Cor. 2.29 M is a manifold, $U \subseteq M$ open subset, $A \subseteq M$ closed subset with $A \subset U$.

Then the following holds:

① $\exists$ C^∞-fct. $\phi : M \longrightarrow [0,1]$ s.t. $\operatorname{supp}(\phi) \subset U$ and $\phi(x) = 1 \quad \forall x \in A$. (Bump function)

② If $f : U \longrightarrow \mathbb{R}^m$ is smooth, then $\exists$ a smooth fct. $\tilde{f} : M \longrightarrow \mathbb{R}^m$ s.t. $\tilde{f}\big|_A = f\big|_A$.

③ Suppose $M \subseteq \mathbb{R}^n$ is a submfd. and $f : M \longrightarrow \mathbb{R}^m$ smooth. Then $\exists$ an open subset $\tilde{U} \subseteq \mathbb{R}^n$ with $M \subset \tilde{U}$ and a smooth fct $\tilde{f} : \tilde{U} \to \mathbb{R}^m$ s.t. $\tilde{f}\big|_M = f$.

Proof

① Set $V := M \setminus A$, which is open. Then $\{U, V\}$ is a open cover of M. By Thm. 2.27, $\exists$ a partition of unity $\tilde{\mathcal{F}} = \{f_k : M \to \mathbb{R} : k \in \mathbb{N}\}$ subordinate to $\{U, V\}$, i.e. for $f_k \in \tilde{\mathcal{F}}$ either $\operatorname{supp}(f_k) \subset U$ or $\operatorname{supp}(f_k) \cap A = \emptyset$.

Let $\phi : M \to [0,1]$ be the sum of all f_k with $\operatorname{supp}(f_k) \subset U$.

Then $\operatorname{supp}(\phi) \subset U$ and $\phi|_A \equiv 1$.

② Choose ϕ as in ① and define

$$\tilde{f}(x) := \begin{cases} f(x)\,\phi(x) & \text{if } x \in U \\ 0 & \text{if } x \in M \setminus U \end{cases}$$

Since $\mathrm{supp}(\phi) \subset U$, the open subsets U and $M \setminus \mathrm{supp}(\phi)$ form an open cover of M and $\tilde{f}$ is smooth on both subsets and hence on all of M. By construction, $\tilde{f}|_A = f$, since $\phi|_A \equiv 1$.

③ $\underset{\subseteq \mathbb{R}^n}{f : M} \to \mathbb{R}^m$ C^∞-map. By definition of smoothness, for every $x \in M$ $\exists$ open neighbhd. $\tilde{U}_x \subseteq \mathbb{R}^n$ of x and a C^∞-fct. $\tilde{f} : \tilde{U}_x \to \mathbb{R}^m$ s.t. $\tilde{f}_x|_{\tilde{U}_x \cap M} = f|_{\tilde{U}_x \cap M}$.

Set $\tilde{U} := \bigcup_{x \in M} \tilde{U}_x$; it is open subset in $\mathbb{R}^n$ and $\mathcal{U} = \{\tilde{U}_x : x \in M\}$ is open cover of $\tilde{U}$.

Thm. 2.27

$\Longrightarrow$ $\exists$ a partition of unity $\{\phi_k : k \in \mathbb{N}\}$ on $\tilde{U}$ subordinate to $\mathcal{U}$.

For each $k \in \mathbb{N}$ choose $\tilde{U}_k \in \mathcal{U}$ s.t. $\mathrm{supp}(\phi_k) \subset \tilde{U}_k$.

and we write $\tilde{f}_k$ for the corresp. fct.

As in (2) we can extend $\tilde{f}_k \phi_k$ by 0 from $\tilde{U}_k$ to a C^∞-fct. $\tilde{U} \to \mathbb{R}$.

Then $\tilde{f} := \sum_{k \in \mathbb{N}} \tilde{f}_k \phi_k$ defines smooth fct. $\tilde{U} \to \mathbb{R}^n$.

Moreover, $\cancel{\tilde{f}(x)}$ for $x \in M$,

$$\tilde{f}(x) = \sum_{k \in \mathbb{N}} \tilde{f}_k(x)\, \phi_k(x) = \sum_{k \in \mathbb{N}} f(x)\, \phi_k(x) = f(x)$$

$\sum_{k \in \mathbb{N}} \phi_k(x) = 1$.

3. The Tangent Bundle

$f : \mathbb{R}^n \to \mathbb{R}^m$ C^∞-map.

