Homework 4:

③
$$\frac{\partial \alpha_j^i}{\partial x^r} + \sum_{l=1}^{k} \alpha_r^l \frac{\partial \alpha_j^i}{\partial z^l} = \frac{\partial \alpha_r^i}{\partial x^j} + \sum_{l=1}^{k} \alpha_j^l \frac{\partial \alpha_r^i}{\partial z^l}$$

4.4. Differential forms

Def. 4.11 M mfd.

① A (differential) k-form on M is a $\binom{0}{k}$-tensor $\omega \in \mathcal{T}_k^0(M)$ s.t. $\omega(x) \in \Lambda^k T_x^* M \quad \forall x \in M$.

② We write $\Omega^k(M)$ for the vector space of k-forms on M, which is also module over $C^\infty(M,\mathbb{R})$. It's a subspace of $\mathcal{T}_k^0(M)$.

Convention: $\Omega^0(M) := C^\infty(M,\mathbb{R})$.

Note for $k > \dim(M)$ we have $\Omega^k(M) = 0$.

Remark • $\Lambda^k T^*M := \bigsqcup_{x\in M} \Lambda^k T_x^*M \subseteq \underbrace{T^*M \otimes \dots \otimes T^*M}_{k}$ is a vector subbundle over M.

• $\Omega^k(M) = \Gamma(\Lambda^k T^*M)$

By Prop. 4.10, we can consider a k-form $\omega \in \Omega^k(M)$ also as a k-linear, alternating map $\omega : \Gamma(TM) \times \dots \times \Gamma(TM) \to C^\infty(M,\mathbb{R})$ that is linear in each entry over $C^\infty(M,\mathbb{R})$.

Def. 4.12 Suppose $f: M \to N$ is a C^∞-map between mfds.
If $\omega \in \Omega^k(N)$, then $f^*\omega$, called the pull-back of ω via f, is a k-form on M given by

$$f^*\omega(x)\left(\xi_1(x), \dots, \xi_k(x)\right) := \omega(f(x))\left(T_x f\, \xi_1(x), \dots, T_x f\, \xi_k(x)\right)$$

$\forall\, \xi_i \in \Gamma(TM)$.

If $\xi_1, \dots, \xi_k \in \Gamma(TM)$, $f^*\omega(\xi_1, \dots, \xi_k) = (\omega \circ f)(Tf \circ \xi_1, \dots, Tf \circ \xi_k)$
which shows that $f^*\omega$ is indeed a smooth tensor field on M.

Remark In general we can pull-back $\binom{0}{k}$-tensors via C^∞-maps.

We have a natural map:

$$\mathrm{Alt} : \mathcal{T}^0_k(M) \longrightarrow \Omega^k(M)$$

$$\mathrm{Alt}(\phi)(x) := \mathrm{Alt}(\phi_x) \qquad \forall x \in M.$$

where $\omega \in \Omega^k(M)$ $\quad \mathrm{Alt}(\omega) = \omega$.

Def. 4.13 If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$ then the wedge product $\omega \wedge \eta \in \Omega^{k+\ell}(M)$ is given by

$$(\omega \wedge \eta)(x) := \omega_x \wedge \eta_x = \frac{(\ell+k)!}{k!\,\ell!} \mathrm{Alt}(\omega_x \otimes \eta_x) \quad \forall x \in M.$$

For $f \in \Omega^0(M) = C^\infty(M,\mathbb{R})$ and $\omega \in \Omega^k(M)$: $f \wedge \omega = f\omega$.

By linearity we can extend this to a linear map

$$\wedge : \Omega^*(M) \times \Omega^*(M) \longrightarrow \Omega^*(M)$$

where $\Omega^*(M) \approx \bigoplus_{k=0}^{\dim(M)} \Omega^k(M)$.

By Prop. 4.7 we have.

Prop. 4.14 The vector space $\Omega^*(M) := \bigoplus_{k\geq 0} \Omega^k(M)$ is an (associative, unital) graded-commutative algebra over the ring $C^\infty(M,\mathbb{R})$ (in particular over $\mathbb{R}$), i.e. it satisfies 1 - 4 of Prop. 4.7, since pointwise.

