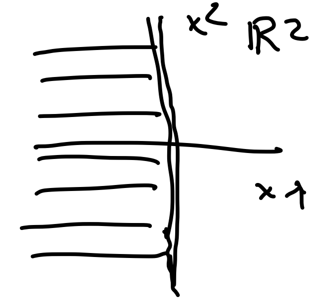
5.3 Manifolds with boundary

Def. 5.6 A n-dimensional (smooth) manifold with boundary is a Hausdorff, second countable topolog. space M equipped with a maximal C^∞-atlas of charts with values in the half space $H^n := \{(x^1, \dots, x^n) \in \mathbb{R}^n : x^1 \leq 0\}$.



- A C^∞-atlas $\mathcal{A} = \{(U_\alpha, u_\alpha) : \alpha \in I\}$ of M with values in H^n is a collection of homeomorphisms $u_\alpha : U_\alpha \to u_\alpha(U_\alpha) \subseteq H^n \subseteq \mathbb{R}^n$, where $U_\alpha \subseteq M$ and $u_\alpha(U_\alpha) \subseteq H^n$ are open subsets s.t.
  - $M = \bigcup_{\alpha \in I} U_\alpha$
  - $u_\beta \circ u_\alpha^{-1} : u_\alpha(U_\alpha \cap U_\beta) \longrightarrow u_\alpha(U_\alpha \cap U_\beta)$.

are smooth, which means that they can be extended to smooth maps defined on open subset of $\mathbb{R}^n$ containing $u_\alpha(U_\alpha \cap U_\beta)$.

- A point $x \in M$ is called a boundary point of M, if $\exists$ a chart (U_α, u_α) s.t. $x \in U_\alpha$ and $u_\alpha(x) \in u_\alpha(U_\alpha) \cap \{0\} \times \mathbb{R}^{n-1}$
$$= u_\alpha(U_\alpha) \cap \partial H^n$$
where $\partial H^n = \{(x^1, \dots, x^n) \in H^n : x^1 = 0\}$.

We write $\partial M = \{x \in M : x \text{ is a boundary point}\}$.

Note that $x \in \partial M \iff \forall$ charts $(U_\alpha, u_\alpha) \in \mathcal{A}$ with $x \in U_\alpha$
$u_\alpha(x) \in \partial H^n$.

- Points $x \in M \setminus \partial M$ are called interior points.

$x \in M \setminus \partial M \iff u_\alpha(x) \in H^n \setminus \partial H^n$ $\forall$ chart (U_α, u_α) with $x \in U_\alpha$.

<u>Prop. 5.7</u> M n-dim. mfd with boundary $\partial M \neq \emptyset$.
Then ∂M is a $(n-1)$-dim. mfd. without boundary.

<u>Proof.</u> An atlas for ∂M is given by

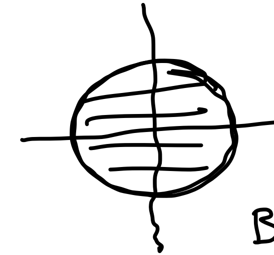
$$\{ (U_\alpha \cap \partial M, u_\alpha|_{U_\alpha \cap \partial M}) : (U_\alpha, u_\alpha) \in \mathcal{A} \}, \text{ where}$$

$\mathcal{A}$ is ~~atlas~~ atlas for M. $(u_\alpha(U_\alpha) \cap \{0\} \times \mathbb{R}^{n-1}$ is open in $\mathbb{R}^{n-1})$.

Ex.

$M = B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ $\quad n = 2$

$\partial M = S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$.



B^2

Ex. Rotatation of B^2 around a circle leads to a mfd with boundary M s.t. $\partial M = T^2$

All concepts such smooth maps, vector fields, differential forms etc. works also dense for mfd. with boundary.

- If $i : \partial M \hookrightarrow M$ is the natural inclusion, then it is smooth and for any k-form ω on M, $i^*\omega$ is a k-form on ∂M.

- $x \in \partial M$, $T_x \partial M \subseteq T_x M$.