For $x \in \mathbb{R}^n$ the derivative of f at x is a linear map

$$D_x f : \mathbb{R}^n \to \mathbb{R}^m .$$

This is best linear approximation of f near x.

To generalize this to maps $f : M \to N$ between manifolds, we need a linear approximation of M at x, which will be called the tangent space $T_x M$ of M at x. Then the derivative will be a linear map $T_x M \to T_{f(x)} N$.

3.1. The tangent space of a submfd. of $\mathbb{R}^n$

$U \subseteq \mathbb{R}^n$ open subset, $f: U \to \mathbb{R}^m$ C^∞-map.

For $x \in U$ we set $T_x U := \{(x,v) : v \in \mathbb{R}^n\}$

$T_x U$ is a vector space, $(x,v)+(x,w) = (x, v+w)$

$$\lambda(x,v) = (x, \lambda v) \quad \lambda \in \mathbb{R}.$$

and call it the tangent space of U at x

(a copy of $\mathbb{R}^n$ with origin at x; evidently, $T_x U \cong \mathbb{R}^n$).

The tangent map of f at x is given by the linear map

$$T_x f : T_x U \longrightarrow T_{f(x)} \mathbb{R}^m \quad T_x f(x,v) = (f(x), D_x f v)$$

Sometimes we just write $T_x f v = D_x f v$ $\quad (T_x U \simeq \mathbb{R}^n, T_{f(x)}\mathbb{R}^m \simeq \mathbb{R}^m)$.

<u>Prop. 3.1</u> $M \subseteq \mathbb{R}^n$ submfd. of dim. $k \le n$ and $x \in M$.
Then the following subsets of $T_x\mathbb{R}^n$ coincide:

① $\{(c(0), c'(0)) \in T_x\mathbb{R}^n \;:\; c : (-\varepsilon,\varepsilon) \to M$ smooth curve, $\varepsilon > 0$, $c(0) = x\}$

$\hookrightarrow$ derivative is taken as a map $c : (-\varepsilon,\varepsilon) \to \mathbb{R}^n$

② $\operatorname{Im}(T_y \varphi)$, where $\varphi : V \to U \subseteq \mathbb{R}^n$ is a local parametrization for M with $\varphi(y) = x$.

③ $\ker(T_x f)$, where $f : \underset{\subseteq \mathbb{R}^n}{U} \to \mathbb{R}^{n-k}$ is a local presentation of M as a zero set of a reg. smooth fct. ($f^{-1}(0) = M \cap U$).

In particular, the subset of $T_x\mathbb{R}^n$ given by any of these equiv. descriptions is a k-dimensional subspace of $T_x\mathbb{R}^n$.

Proof

(2) $\subseteq$ (1) By defn. of ψ, $T_y\psi : T_yV \to T_xU$ is an injective linear map. $\psi(y)=x$

$\Rightarrow \mathrm{Im}(T_y\psi) \subseteq T_xU$ is a k-dim. subspace of $T_xU = T_x\mathbb{R}^n$.

For any $(y,v) \in T_yV$ $\exists\, \varepsilon > 0$ s.t. $y+tv \in V$ for $|t|<\varepsilon$, since V is open.

$\Rightarrow$ $c(t) := \psi(\cancel{x}\, y + tv)$ smooth curve

$c : (-\varepsilon, \varepsilon) \to M$ $c(0) = x$.

Moreover, $c'(0) = D_y\psi\, v$; Hence, $T_y\psi(y,v) = (c(0), c'(0))$.

(1) $\subseteq$ (3) $\quad c : (-\varepsilon, \varepsilon) \to M \quad , \; c(0) = x$

$\exists\, \varepsilon' > 0$ s.t. $c((-\varepsilon', \varepsilon')) \subset M \cap U$ (since U is open).

$\Rightarrow$ $f \circ c : (-\varepsilon', \varepsilon') \to \mathbb{R}^{n-k}$ is smooth and in fact

identically zero.

$\Rightarrow$ $0 = D_0(f \circ c) = D_x f\, c'(0) \Rightarrow (c(0), c'(0)) \in \ker(T_x f)$.

In summary, (2) $\subseteq$ (1) $\subseteq$ (3). Moreover, since $D_x f : \mathbb{R}^n \to \mathbb{R}^{n-k}$

is surjective, $\ker(T_x f) \subseteq T_x \mathbb{R}^n$ is k-dim. Hence, by dim.

reasons, ① = ② = ③.