Prop. 4.15 Let $f: M \to N$ be a C^∞-map between mflds. Then $f^*: \Omega^k(N) \to \Omega^k(M)$ extends to a morphism $f^*: \Omega^*(N) \to \Omega^*(M)$ of (unital) graded algebras:

i.e. f^* is linear, $f^*1 = 1$, $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$ and $f^*(\Omega^k(N)) \subset \Omega^k(M)$.

Moreover, if $g : N \to P$ is another map between mflds, then

$$(g \circ f)^* = f^* \circ g^*$$

Proof: First claim follows from Sec. 4.2 ; since $f^*\omega(x) = (T_x f)^*\omega(f(x))$ $\left[T_x f : T_x M \to T_{f(x)} N \text{ induces } \underline{(T_x f)^* : \Lambda^k T^*_{f(x)} N \longrightarrow \Lambda^k T_x M} \right]$

and the second claim also from there + $T(g \circ f) = Tg \circ Tf$.

$\square$

If (U, u) is chart, then $\{du^{i_1} \wedge \dots \wedge du^{i_k} : 1 \leq i_1 < i_2 < \dots < i_k \leq k\}$ form a basis of $\Lambda^k T^*_x M = \Lambda^k T^*_x U \quad \forall x \in U$.

If $\omega \in \Omega^k(M)$, then

$$\omega|_U = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \cdots i_k} \, du^{i_1} \wedge \cdots \wedge du^{i_k}$$

local coordin. expressie of ω w.r.t (U, u).

for $\omega_{i_1 \cdots i_k} \in C^\infty(U, \mathbb{R})$. $\qquad \omega_{i_1 \cdots i_k} = \omega\left(\frac{\partial}{\partial u^{i_1}}, \ldots, \frac{\partial}{\partial u^{i_k}}\right)$.

Recall that we have an operator:

$$d : C^\infty(M, \mathbb{R}) \longrightarrow \Omega^1(M)$$
$$d(f) = df$$

We can extend this to an operator $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$. for any $k \geq 0$.

Def. 4.16 M mfd., $\omega \in \Omega^k(M)$. Then we define

$$d\omega : \underbrace{\Gamma(TM) \times \ldots \times \Gamma(TM)}_{k+1} \longrightarrow C^\infty(M,\mathbb{R}).$$

$$d\omega(\xi_0, \xi_1, \ldots, \xi_k) := \sum_{i=0}^{k} (-1)^i \, \xi_i \cdot \omega(\xi_0, \ldots, \hat{\xi}_i, \ldots, \xi_k)$$

$$+ \sum_{i<j} (-1)^{i+j} \, \omega([\xi_i, \xi_j], \xi_0, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_k),$$

where $\hat{(\cdot)}$ means we omit this entry.

By linearity, we can extend it to a linear map
$d : \Omega^*(M) \longrightarrow \Omega^*(M)$, called the exterior derivative (on diff. forms).

Lemma 4.17 $d\omega \in \Omega^{k+1}(M)$

Proof • $d\omega$ is alternating : Suppose $\underline{\xi_j = \xi_{j+1}}$

$$d\omega(\xi_0, \dots, \xi_j, \xi_{j+1}, \dots, \xi_k) = \left\{ \begin{array}{l} \underline{(-1)}^j \, \xi_j \cdot \underline{\omega(\xi_0, \dots, \hat{\xi}_j, \xi_{j+1}, \dots, \xi_k)} \\ + \underline{(-1)}^{j+1} \, \xi_{j+1} \cdot \underline{\omega(\xi_0, \dots, \xi_j, \hat{\xi}_{j+1}, \dots, \xi_k)} \end{array} \right\} = 0$$

