4.2 Review: Multilinear algebra

Suppose $V_1, \dots, V_r$ are (real) finite-dim. vector spaces.

For a vector sp. W we write $L(V_1, \dots, V_r; W)$ for the vector space r-linear maps $V_1 \times \dots \times V_r \longrightarrow W$.

<u>Def. 4.4</u> $V_1, \dots, V_r$ vector sp.

1) The tensor product of $V_1, \dots, V_r$ is

$$V_1 \otimes \dots \otimes V_r := L(V_1^*, \dots, V_r^*; \mathbb{R}).$$

② For $(v_1, \dots, v_r) \in V_1 \times \dots \times V_r$ we write

$$v_1 \otimes \dots \otimes v_r \in V_1 \otimes \dots \otimes V_r$$

for the $v_1 \otimes \dots \otimes v_r : (\lambda_1, \dots, \lambda_r) \longmapsto \prod_{i=1}^{r} \lambda_i(v_i)$.

Note that the map

$$\otimes : V_1 \times \dots \times V_r \longrightarrow V_1 \otimes \dots \otimes V_r$$
$$(v_1, \dots, v_r) \longmapsto v_1 \otimes \dots \otimes v_r$$

is a r-linear map, i.e. $\otimes \in L(V_1, \dots, V_r ; V_1 \otimes \dots \otimes V_r)$.

Some properties of the tensor product:

- **Universal property:** For any r-linear map $f: V_1 \times \dots \times V_r \to W$ into a vector sp. W $\exists !$ linear map $\tilde{f}: V_1 \otimes \dots \otimes V_r \to W$ s.t. $f = \hat{f} \circ \otimes$.

$$\begin{tikzcd} V_1 \times \dots \times V_r \arrow[r, "f"] \arrow[d, "\otimes"] & W \\ V_1 \otimes \dots \otimes V_r \arrow[ur, dashed, "\tilde{f}"'] & \end{tikzcd}$$

$\Big(f(v_1, \dots, v_r) = \tilde{f}(v_1 \otimes \dots \otimes v_r) \Big)$

In particular, $f \longmapsto \hat{f}$ defines an isomorphism $L(V_1, \dots, V_r; W) \cong L(V_1 \otimes \dots \otimes V_r, W)$.

[Notation: For V, W vector sp., $L(V, W)$ denotes vector sp. of linear maps $V \to W$]

- <u>Associativity and Distributivity</u> : $\cdot\ (V_1 \otimes V_2) \otimes V_3 \simeq V_1 \otimes V_2 \otimes V_3$

  (natural isomorphism). $\cdot\ (V_1 \oplus V_2) \otimes V_3 \simeq V_1 \otimes V_3 \oplus V_2 \otimes V_3$

- <u>Basis</u> : If $\{e_{i,j}\}_{1 \leq j \leq n_i}$ is a basis of V_i, $i = 1, \dots, r$. with $n_i = \dim(V_i)$.

then $\{(e_{1,j_1} \otimes \dots \otimes e_{r,j_r})\}_{1 \leq j_i \leq n_i,\ 1 \leq i \leq r}$

is a basis for $V_1 \otimes \dots \otimes V_r$.

$\Longrightarrow \dim(V_1 \otimes \dots \otimes V_r) = \prod_{i=1}^{r} \dim(V_i)$.

- $\exists$ canonical isomorphism:
  - $V_1^* \otimes V_2^* \simeq (V_1 \otimes V_2)^*$
    $\lambda_1 \otimes \lambda_2 \longmapsto (v_1 \otimes v_2 \longmapsto \lambda_1(v_1)\lambda_2(v_2))$.
  - $V_1^* \otimes V_2 \simeq L(V_1, V_2)$
    $\lambda_1 \otimes v_2 \longmapsto (v_1 \mapsto \lambda_1(v_1)\, v_2)$.

