Tutorial 5

① $E \xrightarrow{p} M$, $F \xrightarrow{q} M$ vector bundles over M of rank k resp. ℓ.

$$\begin{array}{l} E \oplus F := \bigsqcup_{x \in M} E_x \oplus F_x \\ \quad \downarrow \pi \\ \quad M \end{array} \qquad (x, e_x, f_x) \simeq (e_x, f_x)$$

Fix $x \in M$ $\exists$ open neighbh. U of x and diffeom.

$$\begin{array}{ccc} \varphi^E : p^{-1}(U) & \xrightarrow{\sim} & U \times \mathbb{R}^k \\ p \searrow & & \swarrow pr_1 \\ & U & \end{array} \qquad \begin{array}{ccc} \varphi^F : q^{-1}(U) & \xrightarrow{\sim} & U \times \mathbb{R}^\ell \\ q \searrow & & \swarrow pr_1 \\ & U & \end{array} .$$

and restrictions to fibers are linear isomorphism $E_y \simeq \mathbb{R}^k$, $F_y \simeq \mathbb{R}^\ell$ $\forall y \in U$.

- $\pi^{-1}(U) = \{(e_y, f_y) \in E_y \oplus F_y : y \in U\}$

- $\underset{1}{\varphi}^{E\oplus F}(e_y, f_y) = \Big(y, \underbrace{\big(pr_2 \circ \varphi^{E}(e_y), pr_2 \circ \varphi^{F}(f_y)\big)}_{\in \mathbb{R}^k \times \mathbb{R}^e}\Big)$

$$
\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\varphi^{E\oplus F}"] \arrow[dr, "\pi"'] & U \times \mathbb{R}^k \times \mathbb{R}^e \arrow[d, "pr_1"] \arrow[r, "u \times id"] & u(U) \times \mathbb{R}^k \times \mathbb{R}^e \subseteq \mathbb{R}^n \\
 & U &
\end{tikzcd}
$$

Equipp $E \oplus F$ with the topology generated by $\pi^{-1}(U)$'s

$\Longrightarrow \varphi^{E\oplus F}$ homeomorphism.

$$E \otimes F := \bigsqcup_{x \in M} E_x \otimes F_x \xrightarrow{\pi} M$$

vector bundle

Fix $x \in M$, (U, u) chart around $x \in M$ s.t. $\exists$ trivialization

$$\varphi_\alpha^E : p^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^k \qquad \varphi_\alpha^F : q^{-1}(U) \longrightarrow U \times \mathbb{R}^e \quad \text{over } U$$

of E and F.

$$(y, e_y \otimes f_y) \longmapsto (y, pr_2(\varphi_\alpha^E(y, e_y)) \otimes pr_2(\varphi_\alpha^F(y, f_y)))$$

$$\varphi_\alpha^E \otimes \varphi_\alpha^F : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k \otimes \mathbb{R}^e$$

$$\begin{tikzcd} \pi^{-1}(U) \arrow[dr, "\pi"'] & & U \times \mathbb{R}^k \otimes \mathbb{R}^e \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

$$u \times id \circ \varphi_\alpha^E \otimes \varphi_\alpha^F : \pi^{-1}(U) \longrightarrow u(U) \times \mathbb{R}^k \otimes \mathbb{R}^e .$$

$u(U) \subseteq \mathbb{R}^n$

(2) $E \subseteq TM$ smooth distribution of rank k of a manifold of dim. n.

(a) Locally around any $x \in M$ $\exists$ 1-forms $\omega^1, \dots, \omega^{n-k}$ s.t.
for any local vector field ξ : ξ is a section of E
$\Longleftrightarrow$ $\omega^i(\xi) = 0$ $i = 1, \dots, n-k$.

By smoothness of E, for any $x \in M$ $\exists$ a local frame $\xi_1, \dots, \xi_k$ of E defined on a neighbhd. U of x. Extend to local frame of TM : $\xi_1, \dots, \xi_n$.

Define $\omega^i(\xi^{n-i+1}) = 1$ and $\omega^i(\xi^j) = 0$ $j \neq n-i+1$.

