.$$

It is also called a local Gauß map defined on U.

Fix $x \in M$. Then $T_x\nu : T_xM \to T_{\nu(x)}S^n = \nu(x)^\perp = T_xM$

is a linear map $T_x\nu := L_x$ $T_xM \to T_xM$, called the Weingarten map at x (defined by ν).

Note that $L: x \mapsto L_x \in \mathrm{Hom}(T_xM, T_xM) \simeq T_x^*M \otimes T_xM$ is a $\binom{1}{1}$-tensor on M.

<u>Prop. 6.7</u> $(M,g) \subseteq (\mathbb{R}^{n+1}, g)$ hypersurface, $\nu: U \to S^n$ local Gauss map and L corresp. Weingarten map. Then for any $x \in M$ L_x is symmetric w.r.t. g:

$$\rightsquigarrow \quad g_x(L_x(\xi_x), \eta_x) = g_x(\xi_x, L_x(\eta_x)) \quad \forall \xi_x, \eta_x \in T_xM.$$

<u>Remark</u> Otherwise put, if one uses g to identify the $\binom{1}{1}$-tensor L with a $\binom{0}{2}$-tensor, then $g(L(-), -): \Gamma(TM) \times \Gamma(TM) \to C^\infty(M, \mathbb{R})$.

is symmetric.

<u>Proof.</u> ξ, η are vector fields defined on ~~the~~ neighb.. U around $x \in M$ s.t.

$\xi(x) = \xi_x$ and $\eta(x) = \eta_x$.

Then $g(\nu, \eta)\big|_U \equiv 0$ and differentiating this equation in direction of ξ gives:

$$0 = (\xi \cdot g(\nu, \eta))(x) = \underbrace{g((\xi \cdot \nu)_x, \eta_x)}_{= g(T_x\nu\,\xi_x, \eta_x) = g(L_x(\xi_x), \eta_x)} + g(\nu_x, \overbrace{(\xi \cdot \eta)_x}^{g(\nu_x, T_x\eta\,\xi_x)})$$

where ξ, η are viewed as functions $U \to \mathbb{R}^{n+1}$. $\xi = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^{n+1} \end{pmatrix}$

$$\Longrightarrow \quad g_x(L_x(\xi_x), \eta_x) - g_x(\xi_x, L_x(\eta_x))$$
$$= g\left(\nu(x), T_x\xi\,\eta_x - T_x\eta\,\xi_x\right) \quad (*)$$

Claim: $T_x\xi\,\eta_x - T_x\eta\,\xi_x = [\eta,\xi](x)$.

Note that this implies $T_x\xi\,\eta_x - T_x\eta\,\xi_x \in T_xM$ and hence $(*)$ equals zero.

Indeed, suppose (U,u) is a chart for M and let $v := u^{-1}: u(U) \to U$ and write $\partial_i := \frac{\partial}{\partial u^i}$. Then $\partial_j(v(y)) = T_y v\, e_j = \begin{pmatrix} \frac{\partial v^1}{\partial y^j}(y) \\ \vdots \\ \frac{\partial v^{n+1}}{\partial y^j}(y) \end{pmatrix} \subseteq M \subseteq \mathbb{R}^{n+1}$

$$\partial_i \cdot (\partial_j (v(y))) = \left(\frac{\partial^2 v^1}{\partial y^i \partial y^j}, \dots, \frac{\partial^2 v^{n-1}}{\partial y^i \partial y^j} \right)$$

$$\Rightarrow \quad \partial_i \cdot \partial_j = \partial_j \cdot \partial_i \longleftarrow (*)$$

$$\Rightarrow \quad \xi = \sum \xi^i \partial_i \qquad \eta = \sum \eta^i \partial_i$$

$$\& \quad \xi \cdot \eta - \eta \cdot \xi = \left(\sum \xi^i \partial_i \right) \cdot \left(\sum \eta^j \partial_j \right) - \left(\sum \eta^j \partial_j \right) \cdot \left(\sum \xi^i \partial_i \right)$$

$$= \sum_{i,j} \left(\xi^i (\partial_i \cdot \eta^j) \partial_j - \eta^j (\partial_j \cdot \xi^i) \partial_i + \xi^i \eta^j \big(\underbrace{\partial_i \cdot \partial_j - \partial_j \cdot \partial_i}_{= 0} \big) \right)$$

coordinate formula for Lie bracket.

$$= [\xi, \eta]$$

$(M, g) \subseteq (\mathbb{R}^{n+1}, g)$ hypersurfaces in Euclidean space.