- If $f_i : V_i \to W_i$ are linear maps $i = 1, \dots, r$, then by the univ. property $\exists!$ linear map $f_1 \otimes \dots \otimes f_r$ s.t.

commutes:

$$\begin{array}{ccc} V_1 \times \dots \times V_r & \xrightarrow{\otimes} & V_1 \otimes \dots \otimes V_r \\ \downarrow f_1 \times \dots \times f_r & & \downarrow f_1 \otimes \dots \otimes f_r \\ W_1 \times \dots \times W_r & \xrightarrow{\otimes} & W_1 \otimes \dots \otimes W_r \end{array}$$

Def 4.5 Suppose V is a vector sp. and write

$$L^r(V,\mathbb{R}) := L(\underbrace{V\times\dots\times V}_{r};\mathbb{R}) = \underbrace{V^*\otimes\dots\otimes V^*}_{r}.$$

① A r-lined map $\omega\in L^r(V,\mathbb{R})$ is called alternating, if

$$\omega(v_{\sigma(1)},\dots,v_{\sigma(r)}) = \operatorname{sign}(\sigma)\,\omega(v_1,\dots,v_r)$$

$\forall\, v_1,\dots,v_r\in V$ and $\sigma\in S_r := \{\sigma:\{1,\dots,r\}\to\{1,\dots,r\}$ bijections$\}$

($\Leftrightarrow$ ω vanishes if one inserts an element twice).

Notation: We write $\Lambda^r V^* := L^r_{alt}(V,\mathbb{R}) \subseteq L^r(V,\mathbb{R})$ for the subspace of r-linear alternating maps.

We have a natural projection: $\mathrm{Alt} : L^r(V,\mathbb{R}) \longrightarrow L^r_{alt}(V,\mathbb{R})$, called alternator, given by:

$$\mathrm{Alt}(\omega)(v_1,\dots,v_r) := \frac{1}{r!}\sum_{\sigma\in S_r} \mathrm{sign}(\sigma)\,\omega(v_{\sigma(1)},\dots,v_{\sigma(r)}).$$

Note that, if $\omega \in L^r_{alt}(V,\mathbb{R})$, then $\mathrm{Alt}(\omega) = \omega$.

~~It~~ It follows that, if $r > \dim(V)$, then $\Lambda^r V^* = 0$

by r-linearity and since altern. maps vanish if an element gets inserted twice.

- Also, if $r = \dim(V)$, then $\Lambda^r V^*$ is 1-dim.:

Fix a basis $\mathcal{B} = \{e_1, \dots, e_r\}$ of V, then $\det_{\mathcal{B}} : \underbrace{V \times \dots \times V}_{r} \to \mathbb{R}$ (the determ. of r vectors) is an element of $\Lambda^r V^*$.

If $\omega \in \Lambda^r V^*$, then $\omega(v_1, \dots, v_r) = \det_{\mathcal{B}}(v_1, \dots, v_r)\, \omega(e_1, \dots, e_r)$.

$\Rightarrow \Lambda^r V^* \simeq \mathbb{R} \det_{\mathcal{B}}$.

Notation: $\Lambda^* V^* := \bigoplus_{r \geq 0} \Lambda^r V^*$ with the convention

$\Lambda^0 V^* := \mathbb{R}$ $\Lambda^1 V^* := V^*$

It is a finite-dim. vector space and any linear map $f: V \to W$ induces a linear map $f^*: \Lambda^r W^* \to \Lambda^r V^*$ given by

$$f^*\omega(v_1, \dots, v_r) = \omega(f(v_1), \dots, f(v_r)).$$

which extends to a linear map $f^*: \Lambda^* W^* \to \Lambda^* V^*$.

Note that $(g \circ f)^* = f^* \circ g^*$ for $g: W \to Z$ linear map.