ω is alternating

$$+ \sum_{i \neq j, j+1} \underline{(-1)^{j+i}} \, \omega([\xi_j, \xi_i], \xi_0, \dots, \hat{\xi}_j, \xi_{j+1}, \dots, \hat{\xi}_i, \dots, \xi_k)$$

$$0 = \left\{ + \sum_{i \neq j, j+1} \underline{(-1)^{j+1+i}} \, \omega([\xi_i, \xi_{j+1}], \xi_0, \dots, \hat{\xi}_i, \xi_j, \hat{\xi}_{j+1}, \dots) \right.$$

and $\underbrace{[\xi, \xi]}= 0$

· $d\omega$ is $C^\infty(M,\mathbb{R})$-linear in each entry; by being alternating it is sufficient to check it for one entry:

$f \in C^\infty(M,\mathbb{R})$

$$f\xi_j\cdot\omega(\xi_0,\dots,\hat{\xi}_j,\dots,\xi_k) + (\xi_j\cdot f)\,\omega(\xi_0,\dots,\hat{\xi}_j,\dots,\xi_k)$$

$$f[\xi_0,\xi_j] - (\xi_j\cdot f)\xi_0$$

$$\begin{aligned}
d\omega(f\xi_0,\xi_1,\dots,\xi_k) &= f\xi_0\cdot\omega(\xi_1,\dots,\omega_k) + \sum_{i>0}(-1)^i\,\xi_i\cdot\omega(f\xi_0,\dots,\hat{\xi}_i,\dots,\xi_k) \\
&\quad + \sum_{1\le j}(-1)^j\,\omega([f\xi_0,\xi_j],\xi_1,\dots,\hat{\xi}_j,\dots,\xi_k) \\
&\quad + f\sum_{1\le i<j}(-1)^{i+j}\,\omega([\xi_i,\xi_j],\xi_0,\dots,\hat{\xi}_i,\dots,\hat{\xi}_j,\dots,\xi_k) \\
&= f\,d\omega(\xi_0,\dots,\xi_k).
\end{aligned}$$

Thm. 4.18 M mfd, $d : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfies the following properties

1. For $f \in C^\infty(M, \mathbb{R})$ $df(\xi) = \xi \cdot f \quad \forall \xi \in \Gamma(TM)$.

2. For $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$ we have

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

3. $d \circ d = 0$

4. d is local: If $\omega \in \Omega^*(M)$ vanishes identically on some open subset $U \subseteq M$, then $d\omega|_U$ is also identically zero.

⑤ If (U,u) is a chart for M and $\omega \in \Omega^k(M)$, then

$$d\omega\Big|_U = \sum_{i_1<\dots<i_k} d(\omega_{i_1\dots i_k}) \wedge du^{i_1} \wedge \dots \wedge du^{i_k} =$$

$$= \sum_{i_1<\dots<i_k,\, i_0} \frac{\partial \omega_{i_1\dots i_k}}{\partial u^{i_0}}\, du^{i_0} \wedge du^{i_1} \wedge \dots \wedge du^{i_k}$$

for $\omega\Big|_U = \sum_{i_1<\dots<i_k} \omega_{i_1\dots i_k} du^{i_1} \wedge \dots \wedge du^{i_k}$.

⑥ d is natural: If $f: N \to M$ is a C^∞-map between mfds, then $\underline{d(f^*\omega)} = f^* d\omega$.

Proof:

① ✓ $df(\xi_0) = \xi_0 \cdot f$

② ~~Hence~~ Suppose $\omega|_U = 0$. Then for arbitrary vector fields $\xi_1, \dots, \xi_k$ we have $\underline{\omega(\xi_1, \dots, \xi_k)}|_U = 0$ and also $\underline{\xi_0 \cdot \omega(\xi_1, \dots, \xi_k)}|_U = 0$ for an arbitrary vf. ξ_0.

$\Longrightarrow$ $d\omega|_U = 0$

(In particular, if $\omega|_U = \eta|_U$, then $(\omega - \eta)|_U = 0$ and so $d(\omega-\eta)|_U = d\omega|_U - d\eta|_U = 0$.