$\rightsquigarrow$ unit normal vector field $\nu : U_x \longrightarrow S^n \subseteq \mathbb{R}^{n+1}$

$\rightsquigarrow$ Weingarten map $L_x : T_x M \longrightarrow T_x M$ linear map

$$L_x = T_x \nu$$

$$\underset{\in M}{x} \longrightarrow L_x \in \mathrm{Hom}(T_x M, T_x M) \simeq T_x^* M \otimes T_x M$$

defines a $\binom{1}{1}$-tensor field on M

L_x is symmetric w.r.t. g: $g(L(-), -)$ is symmetric $\binom{0}{2}$-tensor.

Def. 6.8 For $m > 0$

$$T_xM \times T_xM \longrightarrow \mathbb{R}$$
$$(\xi_x, \eta_x) \longmapsto g_x(L_x^{m-1}(\xi_x), \eta_x)$$

is a symmetric bilinear form on T_xM $\forall x \in M$.

It defines symmetric $\binom{0}{2}$-tensor on M, called the m-th fundamental form.

The first fund. form is just g and the 2nd-fund. form $\mathrm{II}: x \longmapsto \mathrm{II}_x$ is given by $\mathrm{II}_x(\xi_x, \eta_x) = g_x(L_x(\xi_x), \eta_x)$.

As a symmetric matrix over $\mathbb{R}$, L_x is orthogonally diagonalizable. Let $\kappa_1(x), \dots, \kappa_n(x)$ be the eigenvalues of L_x, then they are called the principal curvatures of $(M,g) \subseteq (\mathbb{R}^{n+1}, g)$ at x.

The pairwise orthog. eigenvector are called the principal curvature directions at x.

<u>Def. 6.9</u>

① $K(x) := \det(L_x) = \prod_{i=1}^{n} \kappa_i(x)$ is called the Gauß curvature of $(M,g) \subseteq (\mathbb{R}^{n+1}, g)$ at x.

(2) Mean curvature of M at x is given by $H(x) = \frac{1}{n} \operatorname{tr}(L_x)$

$$= \frac{1}{n} \sum_{i=1}^{n} \kappa_i(x).$$

(3) A point $x \in M$ is called umbilic, if $\kappa_1(x) = .. = \kappa_n(x)$ (i.e. L_x is multiple of identity), and flat, if $\kappa_1(x) = .. = \kappa_n(x) = 0$.

(3) A smooth curve $c : I \to M$ is called a curvature line, if $c'(t)$ is an eigenvector of $L_{c(t)}$ $\forall t \in I$.

Ex. $M = \mathbb{R}^n \subseteq \mathbb{R}^{n+1}$ $\{e_1, \dots, e_{n+1}\}$ standard basis in $\mathbb{R}^{n+1}$

. We can have $\nu(x) = (x, e_{n+1}) \in T_x \mathbb{R}^{n+1}$ as a global unit normal vector field.

Gauss map $\nu : M \to S^n$ is constant
$x \longmapsto e_{n+1}$

$\Rightarrow$ $L_x = 0$ $\forall x$ $\Rightarrow$ all curvatures vanish at all $x \in M$.

Ex $S^n_R \subseteq \mathbb{R}^{n+1}$ n-sphere of radius R.
$= \{x \in \mathbb{R}^{n+1} : \|x\| = R\}$

$T_x S^n_R \cong \{v \in \mathbb{R}^{n+1} : \langle x, v \rangle = 0\}$ $\forall x \in S^n_R$.

$\Rightarrow$ $\nu(x) = \frac{1}{R} x$ is global unit normal vector field.

