Tutorial 1

1. Cylinder

$M = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = R^2\}$ $\quad R > 0$

$f: U \longrightarrow \mathbb{R}$ $\quad U := \mathbb{R}^3 \setminus \{(0,0,z) : z \in \mathbb{R}\}$.

$f(x,y,z) = x^2+y^2-R^2$ $\quad f^{-1}(0) = M$.

$D_{(x,y,z)}f = (2x, 2y, 0)$ is surjective $\forall (x,y,z) \in U$.

$\Rightarrow M \subseteq \mathbb{R}^3$ 2-dim. submfd.

- Graph

around points (x,y,z) with $y \neq 0$

$$g(x,z) = \pm\sqrt{R^2 - x^2} \qquad (x,z) \in V = (-R,R) \times \mathbb{R} \subseteq \mathbb{R}^2$$

$$M \cap U_\pm = gr(g) = \{(x, g(x,z), z) \in \mathbb{R}^3\}.$$

$$U_\pm := (-R,R) \times \mathbb{R}_{\substack{>0 \\ <0}} \times \mathbb{R}$$

Similary, for points $(x,y,z) \in M$ with $x \neq 0$

$$g(y,z) = \pm\sqrt{R^2 - y^2} \qquad (y,z) \in V$$

$$U_\pm := \mathbb{R}_{\substack{>0 \\ <0}} \times (-R,R) \times \mathbb{R}$$

Parametrization.

$$\psi : (\alpha, z) \longrightarrow (R\cos\alpha, R\sin\alpha, z)$$

$\alpha \in [0, 2\pi) \times \mathbb{R}$

$$T_{(\alpha,z)}\psi = \begin{pmatrix} -R\sin\alpha & 0 \\ R\cos\alpha & 0 \\ 0 & 1 \end{pmatrix}$$

is injective $\mathbb{R}^2 \to \mathbb{R}^3$ since $\cos\alpha$ and $\sin\alpha$ not zero at the same time.

Trivialization Cylindrical coordinates

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \\ z \end{pmatrix} \quad r > 0 \quad \alpha \in [0, 2\pi)$$

$\neq 0$

$$\phi : \mathbb{R}^3 \setminus \{0\} \longrightarrow \mathbb{R}^3 \qquad \phi(M) = \left\{ \begin{pmatrix} r \\ \alpha \\ z \end{pmatrix} : r = R \right\}.$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} r \\ \alpha \\ z \end{pmatrix}$$

(2) Double cone

- $M = \{(x,y,z) \in \mathbb{R}^3 ; (x^2+y^2)\tan^2\alpha = z^2\}$

$f(x,y,z) = (x^2+y^2)\tan^2\alpha - z^2$

$D_{(x,y,z)} f = (2\tan^2\alpha \, x \, , 2\tan^2\alpha \, y \, , -2z)$

vanishes at $(0,0,0)$; f not regular here.

$\Rightarrow M \setminus \{(0,0,0)\}$ is a submfd. of $\mathbb{R}^3$.

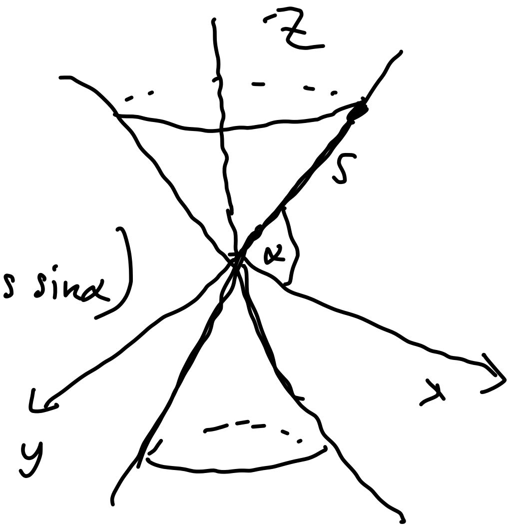
- <u>Graph</u> : $z = \pm \tan\alpha \sqrt{x^2+y^2}$

