TUTORIAL 2

① $SL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) : \det A = 1\} \subseteq GL(n,\mathbb{R}) \underset{\text{open}}{\subseteq} M_n(\mathbb{R})$

② $f : GL(n,\mathbb{R}) \longrightarrow \mathbb{R}$ $\quad f = \det - 1$

is a C^∞-map, regular and $f^{-1}(0) = SL(n,\mathbb{R})$.

$\Longrightarrow$ submf. of $M_n(\mathbb{R})$.

By Prop. 3.1, $T_{Id} SL(n,\mathbb{R})$

$$= \ker(T_{Id} f) = \ker(T_{Id} \det).$$

$T_{Id}GL(n,\mathbb{R}) \simeq M_n(\mathbb{R})$

$\cup$

$T_{Id}SL(n,\mathbb{R})$.

$C \in M_n(\mathbb{R})$

$c(t) = Id + tC$ curve in $GL(n,\mathbb{R})$.

$c(0) = Id \quad c'(0) = C$

$$\boxed{(T_{Id}\det)(C)} = \frac{d}{dt}\Big|_{t=0} \det(Id + tC)$$

$$\det(Id + tC) = \det(e_1, \dots, e_n) + t\left(\det(C_1, e_2, \dots, e_n) + \dots + \det(e_1, \dots, e_{n-1}, C_n)\right)$$

+ terms of higher order in t.

$C_i = Ce_i$ = i-th column of C

$\Longrightarrow$

$$\frac{d}{dt}\Big|_{t=0} \det(\mathrm{Id} + tC) = \frac{d}{dt}\Big|_{t=0} + \big(\det(C_1, e_2, \ldots, e_n) + \ldots + \ldots + \det(e_1, \ldots, C_n)\big)$$

$$= c_{11} + \ldots + c_{nn} = \mathrm{tr}(C).$$

$$\boxed{\det(\mathrm{Id} + tC) = 1 + t\,\mathrm{tr}(C) + \text{higher order terms in } t}$$

$\Longrightarrow$ $T_{\mathrm{Id}} SL(n, \mathbb{R}) = \{ C \in M_n(\mathbb{R}) : \mathrm{tr}(C) = 0 \}$.

$\|$

$\ker(T_{\mathrm{Id}} \det)$ $\rightsquigarrow$ ~~ker~~

$T_{\mathrm{Id}} \det \cancel{A}\, C = \mathrm{tr}(C)$.

Alternatively, you may argue like this:

If C upper triangular matrix: $C = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ & \ddots & \vdots \\ 0 & & c_{nn} \end{pmatrix}$

$$\frac{d}{dt}\Big|_{t=0} \det(\mathrm{Id} + tC) = \frac{d}{dt}\Big|_{t=0} \det \begin{pmatrix} 1+tc_{11} & c_{12} & \cdots & c_{1n} \\ & \ddots & & \vdots \\ 0 & & \cdots & 1+tc_{nn} \end{pmatrix} =$$

$$= \frac{d}{dt}\Big|_{t=0} (1+tc_{11}) \cdots (1+tc_{nn}) = c_{11} + \cdots + c_{nn}$$
$$= \operatorname{tr}(C).$$

Any $C \in M_n(\mathbb{R})$ is triangulizable over $\mathbb{C}$, i.e. $\exists A \in GL(n, \mathbb{C})$ s.t. $A \cdot C \cdot A^{-1}$ is triangular.

$$\frac{d}{dt}\Big|_{t=0} \det(\mathrm{Id} + tC) = \frac{d}{dt}\Big|_{t=0} \overbrace{\det\big(A(\mathrm{Id} + tC)A^{-1}\big)}^{\det(\mathrm{Id} + tACA^{-1})} = \operatorname{tr}(ACA^{-1})$$
$$= \operatorname{tr}(C).$$

(b) Fix $A \in SL(n,\mathbb{R})$:

wanted to write this

$$\mathrm{conj}_A(B) = A^{-1} B A \qquad (ABA^{-1} = \mathrm{conj}_A\,.)$$

$$\mathrm{conj}_A : SL(n,\mathbb{R}) \to SL(n,\mathbb{R})$$

(*) $\mathrm{conj}_A : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$ is a linear map and so also smooth.

$$T_{Id}\,\mathrm{conj}_A : T_{Id} SL(n,\mathbb{R}) \to T_{Id} SL(n,\mathbb{R})$$

$$T_{Id}\,\mathrm{conj}_A(C) = A^{-1} C A$$ by linearity of (*).