An orientation on M (defined as for mf. without boundary).
induces an orientation on ∂M:

Suppose $\mathcal{B} = \{(U_\alpha, u_\alpha) ; \alpha \in I\}$ is an oriented atlas for M and consider $u_\beta \circ u_\alpha^{-1} : u_\alpha(U_{\alpha\beta}) \to u_\beta(U_{\alpha\beta})$ for two charts in $\mathcal{B}$. As observed, $u_\alpha(U_{\alpha\beta}) \cap \{0\} \times \mathbb{R}^{n-1}$ is mapped to $u_\beta(U_{\alpha\beta}) \cap \{0\} \times \mathbb{R}^{n-1}$.

$\Rightarrow$ At a point $x = (0, x^2, \ldots, x^n) \in u_\alpha(U_{\alpha\beta})$ the derivative $D_x(u_\beta \circ u_\alpha^{-1})$ has the following form

$$D_x(u_\beta \circ u_\alpha^{-1}) = \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ v & & A & \end{pmatrix} \qquad \begin{array}{l} \lambda \in \mathbb{R} \\ v \in \mathbb{R}^{n-1} \\ A \in M_{n \times n}(\mathbb{R}). \end{array}$$

$u_\beta \circ u_\alpha^{-1}$ maps interior points to interior points, i.e. points with negative x^1 to points with negative x^1, which implies $\lambda > 0$. Since $\det(D_x(u_\beta \circ u_\alpha^{-1})) > 0$, this implies $\det(A) > 0$, which describes the derivative of the transition maps of the atlas on ∂M induced by $\mathcal{B}$.

Hence, the atlas on ∂M induced by $\mathcal{B}$ is also oriented.

Suppose M is a mfd. with boundary of dim. n.

If $\omega \in \Omega_c^{n-1}(M)$, then ω vanishes on the open subset $M \setminus \mathrm{supp}(\omega)$ and so does $d\omega$ (by Thm. 4.28), which implies that $\mathrm{supp}(d\omega) \subset \mathrm{supp}(\omega)$. In particular, $d\omega \in \Omega_c^n(M)$.

Thm. 5.8 (Stokes Theorem).

Suppose M is an oriented n-dim. mfld. with boundary ∂M.

For any $\omega \in \Omega_c^{n-1}(M)$ we have:

$$\int_M d\omega = \int_{\partial M} \omega \quad \left(= \int_{\partial M} i^*\omega \right).$$

In particular, if M is a manifold without boundary ($\partial M = \emptyset$),

then $\int_M d\omega = 0$.

Proof Assume $\omega \in \Omega_c^{n-1}(M)$

- Let (U_i, u_i) $i = 1, \dots, \ell$ be chart of an oriented atlas of M s.t. $\mathrm{supp}(\omega) \subseteq U_1 \cup \dots \cup U_\ell$ and $f_i : M \to [0,1]$ $i = 1, \dots, \ell$ smooth fcts s.t. $\mathrm{supp}(f_i) \subset U_i$ and $\sum_{i=1}^{\ell} f_i \Big|_{\mathrm{supp}(\omega)} = 1$.

- Then $\left(U_i \cap \partial M, u_i\big|_{U_i \cap \partial M}\right)$ and $f_i\big|_{\partial M}$ can be used to compute $\int_{\partial M} \omega$: $\int_{\partial M} \omega = \sum_{i=1}^{\ell} \int_{U_i \cap \partial M} f_i \omega$

- Also, $\omega = \sum_{i=1}^{\ell} f_i \omega$ implies $d\omega = \sum_{i=1}^{\ell} d(f_i \omega)$ and $\mathrm{supp}(d(f_i\omega)) \subset \mathrm{supp}(f_i \omega) \subseteq U_i$

Hence, $\int_M d\omega = \sum_{i=1}^{\ell} \int_{U_i} d(f_i \omega)$

It suffices to show that $\int_{U_i} d(f_i\omega) = \int_{\partial U_i} f_i \omega \quad \forall i$.