(5) We first prove a special case of (2) : $\omega \in \Omega^k(M)$, $f \in C^\infty(M,\mathbb{R})$.

$$d(f\omega)(\xi_0, \dots, \xi_k) = \sum_{i=0}^{k} (-1)^i \xi_i \cdot f\omega(\xi_0, \dots, \hat{\xi}_i, \dots, \xi_k)$$

$$+ \sum_{i<j} (-1)^{i+j} f\omega([\xi_i, \xi_j], \dots, \hat{\xi}_i, \dots, \hat{\xi}_j, \dots)$$

$$= \sum_{i=0}^{k} (-1)^i \Big(\overbrace{(\xi_i \cdot f)}^{df(\xi_i)} \omega(\xi_0, \dots, \hat{\xi}_i, \dots) + f\, \xi_i \cdot \omega(\xi_0, \dots, \hat{\xi}_i, \dots, \xi_k) \Big)$$

$$+ f \sum_{i<j} (-1)^{i+j} \omega([\xi_i, \xi_j], \dots, \hat{\xi}_i, \dots, \hat{\xi}_j).$$

$\Longrightarrow$ $d(f\omega) - f\,d\omega = df \wedge \omega$ (since ω is alternating).

2nd claim: $d(du^{i_1} \wedge \dots \wedge du^{i_k}) = 0$

It follows from the definition, since insertion of $k+1$ coordinate vf. in $d(du^{i_1} \wedge \dots du^{i_k})$ ~~...~~ is zero. This is the case since $du^{i_1} \wedge \dots \wedge du^{i_k}$ is constant upon insertion of k coord. vf. and $[\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}] = 0$.

$$\Rightarrow d\Big(\sum_{i_1 < \dots i_k} \omega_{i_1 \dots i_k} \underline{du^{i_1} \wedge \dots \wedge du^{i_k}}\Big) = \sum_{i_1 < i_k} \underline{d(\omega_{i_1 \dots i_k})} \wedge du^{i_1} \wedge \dots \wedge du^{i_k}$$

$$= \sum_{i_1 < \dots < i_k, i_0} \frac{\partial \omega_{i_1 \dots i_k}}{\partial u^{i_0}} du^{i_0} \wedge \dots \wedge du^{i_k}.$$

<u>Thm. 4.18</u> M mfd., $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ exterior derivative.

Then the following holds,

① $df(\xi) = \xi \cdot f \quad \forall \xi \in \Gamma(TM)$.

② For $\omega \in \Omega^k(M)$, $\eta \in \Omega^\ell(M)$ we have $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$

③ $d \circ d = 0$

④ d is a local operator: If $\omega \in \Omega^k(M)$ is vanishing identically on an open subset $U \subseteq M$, then $d\omega|_U$ vanishes identically.

⑤ If (U, u) is a chart for M and $\omega \in \Omega^k(M)$, then

$$d\omega|_U = \sum_{i_1 < i_2 < \dots < i_k} d(\omega_{i_1 \dots i_k}) \wedge du^{i_1} \wedge \dots \wedge du^{i_k}$$

where $\omega|_U = \sum_{i_1<\dots<i_k} \omega_{i_1\dots i_k} du^{i_1}\wedge\dots\wedge du^{i_k}$.

6 d is a natural operator: If $f: N \to M$ is C^∞-map between mfds., then $d f^*\omega = f^* d\omega$.

Proof

1 ✓, 4 ✓, 5 ✓

2 By 4 we can prove this in local coordinates:

$$\omega|_U = \sum_{i_1<\dots<i_k} \omega_{i_1\dots i_k} du^{i_1}\wedge\dots\wedge du^{i_k} \qquad \eta|_U = \sum_{j_1<\dots<j_e} \omega_{j_1\dots j_e} du^{j_1}\wedge\dots\wedge du^{j_e}$$