$\Rightarrow$ Gauss map $\nu : S^n_R \to S^n_1 = S^n$

$\Rightarrow$ $\forall x \in S^n_R \quad L_x = \frac{1}{R} \mathrm{Id}$.

$\Rightarrow$ every point $x \in S^n_R$ is umbilic, since $k_i(x) = \frac{1}{R} \quad i = 1, .., n$.
and all directions are princ. curv. directions.

$\Rightarrow$ Gauss curvature $K = \frac{1}{R^n}$

Mean curvature $H = \frac{1}{R}$.

$\Rightarrow$ $\mathrm{II} = \frac{1}{R} g(-, -)$

(3) Cylinder in $\mathbb{R}^{n+1}$ of radius $R > 0$.

$$M = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \ , \ \|x\| = R\} \subseteq \mathbb{R}^{n+1}$$

$$\nu(x,t) = \frac{1}{R}(x,0) \in \mathbb{R}^n \times \mathbb{R} \simeq T_x \mathbb{R}^n \times T_t \mathbb{R}$$ global unit normal v.f.

$$T_{(x,t)} M = \nu(x,t)^\perp = \{(y,s) \in \mathbb{R}^n \times \mathbb{R} \ , \ \langle x, y\rangle = 0\}$$

Gauß map is restriction of linear map $\frac{1}{R} \mathrm{Id} \times 0 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}$ to $S^{n-1} \times \mathbb{R} = M$

$$\Rightarrow \quad L_{(x,t)} = \frac{1}{R} \mathrm{Id} \times 0$$

$\Rightarrow$ principal curv. are all constant (in x), $n-1$ of them are equal to $\frac{1}{R}$ and an is equal to 0.

In particular, $K(x) = 0 \ \forall x \in M$ and $H(x) = \frac{n-1}{nR}$.

<u>Remark.</u> $(M, g) \subseteq (\mathbb{R}^{n+1}, g)$

Properties / Quantities / objects of / on $(M, g) \subseteq (\mathbb{R}^{n+1}, g)$ are called extrinsic, if they depend on the embedding $\iota : M \hookrightarrow \mathbb{R}^{n+1}$ and intrinsic, if they depend only on (M, g) as a Riem. mfd. (i.e. they are invariant under isometries of (M, g)).

① Group of Isometries of $(\mathbb{R}^{n+1}, g) \hookrightarrow$ Euclid. metric.
equals $Euc(n+1)$, the so called
Euclidean group of motions.

$$Euc(n+1) = \{ F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} : F(x) = Ax + b \text{ for } A \in O(n+1),\ b \in \mathbb{R}^{n+1} \}$$

$F \in Euc(n+1) \Rightarrow F$ is smooth

$$T_x F = A \qquad \forall x \in \mathbb{R}^{n+1}$$

$\Rightarrow Euc(n+1) \subseteq Isom(\mathbb{R}^{n+1}, g)$. In fact, one has
$Euc(n+1) = Isom(\mathbb{R}^{n+1}, g)$.

(2) $M, N \subseteq \mathbb{R}^{n+1}$ hypersurfaces and $F \in \mathrm{Euc}(n+1)$ s.t.

$F(M) = N$.

$\Rightarrow F\big|_M : M \to N$ is an isometry.

But not all isometries between hypersurfaces arises in this way (i.e. are induced from isometries of $(\mathbb{R}^{n+1}, g)$).

For example: $M = \{(x, 0, z) : |x| < \pi, \ |z| < 1\} \subseteq \mathbb{R}^3$

$f : M \to \mathbb{R}^3$

$f(x, 0, z) = (\cos x, \sin x, z)$

M is open rectangle in (x,z)-plane and $f: M \to f(M)$ bijection to open subset of a cylinder (parametrisation of $f(M)$.)