(c) $\mathrm{Ad}: SL(n,\mathbb{R}) \longrightarrow GL(T_{\mathrm{Id}} SL(n,\mathbb{R})) \subseteq \mathrm{Hom}(T_{\mathrm{Id}} SL(n,\mathbb{R}), T_{\mathrm{Id}} SL(n,\mathbb{R}))$.

$$\underset{A}{\cup} \longmapsto T_{\mathrm{Id}}\, \mathrm{conj}_A$$

Short detour: $\mu: M_n(\mathbb{R}) \times M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$ multipl. is smooth.

$\Longrightarrow$ $A \mapsto \lambda_A$ $\quad \lambda_A(B) = \mu(A,B)$

$A \mapsto \rho_A$ $\quad \rho_A(B) = \mu(B,A)$

$\hookrightarrow$ are smooth.

Also, inversion is smooth $i: A \mapsto A^{-1}$:

$F: GL(n,\mathbb{R}) \times GL(n,\mathbb{R}) \longrightarrow GL(n,\mathbb{R}) \times GL(n,\mathbb{R})$

$F(A,B) = (A, AB)$ is smooth.

$T_{Id}F(C,D) = (C, C+D)$ is a linear isomorphism.

$\Rightarrow$ locally around (Id, Id), F admits a smooth inverse $\tilde{F}$. By definition, $\tilde{F}(A, Id) = (A, i(A)) = (A, A^{-1})$ and hence i is smooth.

Also, $\forall$ $C \in M_n(\mathbb{R})$, $c(t) = Id + tC$ curve $\in GL(n,\mathbb{R})$

$$c(t) \cdot c(t)^{-1} = Id.$$

Differentiating gives: $0 = \frac{d}{dt}\Big|_{t=0} c(t) \cdot c(t)^{-1} = c'(0) \cdot c(0)^{-1} + c(0) \cdot \underline{\frac{d}{dt}\Big|_{t=0} c(t)^{-1}}$.

$$\Longrightarrow \underline{\frac{d}{dt}\Big|_{t=0} c(t)^{-1}} = -\, c(0)^{-1} c'(0)\, c(0)^{-1} = \underline{-c'(0)}.$$

Hence, $Ad : SL(n,\mathbb{R}) \longrightarrow GL(T_{Id} SL(n,\mathbb{R}))$

$$A \longmapsto T_{Id}\,conj_A \;(C \mapsto A^{-1} C A)$$

$$conj_A = \lambda_{A^{-1}} \circ \rho_A$$

$$T_{Id} Ad : T_{Id} SL(n,\mathbb{R}) \longrightarrow T_{Id}\, GL(T_{Id} SL(n,\mathbb{R}))$$

$$\simeq Hom\,(T_{Id} SL(n,\mathbb{R}), T_{Id} SL(n,\mathbb{R}))$$

$$(T_{Id} Ad)(C)(D) = \frac{d}{dt}\Big|_{t=0} Ad(c(t))(D) = \frac{d}{dt}\Big|_{t=0} \underline{c(t)^{-1}}\, \underline{D\, c(t)}$$

$$= -c'(0) \underbrace{D c(0)}_{= \mathrm{Id}} + \underbrace{c(0)^{-1}}_{= \mathrm{Id}} D c'(0))$$

$$\overset{(*)}{=} -CD + DC = -[C, D]. \quad \text{commutator of matrices.}$$

Remark. ~~But the~~ Taking $\mathrm{conj}_A : B \to A.BA^{-1}$ (not $A^{-1}BA$) leads for (*) then just $[C, D]$.

$(T_{\mathrm{Id}} SL(n, \mathbb{R}), [\ ,\ ])$ is the Lie algebra of $SL(n, \mathbb{R})$.

$$[\ ,\ ] : T_{\mathrm{Id}} SL(n, \mathbb{R}) \times T_{\mathrm{Id}} SL(n, \mathbb{R}) \longrightarrow T_{\mathrm{Id}} SL(n, \mathbb{R})$$

bilinear, skew-symmetric and $[C, [D, E]] + [D, [E, C]] + [E, [C, D]] = 0$

Jacobi identity.