$I = (i_1, \dots, i_k)$ $\qquad du^I = du^{i_1} \wedge \dots \wedge du^{i_k}$

$J = (j_1, \dots, j_l)$

$$\Longrightarrow \quad \omega \wedge \eta\big|_U = \sum_{I,J} \omega_I \eta_J \, \underline{du^I \wedge du^J}$$

- $d(\omega_I \eta_J) = (d\omega_I)\eta_J + \omega_I \, d\eta_J .$

$$\underline{d(\omega \wedge \eta)}\big|_U = \sum_{I,J} (d\omega_I)\eta_J \wedge du^I \wedge du^J + \sum_{I,J} \omega_I \, \underline{d\eta_J} \wedge du^I \wedge du^J .$$

$$= \underline{\sum_I d\omega_I \wedge du^I} \wedge \overline{\sum_J \eta_J \, du^J} + (-1)^k \sum_I \omega_I du^I \wedge \sum_J d\eta_J \wedge du^J$$

$$= \underline{(d\omega \wedge \eta + (-1)^k \omega \wedge d\eta)}\big|_U .$$

3 We can prove this in local coordinates:

We already know $\underline{d(du^{i_1} \wedge \dots du^{i_k})} = 0$

Hence 2 and 4 imply:

$$d^2\omega\big|_U = \sum_{i_1 < \dots i_k} d^2\omega_{i_1 \dots i_k} \wedge du^{i_1} \wedge \dots \wedge du^{i_k} = 0$$

It remains to show that $d^2 f = 0$ for $f \in C^\infty(M, \mathbb{R})$.

$$d(\underline{df})(\xi_0, \xi_1) = \xi_0 \cdot df(\xi_1) - \xi_1 \cdot df(\xi_0) - df([\xi_0, \xi_1])$$
$$= \underline{\xi_0 \cdot (\xi_1 \cdot f) - \xi_1 \cdot (\xi_0 \cdot f) - \underline{[\xi_0, \xi_1]} \cdot f}$$
$$= 0 .$$

⑥ $g \in C^\infty(M, \mathbb{R})$ $\quad f : N \longrightarrow M$

$f^* g = g \circ f$ and hence $\underline{d(f^* g)}(\xi) = \xi \cdot (g \circ f) =$

$= (Tf\,\xi) \cdot g = dg(Tf\,\xi)$

$= \underline{(f^* dg)}(\xi)$. $(*)$

Suppose (U, u) is a chart for M s.t. $f(N) \cap U \neq \emptyset$.

For $\omega|_U = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} \underline{du^{i_1} \wedge \dots \wedge du^{i_k}}$ we

Prop. 4.15

have $(f^* \omega)\Big|_{f^{-1}(U)} = \sum_{i_1 < \dots < i_k} f^* \omega_{i_1 \dots i_k} f^* du^{i_1} \wedge \dots \wedge f^* du^{i_k}$

$(*) \Longrightarrow \quad d(f^* \omega_{i_1 \dots i_k}) = f^* d\omega_{i_1 \dots i_k}$ and $d(f^* u^{i_j}) = f^* du^{i_j}$

$$\Longrightarrow \quad \underline{d(f^* \omega}\big|_{f^{-1}(U)}) = \sum f^* d\omega_{i_1 \dots i_k} \wedge f^* du^{i_1} \wedge \dots \wedge f^* du^{i_k}$$

$$= \underline{f^* (d\omega}\big|_U) \,.$$

$\square$

4.5 Lie derivatives

Suppose $f: M \to N$ local diffeom. between mfds.

$\Rightarrow$ $T_x f : T_x M \to T_{f(x)} N$ is an isomorphism.

$$T_x^* f = (T_x f)^* : T_{f(x)}^* N \to T_x^* M$$
$$\omega \mapsto \omega \circ T_x f$$

$$\left((T_x f)^{-1} \right)^* = (T_x^* f)^{-1} : T_x^* M \to T_{f(x)}^* N .$$

Def. 4.19 $\phi \in \mathcal{T}^p_p(N)$, $f: M \to N$ local diffeom.