<u>Claim</u> f is an isometry between (M,g) and $(f(M),g)$.

$$T_{(x,0,z)} f\,(v,0,w) = (-\sin(x)v,\ \cos x\, v,\ w)$$

$$\langle(-\sin(x)v_1,\ \cos x\, v_1,\ w_1),\ (-\sin x\, v_2,\ \cos x\, v_2,\ w_2)\rangle$$

$$= \underbrace{(\sin(x)^2 + \cos(x)^2)}_{=1} v_1 v_2 + w_1 w_2 = \langle (v_1, w_1), (v_2, w_2)\rangle$$

But f can't be the restriction of an Euclid. motion of $\mathbb{R}^{n+1}$, since it would need to preserve distances:

$$\mathrm{dist}\left(f(\pi/2,0,0), f(0,0,0)\right) = \| f(\pi/2,0,0) - f(0,0,0) \|$$
$$= 1 \neq \pi/2 = \mathrm{dist}\left((\pi/2,0,0),(0,0,0)\right)$$

- All curv. quantities we defined are invariant or behave naturally under orientation preserving Euclidean motions: If $M \subseteq \mathbb{R}^{n+1}$ hypersurface, $F(x) = Ax + b$ with $\det(A) = 1$, then $f := F\big|_M : M \to f(M) =: \tilde{M}$ is a diffeomorphism.

If ν is a unit norm. vector field on $\tilde{U} \subseteq \tilde{M}$, then

$f^* \nu$ is ——— " ——— on $U := f^{-1}(\tilde{U}) \subseteq M$;

since $T_x f = A : T_x \mathbb{R}^{n+1} \to T_{f(x)} \mathbb{R}^{n+1}$ orthog. linear map

that maps $T_x M$ to $T_{f(x)} \tilde{M}$.

$$\Rightarrow L_x = (T_x f)^{-1} \circ \tilde{L}_{f(x)} \circ T_x f = A^{-1} \circ \tilde{L}_{f(x)} \circ A$$

$\Rightarrow L_x$ and $\tilde{L}_{f(x)}$ have the same eigenvalues.

(i.e. princ. curv. of M at x are the same

as princ. curv. of $\tilde{M}$ at $f(x)$.)

$\Rightarrow$ If ξ_x is an eigenv. of L_x then $A\xi_x$ is one for $\tilde{L}_{f(x)}$.

$\Rightarrow$ principal curv. directions are compatible with Euclidean motions.

However, these objects / quantities are not intrinsic as the examples of $\mathbb{R}^2 \subseteq \mathbb{R}^3$ and the cylinder $S^1 \times \mathbb{R} \subset \mathbb{R}^3$ shows with the exception of maybe the Gauß curvature.
In fact, the Gauß curv. is intrinsic (up to sign for n odd).
We will see this if $n = 2$ „Theorema Egregium".

Lemma 6.10 $(M, g) \subseteq (\mathbb{R}^{n+1}, g_{euc} = \langle , \rangle)$ hypersurface.

If $c : I \to M$ a smooth curve, $I \subset \mathbb{R}$ open interval, $0 \in I$ and ν a local unit normal v.f. defined locally around $c(0) =: x$, then

$$\mathrm{II}\big(c'(0), c'(0)\big) = - \langle \nu(x), c''(0) \rangle$$

where the second deriv. c'' is taken as a curve $c : I \to M \subseteq \mathbb{R}^{n+1}$.

Proof. $\nu \circ c$ defines a smooth curve in $\mathbb{R}^{n+1}$ defined on interval I' containing 0 and $(\nu \circ c)'(0) = T_{c(0)} \nu \, c'(0) = L_{c(0)} c'(t)$.

c has values in $M \implies \langle \underline{\nu(c(t))}, c'(t) \rangle = 0 \quad (*)$

Differ. $(*)$ at $t=0$: $0 = \langle L_{c(0)} c'(0), c'(0) \rangle$

$\mathrm{II}(c'(0), c'(0)) = \langle L_{c(0)} c'(0), c'(0) \rangle + \underline{\langle \nu(c(0)), c''(0) \rangle}$

<u>Def. 6.11</u> $(M, g) \subseteq (\mathbb{R}^{n+1}, g_{euc})$ hypersurf., $x \in M$.

$$K_{nor}(x)(\xi_x) := \mathrm{II}(\xi_x, \xi_x) \qquad \xi_x \in S^{n-1} \subseteq T_x M$$

is called the normal curvature at x in direction $\xi_x \in S^{n-1} \subseteq T_x M$.