Without loss of generality, we can assume $\mathrm{supp}(\omega)$ is contained in domain of one single chart (U, u).

Then $\omega = \sum_{i=1}^{n} \omega_i \, du^1 \wedge \ldots \wedge \widehat{du^i} \wedge \ldots \wedge du^n$

for smooth fcts $\omega_i : M \to \mathbb{R}$ with compact support in U.

- The tangent space $T_x \partial M$ for $x \in \partial M$ is spanned $\frac{\partial}{\partial u^i}$, $i \geq 2$.

$\Rightarrow du^1\big|_{\partial M} \equiv 0$ and so $\omega\big|_{\partial M} = \underline{\omega_1} \, du^2 \wedge \dots \wedge du^n$

$$\Rightarrow \int_{\partial M} \omega = \int_{\partial u(U)} \underline{\omega_1 \circ u^{-1}} = \int \omega_1 \circ u^{-1} \quad \leftarrow$$

↑ $\{0\} \times \mathbb{R}^{n-1}$

ω_1 has compact support in U.

- By Thm. 4.18, $d\omega = \sum_{i=1}^{n} \frac{\partial \omega^i}{\partial u^i} \underline{\underline{du^i}} \wedge du^1 \wedge \dots \wedge \widehat{du^i} \wedge \dots \wedge du^n$

$$= \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial u^i} \, du^1 \wedge \dots \wedge du^n .$$

$$\Longrightarrow \int_M d\omega = \sum_{i=1}^{n} (-1)^{i-1} \int_{u(U)} \frac{\partial (\omega_i \circ u^{-1})}{\partial x^i} =$$

$$= \sum_{i=1}^{n} \int_{[-\infty, 0] \times \mathbb{R}^{n-1}} \frac{\partial (\omega_i \circ u^{-1})}{\partial x^i} \qquad (*)$$

$[-\infty, 0] \times \mathbb{R}^{n-1} = H^n$

ω_i has compact support in U

Fubini: Thm for integrals allows to decompose $(*)$ into integrals over the individual coordinates, where the order of integration doesn't matter.

$$\Longrightarrow \int_M d\omega = \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{0} \frac{\partial(\omega_1 \circ u^{-1})}{\partial x^1} dx^1 \right) dx^2 \dots dx^n$$

$$+ \sum_{i=1}^{n} (-1)^{i-1} \int_{[-\infty,0] \times \mathbb{R}^{n-2}} \left(\int_{-\infty}^{\infty} \frac{\partial(\omega_i \circ u^{-1})}{\partial x^i} dx^i \right) dx^1 \dots \widehat{dx^i} \dots dx^n$$

(the inner integrals in the sum are 0)

$$= \int_{\mathbb{R}^{n-1}} (\omega_1 \circ u^{-1})(0, x^2, \dots, x^n) \, dx^2 \dots dx^n$$

FTC + ω_i have compact support

$$= \int_{\partial M} \omega .$$

$\square$.

5.4 De Rham cohomology

- $\Omega(M) := \bigoplus_{k \in \mathbb{N}} \Omega^k(M)$ $\qquad \Omega^k(M) = \{0\}$ for $k > \dim(M) := n$

↗ graded vector space

- graded-commutative algebra w.r.t. to $\wedge$:

$$\Omega^k(M) \wedge \Omega^\ell(M) \subseteq \Omega^{k+\ell}(M)$$

$$\omega \wedge \eta = (-1)^{k\ell}\, \eta \wedge \omega \qquad \omega \in \Omega^k(M),\ \eta \in \Omega^\ell(M)$$

Moreover, we have a linear map $d : \Omega(M) \longrightarrow \Omega(M)$ which is a graded derivation of degree 1 of $(\Omega(M), \wedge)$.

$$0 \xrightarrow{0} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \to \dots \xrightarrow{d} \Omega^{\dim(M)}(M) \xrightarrow{0} 0$$

By Thm. 4.18 : $\underline{d \circ d = 0}$

<u>Def 5.9</u> $\omega \in \Omega^*(M)$

① ω is closed, if $d\omega = 0$

② ω is exact, if $\exists \eta \in \Omega^*(M)$ s.t. $\omega = d\eta$.