· Parametrization

Rotate $s \longmapsto (s\cos\alpha, 0, s\sin\alpha)$

$$\psi : (\varphi, s) \longmapsto (s\cos\alpha\cos\varphi, s\cos\alpha\sin\varphi, s\sin\alpha)$$

$$T_{(\varphi,s)}\psi = \begin{pmatrix} -s\cos\alpha\sin\varphi & \cos\alpha\cos\varphi \\ s\cos\alpha\cos\varphi & \cos\alpha\sin\varphi \\ 0 & \sin\alpha \end{pmatrix}$$



It is injective $\mathbb{R}^2 \to \mathbb{R}^3$.

Trivialization: Spherical coordinates

(3) $\mathrm{Hom}_r(\mathbb{R}^n, \mathbb{R}^m) \subseteq \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \simeq \mathbb{R}^{n \cdot m}$

$T_0 \in \mathrm{Hom}_r(\mathbb{R}^n, \mathbb{R}^m)$

$$(*) \quad \begin{aligned} \mathbb{R}^n &= E \oplus E^\perp \qquad & E^\perp &= \ker(T_0) \qquad & \dim(F) &= r = \dim(E). \\ \mathbb{R}^m &= F \oplus F^\perp \qquad & F &= \mathrm{Im}(T_0) & & \end{aligned}$$

With respect to $(*)$ we can identify $T \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ with a matrix

$$T = \begin{array}{c c} & \begin{array}{cc} r & n-r \end{array} \\ \begin{array}{c} r \\ m-r \end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array}$$

$A \in \mathrm{Hom}(E, F)$
$B \in \mathrm{Hom}(E^\perp, F)$
$C \in \mathrm{Hom}(E, F^\perp)$
$D \in \mathrm{Hom}(E^\perp, F^\perp)$.

- $U := \left\{ T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A \text{ invertible} \right\}$ open subset,

$\subseteq \mathbb{R}^{n \cdot m}$ because $GL(r) \subseteq \mathrm{Hom}(\mathbb{R}^r, \mathbb{R}^r)$ open,

$T_0 \in U$

$\quad ||$

$\begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad A_0$ invertible.

Fix $T \in U$. $\quad T \in \mathrm{Hom}_r(\mathbb{R}^n, \mathbb{R}^m) \iff \dim(\ker(T)) = n - r$

$$T \begin{pmatrix} v \\ w \end{pmatrix} = 0 \iff \begin{pmatrix} Av + Bw \\ Cv + Dw \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$(v, w) \in E \oplus E^{\perp} = \mathbb{R}^n$

$$\iff v = -A^{-1} B w$$
$$CA^{-1} B w = D w$$

$\Rightarrow \dim(\ker(T)) = n-r \iff \forall w \in E^\perp \quad CA^{-1}Bw = Dw$

$\iff CA^{-1}B = D.$

$$g : \underbrace{\left\{ \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \ : \ A \text{ is invertible} \right\}}_{\underline{\mathcal{U}}}$$

open subset of the vector subspace

$\left\{ \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \right\}$

$\subseteq \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$

$\longrightarrow CA^{-1}B \in \mathrm{Hom}(E^\perp, F^\perp).$

smooth and $gr(g) = U \cap \mathrm{Hom}_r(\mathbb{R}^n, \mathbb{R}^m).$

$\overset{\shortparallel}{\begin{pmatrix} A & B \\ C & CA^{-1}B \end{pmatrix}} \quad \left(\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}, g\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \right) = \Big(A$

In particular, if $r = \min\{n,m\}$, then $\dim(\mathrm{Hom}_r(\mathbb{R}^n, \mathbb{R}^m)) = n \cdot m$.

$\Rightarrow$ $\mathrm{Hom}_r(\mathbb{R}^n, \mathbb{R}^m) \subseteq \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ open subset.

(4) Grassmannian variety / manifold.

$Gr(r,n) = \{W : W \subseteq \mathbb{R}^n$ is a r-dim. subspace of $\mathbb{R}^n\}$

$=$ set of all ~~~~ r-dim. subspaces of $\mathbb{R}^n$.