Def. 4.6 $\omega \in \Lambda^r V^*$, $\eta \in \Lambda^s V^*$. Then their wedge product $\omega \wedge \eta \in \Lambda^{r+s} V^*$ is given by:

$$\omega \wedge \eta (v_1, \ldots, v_{r+s}) := \frac{(r+s)!}{r!\, s!} \operatorname{Alt}(\omega \otimes \eta)(v_1, \ldots, v_{r+s})$$

$$= \frac{1}{r!\, s!} \sum_{\sigma \in S_{r+s}} \operatorname{sign}(\sigma)\, \omega(v_{\sigma(1)}, \ldots, v_{\sigma(r)})\, \eta(v_{\sigma(r+1)}, \ldots, v_{\sigma(r+s)}).$$

By linearity, we can extend this to $\Lambda^* V$:

$$\sum \omega_i \wedge \sum \eta_j := \sum_{i,j} \omega_i \wedge \eta_j \qquad \omega_i, \eta_j \in \Lambda^i V^*.$$

Prop. 4.7 The vector space $\Lambda^* V^* := \bigoplus_{r \geq 0} \Lambda^r V^*$ equipped with $\wedge$ is an (associative, unital) graded-commutative algebra; i.e.:

① $(\omega \wedge \eta) \wedge \varrho = \omega \wedge (\eta \wedge \varrho) \qquad \omega, \eta, \varrho \in \Lambda^* V^*$

② $1 \in \mathbb{R} = \Lambda^0 V^*$ satisfies $\underline{1 \wedge \omega = \omega \wedge 1 = \omega} \qquad \forall \omega \in \Lambda^* V^*$.

③ $\Lambda^r V^* \wedge \Lambda^s V^* \subseteq \Lambda^{r+s} V^*$ (graded algebra).

④ $\omega \in \Lambda^r V^*, \eta \in \Lambda^s V^*$: $\omega \wedge \eta = (-1)^{rs} \eta \wedge \omega$

(graded-commutative).

Moreover, for any linear map $f: V \to W$, the map $f^*: \Lambda^* W^* \to \Lambda^* V^*$ is a graded algebra morphism:

$$f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$$
$$f^* \Lambda^r W^* \subset \Lambda^r V^*.$$
$$f^* 1 = 1.$$

Prop. 4.8

1. If $\omega_1, \dots, \omega_r \in V^*$ and $v_1, \dots, v_r \in V$, then

$$\omega_1 \wedge \dots \wedge \omega_r (v_1, \dots, v_r) = \det\left(\left(\omega_i (v_j) \right)_{1 \leq i,j \leq r} \right)$$

In particular, $\omega_1, \dots, \omega_r$ are linearly indep. $\Leftrightarrow$ $\omega_1 \wedge \dots \wedge \omega_r \neq 0$.

② If $\{\lambda_1, \dots, \lambda_n\}$ is a basis of V^*, then

$$\{\lambda_{i_1} \wedge \dots \lambda_{i_r} : 1 \leq i_1 < i_2 < \dots < i_r \leq n\}$$

is a basis of $\Lambda^r V^*$.

4.3 Tensors

M mfd.

- Fix $x \in M$ consider $\overbrace{T_xM \otimes \dots \otimes T_xM}^{p} \otimes \overbrace{T_x^*M \otimes \dots \otimes T_x^*M}^{q} =$

$= L(\underbrace{T_x^*M, \dots, T_x^*M}_{p}, \underbrace{T_xM, \dots, T_xM}_{q}; \mathbb{R})$. $(*)$

and denote by $(TM)^{\otimes p} \otimes (T^*M)^{\otimes q}$ the disjoint union of $(*)$ over all $x \in M$.

- We have a natural projection $\pi : \underline{TM^{\otimes p} \otimes T^*M^{\otimes q}} \to M$.

It admits the structure of a smooth vector bundle over M

(induced from the vector bundle struc. on TM and T^*M).

<u>Def. 4.9</u>

① A (smooth) $\binom{p}{q}$-tensor on M is a (smooth) section of $\pi : TM^{\otimes p} \otimes (T^*M)^{\otimes q} \longrightarrow M$.

② We write $\mathcal{T}^p_q(M)$ for the vector space of $\binom{p}{q}$-tensors on M, which is also a module over $C^\infty(M, \mathbb{R})$.