If $\xi_x \in S^{n-1} \subseteq T_x M$ is a normed eigenvector of L_x corresp. to the eigenvalue $\kappa_i(x)$, then $\mathrm{II}(\xi_x, \xi_x) = \kappa_i(x)$

Suppose $\xi_x \in T_x M$, $g_x(\xi_x, \xi_x) = 1$ and let ν be local unit normal v.f. around $x \in M$.

Consider the affine plane

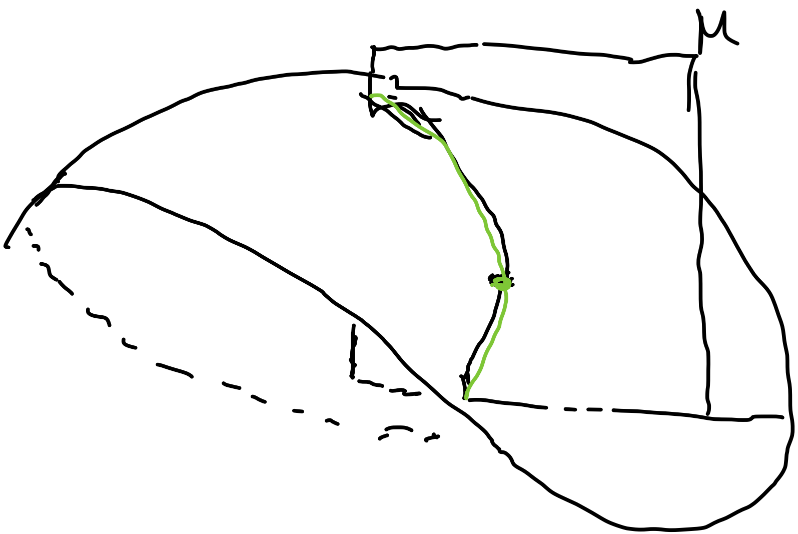
$$P = \{ x + t\nu(x) + s\xi_x \; : \; (t,s) \in \mathbb{R}^2 \}$$

through x generated by $\nu(x)$ and ξ_x.

Let $f : V \to \mathbb{R}$ be a smooth reg. fct., $V \subseteq \mathbb{R}^{n+1}$ open neigh. of $x \in \mathbb{R}^{n+1}$ s.t. $f^{-1}(0) = M \cap V$

Consider $f|_{V \cap P}$, $V \cap P \subset P$ open subset in P, then $F(s,t) := f(x + t\nu(x) + s\xi_x) = 0$

is the intersection of M with P locally around x.



We have $\frac{\partial F}{\partial t}(0,0)$

$= T_x f\, \nu(x) \neq 0$

Implicit Fct Thm.

$\Longrightarrow$ locally around x, $F(s,t)=0$

is given by a smooth curve

$c : s \longmapsto x + t(s)\,\nu(x) + s\xi_x$

with $c(0) = x$ and $c'(0)$

$= t'(0)\,\nu(x) + \xi_x$

$= \xi_x$.

Without loss of generality assume c is param. by arc length

$$(\|c'(s)\| = 1 \quad \forall s)$$

$$\hookrightarrow \langle c'(s), c''(s) \rangle = 0$$

$$\Rightarrow \underline{c''(s) = -\kappa(s)\nu(x)}.$$

$$\Rightarrow \mathrm{II}(\xi_x, \xi_x) = -\langle \nu(x), c''(0) \rangle = \kappa(s)$$

$$\mathrm{II}(\xi_x, \xi_x) = \kappa_{nor}(x)(\xi_x)$$

Hence, we see that the normal curv. of M at x & equals the curv. at x of the curve one gets from intersecting M with normal planes through x.

<u>Exercise</u> Suppose $(M, g) \subseteq (\mathbb{R}^{n+1}, g)$ is a orientable, connected hypersurface.

Then every point $x \in M$ is umbilic $\iff$ M is contained in a sphere or an affine plane.