The pullback of ϕ with respect to f is the $\binom{p}{q}$-tensor

$f^*\phi \in T^p_q(M)$ on M given by

$$f^*\phi(x)\left(\omega^1,\dots,\omega^p,\xi_1,\dots,\xi_q\right) := \phi(f(x))\left(\left(T_x^*f\right)^{-1}\omega^1,\dots,\left(T_x^*f\right)^{-1}\omega^p,\; T_xf\xi_1,\dots,T_xf\xi_k\right)$$

for $\omega^i \in T_x^*M$, $\xi_j \in T_xM$.

(Evidently, this is a $\binom{p}{q}$-tensor on M).

Applied to the flow of a vector fields we get:

<u>Def. 4.20</u> $\xi \in T(TM)$, $\phi \in \mathcal{T}^p_q(M)$, M mfd. Then the Lie derivative $\mathcal{L}_\xi \phi \in \mathcal{T}^p_q(M)$ of ϕ along ξ is given by

$$\mathcal{L}_\xi \phi(x) := \frac{d}{dt}\Big|_{t=0} \underbrace{(Fl^\xi_t)^* \phi(x)}_{\underline{T_xM^{\otimes p} \otimes T_x^*M^{\otimes q}}} \quad \forall x \in M$$

<u>Some properties:</u> $\xi \in \Gamma(TM)$.

- $f \in \Omega^0(M) = C^\infty(M, \mathbb{R})$, then $\mathcal{L}_\xi f(x) = \frac{d}{dt}\Big|_{t=0} \overbrace{Fl_t^{\xi *} f}^{(f \circ Fl_t^\xi)(x)}(x)$

$$= T_x f\, \xi(x) = df(\xi)(x).$$

$$\mathcal{L}_\xi f = df(\xi)$$

- $\eta \in \Gamma(TM)$: $\mathcal{L}_\xi \eta = [\xi, \eta]$

- $\phi \in \mathcal{T}^p_q(M)$, $\psi \in \mathcal{T}^r_s(M)$; $\mathcal{L}_\xi(\phi \otimes \psi) = \mathcal{L}_\xi \phi \otimes \psi + \phi \otimes \mathcal{L}_\xi \psi$.

(This follows from $f^*(\phi \otimes \psi) = f^*\phi \otimes f^*\psi$ for any local diffeo. f of M and from bilinearity $\otimes : (\phi, \psi) \longmapsto \phi \otimes \psi$.)

- For $\omega \in \Omega^k(M)$, $\mu \in \Omega^l(M)$

$$\mathcal{L}_\xi(\omega \wedge \mu) = \mathcal{L}_\xi \omega \wedge \mu + \omega \wedge \mathcal{L}_\xi \mu$$

Moreover note that for $\phi \in \mathcal{T}^p_q(M)$ and $\omega^1, \dots, \omega^p \in \Gamma(T^*M)$ and $\xi_1, \dots, \xi_q \in \Gamma(TM)$ the full contraction:

$$\phi \otimes \omega^1 \otimes \dots \otimes \omega^p \otimes \xi_1 \otimes \dots \otimes \xi_q \xrightarrow{C} \phi(\omega^1, \dots, \omega^p, \xi_1, \dots, \xi_q).$$

is linear and commutes with pullbacks by local diffeom. f

$$(*) \qquad C(f^*(\phi \otimes \omega^1 \otimes \dots \otimes \xi_q)) \overset{\text{def}}{=} C(\overline{f^*\phi \otimes f^*\omega^1 \otimes \dots \otimes f^*\xi_q}))$$

$$\rightarrow = f^*(\phi(\omega^1, \dots, \xi_q)) = \phi(\omega^1, \dots, \xi_q) \circ f$$

$\Longrightarrow$)

$$\mathcal{L}_\eta(\phi \otimes w^1 \otimes \cdots \otimes \xi_q) = \mathcal{L}_\eta \phi \otimes w^1 \otimes \cdots \otimes \xi_q$$

$$+ \sum_{i=1}^{p} \phi \otimes \cdots \otimes \mathcal{L}_\eta w^i \otimes \cdots \otimes \xi_1 \otimes \cdots \otimes \xi_q$$

$$+ \sum_{j=1}^{q} \phi \otimes w^1 \otimes \cdots \otimes w^p \otimes \cdots \otimes \mathcal{L}_\eta \xi_j \otimes \cdots \otimes \xi_q .$$