Def 3.2 $M \subseteq \mathbb{R}^n$ a k-dim. submfd.

For $x \in M$ the tangent space T_xM of M at x is the k-dimensional subspace of $T_x\mathbb{R}^n$ defined by any of the three equiv. descriptions of Prop. 3.1.

Ex. If $U \subseteq \mathbb{R}^n$ is open subset, then $id_U : U \to U$ is a global parametrization of and so $T_xU = T_x\mathbb{R}^n$, which justifies our definition of T_xU for an open subset at beginning.

Ex $S^n \subseteq \mathbb{R}^{n+1}$

Recall that $f^{-1}(0) = S^n$, $f : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{R}$

$f(x) = \langle x, x \rangle - 1$.

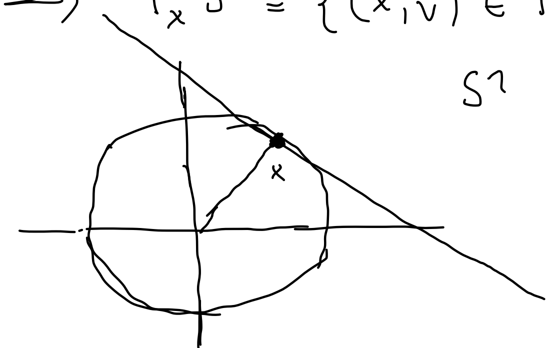
$T_x S^n$

$T_x f : T_x \mathbb{R}^{n+1} \longrightarrow T_0 \mathbb{R} = \mathbb{R}$ $\qquad D_x f\, v = 2 \langle x, v \rangle$

$(x, v) \longmapsto (0, D_x f)$ $\qquad v \in \mathbb{R}^{n+1}$

$\Longrightarrow T_x S^n = \{ (x, v) \in T_x \mathbb{R}^{n+1} : \langle x, v \rangle = 0 \}$

S^1



$T_x S^1$

Ex. $GL(n,\mathbb{R}) \subseteq M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ general linear group.

open subset.

$\Rightarrow$ $A \in GL(n,\mathbb{R})$ $\quad T_A GL(n,\mathbb{R}) = \{(A,B) : B \in M_n(\mathbb{R})\}$

$\simeq M_n(\mathbb{R}) \simeq \mathbb{R}^{n^2}$.

$O(n) = \{A \in GL(n,\mathbb{R}) : A^{-1} = A^t\}$ orthog. group.

$O(n) = f^{-1}(0)$ $\quad f(A) = AA^t - \mathrm{Id}$. $\quad f : GL(n,\mathbb{R}) \to M_n^{sym}(\mathbb{R}) \simeq \mathbb{R}^{\frac{(n+1)n}{2}}$

$T_A f(A,B) =$ $\quad T_A f : T_A GL(n,\mathbb{R}) \to T_0 M_n^{sym}(\mathbb{R}) = M_n^{sym}(\mathbb{R})$.

$= (f(A), AB^t + BA^t)$ $\quad A \in O(n)$

$= (0, AB^t + BA^t)$.

$\Longrightarrow$ for $A \in O(n)$, $T_A O(n) = \{(A, B) \in T_A GL(n, \mathbb{R}) :$
$B^t = -A^{-1} BA\}$.

In particular, $T_{Id} O(n) = \{(Id, B) \in T_{Id} GL(n, \mathbb{R}) : B^t = -B\}$
$\simeq M_n^{skew}(\mathbb{R})$

$\hookrightarrow$ skew-symmetric $n \times n$ matrices

$T_{Id} O(n) := \mathfrak{o}(n)$

inherits from the group structure on $O(n)$ a Lie algebra structure (given by the commutator of matrices).

It is called the Lie algebra of $O(n)$.

Ex. $M \subseteq \mathbb{R}^n$, $N \subseteq \mathbb{R}^m$ two submfd. Then $M \times N \subseteq \mathbb{R}^n \times \mathbb{R}^m$ is a submfd. of $\mathbb{R}^{n+m}$.

$$T_{(x,y)}(M \times N) = T_x M \times T_y N \subseteq T_x \mathbb{R}^n \times T_y \mathbb{R}^m .$$

Ex $T^n = \underbrace{S^1 \times \dots \times S^1}_{n \text{ times}}$

$$T_z T^n = T_{z_1} S^1 \times \dots \times T_{z_n} S^n \qquad z = (z_1, \dots, z_n) \in S^1 \times \dots \times S^1 .$$