$\rightsquigarrow$ gives a local frame of T^*M

Suppose ξ is a section of TU:

$$\xi = \sum_{l=1}^{n} f^l \xi_l \qquad f^l \in C^\infty(U, \mathbb{R}).$$

$$\omega^j(\xi) = \sum_{l=1}^{n} f^l \underline{\omega^j(\xi_l)} = f^{n-j+1}$$

$$\Longrightarrow \quad \xi \in T(E) \Longleftrightarrow \omega^j(\xi) = 0 \text{ for } j = 1, \dots, k.$$

(5) E is involutive $\Longleftrightarrow$ for 1-forms ω in (a)

$\exists$ 1-forms $\mu^{i,j}$ for $i,j = 1, \dots, n-k$ s.t.

$$d\omega^i = \sum_{j=1}^{n-k} \mu^{i,j} \wedge \omega^j.$$

E is involutive $\Longleftrightarrow$ $[\xi_l, \xi_m]$ is a smooth section of E for all $l, m \in \{1, \dots, k\}$.

By ⓐ, $[\xi_l, \xi_m] \in \Gamma(E) \Longleftrightarrow \omega^i([\xi_l, \xi_m]) = 0$ $\forall i \in \{1, \dots, n-k\}$ $\forall l, m \in \{1, \dots, k\}$.

$$0 = \omega^i([\xi_l, \xi_m]) = \underline{-d\omega^i(\xi_l, \xi_m)} + \xi_l \cdot \overset{=0}{\omega^i(\xi_m)} - \xi_m \cdot \underset{=0}{\omega^i(\xi_l)}$$

$$= \underline{-d\omega^i(\xi_l, \xi_m)}.$$

$\Longleftrightarrow$ $d\omega^i = \sum_{j=1}^{n-k} \mu^{i,j} \wedge \omega^j$ (any $\overset{\text{could}}{\text{2-form}}$ $\sum_{1 \leq i,j \leq n} f^{i,j} \omega^i \wedge \omega^j$

(c) $\Omega_E(M) := \{\omega \in \Omega(M) : \omega|_E = 0\}$

is an ideal in $(\Omega(M), \wedge)$.

$\omega, \mu \in \Omega_E(M)$, $\omega + \mu \in \Omega_E(M)$,

$\lambda\omega \in \Omega_E(M)$ $\lambda \in \mathbb{R}$.

$\alpha \in \Omega^q(M)$, $\omega \in \Omega^p_E(M)$: $\omega \wedge \alpha(\xi^1, \dots, \xi^{p+q})$

$$= \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \operatorname{sign}(\sigma)\, \omega(\xi_{\sigma(1)}, \dots, \xi_{\sigma(p)})\, \alpha(\xi_{\sigma(p+1)}, \dots, \xi_{\sigma(p+q)})$$

$$= 0$$

for $\xi_1, \dots, \xi^{p+q} \in \Gamma(E)$.

(d) $\Omega_E(M)$ is a diff. ideal $\iff$ E is involutive.

$\left(d\,\Omega_E(M) \subseteq \Omega_E(M) \right)$.

$\omega \in \Omega^p_E(M)$ $\qquad \xi_0, \ldots, \xi_p$ sections of E

$$d\omega(\xi_0, \ldots, \xi_p) = \sum_i (-1)^i \, \xi_i \cdot \overset{=0}{\omega(\xi_0, \ldots, \hat{\xi}_i, \ldots, \xi_p)}$$
$$+ \sum_{i<j} (-1)^{i+j} \underbrace{\omega([\xi_i, \xi_j], \xi_0, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_p)}_{\substack{\neq 0 \\ (*)}}.$$

If E is involutive, $(*)$ equals zero, since $[\xi_i, \xi_j]$ is again a section of E and so $d\omega \in \Omega_E(M)$.

Conversely, if $\Omega_E(M)$ is diff. ideal. Extend $\omega^1, \dots, \omega^{n-k}$ to be diff. forms on all of M.

Then $\omega^i \in \Omega_E(M)$ $\forall i = 1, \dots, n-k$ by (a).

By assumption, $d\omega^i \in \Omega_E(M)$.

$$\underline{0} = d\omega^i(\xi, \eta) = \underbrace{\xi \cdot \omega^i(\eta)}_{=0} - \underbrace{\eta \cdot \omega^i(\xi)}_{=0} - \underline{\omega^i([\xi, \eta])}$$

$\xi, \eta \in \Gamma(E)$

By (a), E is also involutive, since $[\xi, \eta]$ is again a section of E.