$G(1,n) = \mathbb{R}P^{n-1}$ (projective).

$Gr(r,n)$ is a submfld. of $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ of dim. $r(n-r)$.

Idea: We identify r-dim. subspaces of $\mathbb{R}^n$ with the orthog. projections onto them. In this way we realise $Gr(r,n)$ as a subset $\mathrm{Hom}_r(\mathbb{R}^n,\mathbb{R}^n)$.

$E_0 \subseteq \mathbb{R}^n$ subspace of dim. r, $\mathbb{R}^n = E_0 \oplus E_0^\perp$ (*)

$P_0 : \mathbb{R}^n \to \mathbb{R}^n$ orthog. proj. to E_0, which looks with respect to (*) just as
$$P_0 = \begin{array}{cc} & \begin{array}{cc} r & n-r \end{array} \\ r & \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix} \end{array}.$$

Open neighbh. of $P_0 \in \mathrm{Hom}_r(\mathbb{R}^n,\mathbb{R}^n)$ is given by
$$U := \left\{ \begin{pmatrix} A & B \\ C & CA^{-1}B \end{pmatrix} : A \text{ invertible} \right\}.$$

Recall a linear map $P: \mathbb{R}^n \to \mathbb{R}^n$ is an orthog. projection to some subspace of $\mathbb{R}^n$ $\Longleftrightarrow$ $P^2 = P$ and $P^t = P$.

When is $T \in U$ an orthogon. projection?

$$T = T^t \qquad \begin{pmatrix} A & B \\ C & \underline{CA^{-1}B} \end{pmatrix} = \begin{pmatrix} A^t & C^t \\ B^t & \underline{(CA^{-1}B)^t} \end{pmatrix}$$

$$\Longrightarrow A = A^t, \quad B = C^t \quad \left(\Longrightarrow (CA^{-1}B)^t = CA^{-1}B.\right)$$

$$T^2 = T \qquad \begin{pmatrix} A^2 + BC & AB + BCA^{-1}B \\ CA + CA^{-1}BC & CB + (CA^{-1}B)^2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & CA^{-1}B \end{pmatrix}$$

$$\Longrightarrow \left.\begin{matrix} A^2 + BC = A \\ (AA^t + CC^t = A) \end{matrix}\right\} \text{ If this is satisfied all other identities are :}$$

$$AB + \underbrace{(A - A^2)}_{= BC} A^{-1} B = AB + B - AB = B.$$

$$f : \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n) : A \in GL(r) \right\}$$

$$\longrightarrow \Big(\underbrace{A^t A + C^t C - A}_{\in \mathrm{Hom}(\mathbb{R}^r, \mathbb{R}^r)}, \underbrace{B - C^t}_{\in \mathrm{Hom}(\mathbb{R}^{n-r}, \mathbb{R}^r)}, \underbrace{D - CA^{-1}B}_{\in \mathrm{Hom}(\mathbb{R}^{n-r}, \mathbb{R}^{n-r})} \Big).$$

<u>Parametrisation:</u>

$$A = A^t A + C^t C \qquad \underset{\displaystyle A^{-1}}{\overset{\displaystyle (A^{-1})^t}{\|}} = \mathrm{Id} + (CA^{-1})^t (CA^{-1})$$

$$\Rightarrow A = (1 + Z^t Z)^{-1} \qquad Z = CA^{-1}.$$

Parametrisation:

$$\varphi : \mathrm{Hom}(\mathbb{R}^r, \mathbb{R}^{n-r}) \ni Z \longmapsto \begin{pmatrix} \underbrace{(1 + Z^t Z)^{-1}}_{=:A} & B := C^t \\ \underbrace{ZA}_{= Z(1+Z^tZ) =: C} & CA^{-1}B \end{pmatrix}$$

$$\Rightarrow \quad \dim = r^2 + r(n-r) + (m-r)r$$

$$= r(n+m-2r+r) = r(n+m-r$$