If $\phi \in \mathcal{T}^p_q(M)$, $\psi \in \mathcal{T}^r_s(M)$, then $\phi \otimes \psi$ defined by

$$(\phi \otimes \psi)(x) := \phi(x) \otimes \psi(x) \quad \forall x \in M.$$

is a $\binom{r+p}{s+q}$ - tensor on M $\big(x \longmapsto (\phi_x, \psi_x) \xrightarrow{\otimes} \phi_x \otimes \psi_x$
composition of smooth maps$\big)$.

Suppose (U, u) is a chart, then $\frac{\partial}{\partial u^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial u^{i_p}} \otimes du^{j_1} \otimes \dots \otimes du^{j_q}$
$\in T^p_q(U)$

form a basis of $\pi^{-1}(x) = T_x M^{\otimes r} \otimes T_x^* M^{\otimes q}$, $\forall x \in U$.

Hence any section ϕ of π can be written as:

$$(*) \quad \phi\big|_U = \sum_{\substack{i_1, \dots, i_p \\ j_1, \dots, j_q}} \phi^{i_1 \dots i_p}_{j_1 \dots j_q} \frac{\partial}{\partial u^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial u^{i_p}} \otimes du^{j_1} \otimes \dots \otimes du^{j_q}$$

for real-valued fct $\phi^{i_1 \dots i_p}_{j_1 \dots j_q}$ on U. Smoothness of ϕ is equiv. to smoothness of $\phi^{i_1 \dots i_p}_{j_1 \dots j_q}$ for any coord.

They are called the local coordinate expression of ϕ w.r. to (U, u).

<u>Remark</u> We have locally many tensors and via partitions of unity also globally.

Given $\phi \in T^p_q(M)$ we can construct a map, also denoted by ϕ, given by :

$$\phi : \underbrace{\Gamma(T^*M) \times \dots \times \Gamma(T^*M)}_{p} \times \underbrace{\Gamma(TM) \times \dots \times \Gamma(TM)}_{q} \longrightarrow C^\infty(M,\mathbb{R}) \qquad (*)$$

$$(\omega^1, \dots, \omega^p, s_1, \dots, s_q) \longmapsto \underline{\left(x \longmapsto \phi_x(\omega^1(x), \dots, \omega^p(x), s_1(x), \dots, s_q(x)\right)}.$$

By construction, this map is $\underline{C^\infty(M,\mathbb{R})}$-linear in each entry and by $(*)$ $\phi(\omega^1, \dots, \omega^p, s_1, \dots, s_q)$ is indeed in $C^\infty(M,\mathbb{R})$, since locally on the domain U of a chart, it is given by

$\underline{\sum \phi^{i_1 \dots i_p}_{j_1 \dots j_q}}\, \omega^1_{i_1} \dots \omega^p_{i_p}\, \xi^{j_1}_1 \dots \xi^{j_p}_p$ which is a prod of a product of smooth fcts.

<u>Remark</u> A section ϕ of π is smooth $\iff \phi(\omega^1, \dots, \omega^p, \xi_1, \dots, \xi_q)$ is smooth for smooth 1-forms ω^i and vector fields ξ_j.

$$\left(\phi^{i_1 \dots i_p}_{j_1 \dots j_q} = \phi|_U \left(du^{i_1}, \dots, du^{i_p}, \frac{\partial}{\partial u^{j_1}}, \dots, \frac{\partial}{\partial u^{j_q}} \right) \right).$$

R:

<u>Remarks</u>: Special cases.

- $\phi \in \mathcal{T}^0_1(M) = \Gamma(T^*M)$ is a 1-form and we know already $\phi(\xi) : M \to \mathbb{R}$ is C^∞ for $\xi \in \Gamma(TM)$.