(*)

and

$$(\mathcal{L}_\eta \phi)(w^1, \ldots, \xi_q) = \mathcal{L}_\eta(\phi(w^1, \ldots, \xi_q)) - \sum_i \phi(w^1, \mathcal{L}_\eta w^i \ldots, \xi_q)$$

$$- \sum_j \phi(w^1, \ldots, w^p, \ldots \mathcal{L}_\eta \xi_j) .$$

$\mathcal{L}_\eta(w(\xi))$

If $\phi \in \Omega^k(M)$ this yields:

$$\mathcal{L}_\eta \phi(\xi_1, \dots, \xi_k) = \eta \cdot \phi(\xi_1, \dots, \xi_k) - \sum_j \phi(\xi_1, \dots, \overbrace{\mathcal{L}_\eta \xi_j}^{[\eta, \xi_j]}, \dots)$$

<u>Operator on differential forms:</u>

- $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$
- $\mathcal{L}_\xi : \Omega^k(M) \longrightarrow \Omega^k(M)$
- $i_\xi : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$ $\quad \xi \in \Gamma(TM)$.

$$\omega \longmapsto i_\xi \omega = \omega(\xi, \dots)$$

insertion operator.

Def. 4.21 A graded derivation of the algebra $(\Omega^*, \wedge)$ of degree r is a linear map $D : \Omega^*(M) \to \Omega^{*+r}(M)$ s.t. $D(\omega \wedge \eta) =$
$= D(\omega) \wedge \eta + (-1)^{rk} \omega \wedge D\eta \quad \forall \omega \in \Omega^k(M), \eta \in \Omega^\ell(M)$.

Prop. 4.22

1. d is a graded derivation of degree 1
2. For $\xi \in \Gamma(TM)$, $\mathcal{L}_\xi$ is graded derivation of degree 0.
3. For $\xi \in \Gamma(TM)$, i_ξ is a graded derivation of degree -1.

Moreover, if D_1 and D_2 are graded derivations of degree r_1 resp. r_2, then $[D_1, D_2] := D_1 \circ D_2 - (-1)^{r_1 r_2} D_2 \circ D_1$ is a graded derivation of degree $r_1 + r_2$.

Proof. ① and ② ✓ the rest will be discussed in tutorial or I leave as an exercise.

Prop. 4.23 Suppose D is a graded derivation of degree r of $(\Omega^\bullet(M), \wedge)$

① D is a local operator: If $\omega \in \Omega^k(M)$ vanishes identically on an open subset of M, then so does $D\omega$.

② If $\tilde{D}$ is another graded derivation of degree r such that $\tilde{D}(f) = D(f)$ and $\tilde{D}(df) = D(df)$ $\forall f \in C^\infty(M, \mathbb{R})$.

then $D = \tilde{D}$.

Proof. Exercise / Tutorial.

Prop. 4.24 M mfd., $\xi, \eta \in \Gamma(TM)$.

① $[d, \mathcal{L}_\xi] = d \circ \mathcal{L}_\xi - \mathcal{L}_\xi \circ d = 0$

② $[d, i_\xi] = d \circ i_\xi + i_\xi \circ d = \mathcal{L}_\xi$

③ $[d, d] = 2d^2 = 0$

④ $[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_\xi \circ \mathcal{L}_\eta - \mathcal{L}_\eta \circ \mathcal{L}_\xi = \mathcal{L}_{[\xi, \eta]}$.

⑤ $[\mathcal{L}_\xi, i_\eta] = \mathcal{L}_\xi \circ i_\eta - i_\eta \circ \mathcal{L}_\xi = i_{[\xi, \eta]}$

⑥ $[i_\xi, i_\eta] =$

$i_\xi \circ i_\eta + i_\eta \circ i_\xi = 0$

Proof. Exercise / tutorial.