$d^2 = 0 \implies$ any exact form is closed.

- $\ker(d) =: Z(M) \subset \Omega(M)$ subspace of closed diff. forms
$\quad \| \quad \{\omega \in \Omega(M) : d\omega = 0\}$; we write $Z^k(M) = Z(M) \cap \Omega^k(M)$

It is a subalgebra of $(\Omega(M), \wedge)$, since

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \quad \text{for } \omega \in \Omega^k(M).$$

- $\operatorname{Im}(d) =: B(M) \subseteq Z(M) \subseteq \Omega(M)$

is a subspace. It is a two-sided ideal in $Z(M)$:

$$\eta = d\eta' \qquad \eta' \in \Omega^k(M), \quad \omega \in Z(M)$$

$$d(\eta' \wedge \omega) = \underbrace{d\eta'}_{=\eta} \wedge \omega + (-1)^k \wedge \underbrace{d\omega}_{=0} = \eta \wedge \omega$$

$$\Longrightarrow \quad H(M) = Z(M)/B(M) = \bigoplus_{k \geq 0} \underbrace{Z^k(M)/B^k(M)}_{=: H^k(M)}$$

is a graded-commutative (unital, associative) algebra over $\mathbb{R}$.

It is called the de Rham cohomology algebra of M and $H^k(M)$ the k-the de Rham cohomology (space or group) of M.

For $\omega \in Z^k(M)$ we write $[\omega] \in H^k(M)$ for its cohomology class.

Remark:
$$[\omega] \wedge [\eta] := [\omega \wedge \eta]$$
$$[\omega] + [\eta] := [\omega + \eta]$$
$$\lambda [\omega] := [\lambda \omega] \qquad \lambda \in \mathbb{R}.$$

<u>Remark.</u> If M is compact, $H(M)$ is finite-dimensional.
Also true for many other non-compact mflds. but not for all.

- $f: M \to N$ C^∞-map between mflds.

$\Rightarrow f^*: \Omega(N) \to \Omega(M)$ alg. morphism.

Since $f^* \circ d = d \circ f^*$ (Thm. 4.18), $f^*(Z(N)) \subset Z(M)$
and $f^*(B(N)) \subset B(M)$ and hence f^* induces
an algebra morphism:

$$f^\#: H(N) \to H(M) \qquad \left(f^\#(H^k(N)) \subset H^k(M) \right).$$
$$[\omega] \mapsto [f^*\omega]$$

- $(g \circ f)^{\#} = f^{\#} \circ g^{\#}$ for another C^∞-map $g : N \to P$ between mfds.

$\Rightarrow$ If f is a diffeomorphism, then $f^{\#} : H(N) \to H(M)$ is an isomorphism with inverse $(f^{\#})^{-1} = (f^{-1})^{\#}$.

(So diffeom. mfds have isomorphic de Rham cohomology).

In fact, smoothly homotopic mfds M and N have isomorphic de Rham cohomology ($\exists$ C^∞-maps $f : M \to N$ and $g : N \to M$ s.t. $f \circ g$ and $g \circ f$ are smoothly homotopic to the identity).

- In fact, continuously homotopic C^∞-mfds have isomorphic de Rham cohomology (in particular, homeomorphic C^∞-mfds have isomorphic de Rham cohomology).

- de Rham Thm: de Rham cohomology $\cong$ singular cohomology of M with real coefficients

  $\rightsquigarrow$ Can use tools from algebraic topology to compute de Rham cohomology.

<u>Ex.</u> $M = \mathbb{R}^n$ $\quad H^0(M) = \mathbb{R}$

$H^k(M) = \{0\}$ $\quad k > 0$ $\quad$ Any closed form is exact on $\mathbb{R}^n$

⟹ Poincaré Lemma : On any mfld., any closed form is locally exact.