- $\phi \in \mathcal{T}^1_0(M) = \Gamma(TM)$ is a vector field:

$$\phi(\omega) = \omega(\phi) : M \to \mathbb{R} \quad \text{which is } C^\infty \text{ for } \phi \in \Gamma(T^*M).$$

Prop. 4.10 $(*)$ defines a linear isomorphism between $\mathcal{T}^p_q(M)$ and the vector space $W^p_q := L\big(\Gamma(TM)\times\dots\times\Gamma(TM), \Gamma(T^*M), \dots$ $C^\infty(M,\mathbb{R})\dots\times\Gamma(T^*M) ; C^\infty(M,\mathbb{R})\big)$ of $C^\infty(M,\mathbb{R})$-multilinear maps.

Proof. We already know that $\phi \in \mathcal{T}^p_q(M)$ gives rise to an element $\in W^p_q$ and that $(*)$ is linear (and injective).
Conversely, let $\phi : \Gamma(TM)\times\dots\times\Gamma(T^*M) \longrightarrow C^\infty(M,\mathbb{R})$ be $C^\infty(M,\mathbb{R})$-linear. Then we have to show that

$$\phi(x)\big(\omega^1(x),\dots,\xi_q(x)\big) := \phi\big(\omega^1,\dots,\xi_q\big)(x).$$

for 1-forms ω^i and vector fields ξ_j just depends on the values of the ω's and s's at x. In this case,

$x \mapsto \phi_x$ is a $\binom{p}{q}$-tensor.

It is sufficient to show that if any of the 1-forms or vector fields σ vanishes at x, so does $\phi(\omega^1, \ldots, s_q)(x)$.

Suppose first σ vanishes identically on open neighb. U of x and let $f \in C^\infty(M, \mathbb{R})$ s.t. $f\big|_{M \setminus U} \equiv 1$ and $f(x) = 0$ (which exists by Cor. 2.32).

Then $f\sigma = \sigma$ and by $C^\infty(M, \mathbb{R})$-linearity, we have $\phi(\omega^1, \ldots, f\sigma, \ldots s_q)(x) = f(x)\, \phi(\omega^1, \ldots, \sigma, \ldots s_q)(x) = 0$

This implies that for a chart (U, u) with $x \in U$,

$$\phi(\omega^1, \ldots, s_q)\big|_U \quad (\text{in particular at } x), \text{ i.e.}$$

depends on the restrictions of the ω's and s's to U.

We have

$$\phi(\omega^1, \ldots, s_q)\Big|_U = \sum \underbrace{\phi^{i_1 \ldots i_p}_{j_1 \ldots j_q}}_{\phi(du^{i_1}, \ldots, \frac{\partial}{\partial u^{j_q}})} \underline{\omega^1_{i_1} \ldots \omega^p_{i_p} s^{j_1}_1 \ldots s^{j_q}_q} \quad (**).$$

$\Longrightarrow$ If $\sigma(x) = 0$, then so are the local coordinate expression at x and hence $\phi(\omega^1, \ldots, s_q)(x) = 0$ by $(**)$.

Remark Elements $\mathcal{T}^p_0(M)$ (resp. $\mathcal{T}^0_p(M)$) are called p-times contravariant (resp. p-times covariant tensors).

Ex.

A tensor $g \in \mathcal{T}^0_2(M)$ is called a (pseudo-) Riemannian metric on M, if for any $x \in M$ the bilinear form $g_x : T_xM \times T_xM \to \mathbb{R}$ is symmetric and non-degenerate.

If M is connected, the signature (r,s) of $g(x)$ does not depend on x and is ~~reffer~~ referred to as the signature of g.

In particular, g is called a Riemannian metric, if g has sign. $(n,0)$ and Lorentzian if the signature is $(n-1,1)$ or $(1,n-1)$.

$M = \mathbb{R}^n$

Standard inner product gives (flat) Euclidean metric $g = dx^1 \otimes dx^1 + \dots + dx^n \otimes dx^n$.

Standard Lorentzian inner product on $\mathbb{R}^n$ gives (flat) Lorentzian / Minkowski; metric

$$g = -dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + \dots + dx^n \otimes dx^n.$$