Remark d is the unique graded deriv. of degree 1 s.t.

$$Df = df \quad \text{and} \quad D(df) = 0 .$$

$\mathcal{L}_\xi$ is the unique graded deriv. of — " — 0 s.t.

$$\mathcal{L}_\xi f = df(\xi) = \xi . f$$

$$\mathcal{L}_\xi (df) = d(\xi . f) .$$

5. Integration on mfds

Recall the transformation formula for multiple integrals:

Suppose $U \subseteq \mathbb{R}^n$ is an open subset and $\Phi : U \to \phi(U)$ a diffeom. between open subsets of $\mathbb{R}^n$.

Let $f : \Phi(U) \longrightarrow \mathbb{R}$ be a C^∞-fct with compact support

$$\int_{\phi(U)} f = \int_U (f \circ \phi) \, |\det D\phi| \qquad (*) .$$

Looks like the transformation of n-forms ~~on~~ on manifolds of dim n:

Suppose M mfd., $\dim(M) = n$, $\omega \in \Omega^n(M)$ and (U, u) is a chart of M:

Then $\omega|_U = \underline{\omega^U_{1..n}}\, du^1 \wedge .. \wedge du^n$ $\qquad \omega^U_{1..n} = \omega\left(\frac{\partial}{\partial u^1}, .., \frac{\partial}{\partial u^n}\right).$
$\in C^\infty(U, \mathbb{R})$.

Suppose $v : U \to v(U)$ is another chart for M on U.

Then $\omega|_U = \omega^v_{1..n}\, dv^1 \wedge .. \wedge dv^n$

$$\omega^{U}_{1\cdots n}(u^{-1}(y)) = \omega(u^{-1}(y))(T_u u^{-1} e_1, \ldots, T_y u^{-1} e_n)$$

$$\omega^{V}_{1\cdots n}(v^{-1}(z)) = \omega(v^{-1}(y))(T_z v^{-1} e_1, \ldots, T_z v^{-1} e_n)$$

$$y \in u(U) \qquad z \in v(U).$$

Now let $\Phi : u(U) \to v(U)$ be $\phi := v \circ u^{-1}$

$$\Longrightarrow v^{-1} \circ \phi = u^{-1} \quad \text{and} \quad T_y u^{-1} = \underline{T_{\phi(y)} v^{-1} \circ T_y \phi}$$

$$\Longrightarrow \omega^{U}_{1\cdots n}(u^{-1}(y)) = \omega(u^{-1}(y))(T_y \underline{u^{-1}} e_1, \ldots, T_y u^{-1} e_n) \quad \underline{(v^{-1})^*_{\omega}(\phi(y))}$$

$$= \omega(u^{-1}(y))(T_{\phi(y)} v^{-1} \circ T_y \phi e_1, \ldots, T_{\phi(y)} v^{-1} \circ T_y \phi e_n) \quad (T_y \phi e_1, \ldots T_y \phi e_n)$$

$$= \det(D_y\phi)\, \omega(u^{-1}(y))\left(T_{\phi(y)}v^{-1}e_1, \ldots, T_{\phi(y)}v^{-1}e_n\right)$$

$$= \det(D_y\phi)\, \omega^{v}_{1 \cdot\cdot n}(\phi(y)).$$

$$\omega^{U}_{1\cdots n}(u^{-1}(y)) = \det(D_y\phi)\, \omega^{v}_{1\cdots n}(\phi(y)).$$

$$\neq 0 \quad \forall y \in u(U)$$

If we assume U to be connected and hence so is $u(U)$, then sign of $\det(D_y\phi)$ is either always positive or negative.

(*) says that the integral over the local coordinate expression of the fct. $\omega_{1\cdots n}$ $(\omega_{1\cdots n} \circ u^{-1} : u(U) \to \mathbb{R})$

is up to a sign independent of choice of the coor.