Similarly, $(T_{Id}\, GL(n,\mathbb{R}) = M_n(\mathbb{R}), [\ ,\ ])$ is the Lie alg. of $GL(n,\mathbb{R})$ and similarly for any matrix group, i.e. a Lie subgroup of $GL(n,\mathbb{R})$.

② $\mathbb{R}^n$ equipped with an inner product of signature (p,q):

$$\langle x, y\rangle = \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{n} x_i y_i$$

$$\| $$

$$x^t I_{p,q}\, y \qquad I_{p,q} = \begin{pmatrix} \overbrace{1 \ \ddots \ 1}^{p} & \\ & \underbrace{-1 \ \ddots \ -1}_{q} \end{pmatrix}.$$

$\mathbb{R}^4$ with an inner product of sign. $(1,3)$ =

= Minkowski space.

$$O(p,q) := \{ A \in GL(n,\mathbb{R}) \; : \; \langle Ax, Ay \rangle = \langle x, y \rangle \quad \forall x,y \in \mathbb{R}^n \}.$$

(a) $$\langle Ax, Ay \rangle = (Ax)^t I_{p,q} A y = x^t A^t I_{p,q} A y \quad (*)$$

$$(*) = x^t I_{p,q} y \quad \forall x,y \in \mathbb{R}^n$$

$$\iff A^t I_{p,q} A = I_{p,q}.$$

$$\iff A^t I_{p,q} = I_{p,q} A^{-1}$$

$$\iff I_{p,q} A^t I_{p,q} = A^{-1},$$

$$f : GL(n,\mathbb{R}) \to M_n(\mathbb{R})$$

$$A \mapsto \underline{A I_{p,q} A^t - I_{p,q}}$$

$$f^{-1}(0) = O(p,q).$$

$$f(A)^t = (A I_{p,q} A^t)^t - I_{p,q} = A I_{p,q} A^t - I_{p,q} = f(A).$$

$$f : GL(n,\mathbb{R}) \longrightarrow M_n^{sym}(\mathbb{R}) \subseteq M_n(\mathbb{R}).$$

Regularity of f: $A \in O(p,q)$, $S \in M_n^{sym}(\mathbb{R})$

Then set $C = \frac{1}{2} S I_{p,q} A$.

$$T_A f(C) = A I_{p,q} C^t + C I_{p,q} A^t =$$

$$(+) = \frac{1}{2}\Big(A I_{p,q} \underbrace{(S I_{p,q} A)^t}_{A^t I_{p,q} S^t \,=\, A^t I_{p,q} S} + S I_{p,q} \overbrace{A I_{p,q} A^t}^{I_{p,q}} \Big)$$

$$= \frac{1}{2}(S + S) = S.$$

$\Rightarrow$ $O(p,q)$ is a submfd. of $M_n(\mathbb{R})$ of dim:

$$n^2 - \frac{n(n+1)}{2} = \frac{2n^2 - n^2 - n}{2} = \frac{n(n-1)}{2}.$$

(b) $A, B \in O(p,q)$

$$(AB)^{\cancel{t}} I_{p,q} (AB)^t = \underbrace{A \underbrace{B I_{p,q} B^t}_{I_{p,q}} A^t}_{I_{p,q}} = I_{p,q}.$$

$\Rightarrow$ $A \cdot B \in O(p,q)$.

$$A^{-1} I_{p,q} (A^{-1})^t = I_{p,q} A^t \underbrace{I_{p,q} I_{p,q}}_{Id} (A^t)^{-1} = I_{p,q}.$$

$A^{-1} \in O(p,q)$

$\Longrightarrow$ $O(p,q) \subseteq GL(n,\mathbb{R})$ is a subgroup of $GL(n,\mathbb{R})$.

and matrix multiplication on $O(p,q) \times O(p,q) \to O(p,q)$

is smooth as restriction of a smooth map $M_n(\mathbb{R}) \times M_n(\mathbb{R})$
$\to M_n(\mathbb{R})$.

Hence, $O(p,q)$ is a Lie group.

③ c) $T_{Id} O(p,q) = \ker(T_{Id} f) \underset{(+)}{=} \{ C \in M_n(\mathbb{R}) :$

$I_{p,q} C^t + C I_{p,q} = 0$

$\Leftrightarrow \underline{C^t = -I_{p,q} C I_{p,q}} \}$

$\mathbb{R}^4$ $O(1,3)$ or $O(3,1)$

is called Lorentz group.