Frobenius Thm. via diff. forms:

M mfd. and $\omega^1, \dots, \omega^{n-k}$ linearly independent 1-forms.

Then $E := \{ s \in \Gamma(TM) : \omega^i(s) = 0 \ \forall i = 1, \dots, n-k \}$

defines a distribution of rank k.

If $\underline{d\omega^i} = \sum_{j=1}^{n-k} \mu^{ij} \wedge \omega^j$ (equiv. E is involutive),

then for any $x \in M$ $\exists$ an open neighbhd. U of x

and $h^{ij}, f^j \in C^\infty(U, \mathbb{R})$ s.t.

$$\Rightarrow \quad \omega^i = \sum_{j=1}^{n-k} h^{ij} \, df^j \quad ($$

$$\Longrightarrow \quad N = \{ y \in U : f^1(y) = \dots = f^{n-k}(y) = 0 \}$$

is a submanifold, since ω^i's are linearly indep.,

and hence so are the df^j's.

Moreover, $T_yN = \{ \xi \in T_yU = T_yM : df^1(\xi) = \dots = df^{n-k}(\xi) = 0 \}$

$$\subseteq E_y$$

and by dim. reasons $T_yN = E_y \quad \forall y \in N$.

- Transition maps : $\varphi_\alpha^{E\oplus F} \circ (\varphi_\beta^{E\oplus F})^{-1} : U_\alpha \cap U_\beta \times \mathbb{R}^k \times \mathbb{R}^e$

$$\varphi_\alpha^E, \varphi_\beta^E \qquad\qquad \longrightarrow U_\alpha \cap U_\beta \times \mathbb{R}^k \times \mathbb{R}^e$$

$$\varphi_\alpha^F, \varphi_\beta^F$$

$$\varphi_\alpha^{E\oplus F} \circ (\varphi_\beta^{E\oplus F})^{-1}(y, v, w) = (y, \varphi_{\alpha\beta}^E(y)v, \varphi_{\alpha\beta}^F(y)w)$$

$$\varphi_\alpha^E \circ (\varphi_\beta^E)^{-1}(y, v) = \underline{(y, \varphi_{\alpha\beta}^E(y)(v))} \qquad \varphi_{\alpha\beta}^E : U_\alpha \cap U_\beta \to GL(k).$$

and similarly for F.

③ $D_i : \Omega^k(M) \longrightarrow \Omega^{k+r_i}(M)$ $i=1,2$ graded deriv. of degree r_1 resp. r_2.

④ $[D_1, D_2] = D_1 \circ D_2 - (-1)^{r_2+r_1} D_2 \circ D_1$ is a graded deriv. of degree r_1+r_2.

If $\omega \in \Omega^k(M)$, $\underbrace{D_1 \circ \underbrace{D_2(\omega)}_{\Omega^{k+r_2}(M)}}_{\Omega^{k+r_1+r_2}(M)}$ $\Longrightarrow [D_1, D_2] : \Omega^k(M) \to \Omega^{k+r_1+r_2}(M)$.

$[D_1, D_2] : \Omega(M) \to \Omega(M)$ is linear, since composition and addition of linear maps is linear.

$$[D_1, D_2](\omega \wedge \eta) = [D_1, D_2](\omega) \wedge \eta + (-1)^{(r_1+r_2)k}\, \omega \wedge [D_1, D_2](\eta).$$

$\omega \in \Omega^k(M)$

$$[D_1, D_2](\omega \wedge \eta) = D_1 \circ \underbrace{D_2(\omega \wedge \eta)}_{D_2\omega \wedge \eta + (-1)^{r_2 k}\, \omega \wedge D_2 \eta} - (-1)^{r_1 r_2}\, \underline{D_2 \circ D_1(\omega \wedge \eta)}.$$

$$\underbrace{D_1(D_2\omega \wedge \eta)}_{D_1 D_2 \omega \wedge \eta + (-1)^{r_1(r_2+k)} D_2 \omega \wedge D_1 \eta} + \underbrace{(-1)^{r_2 k} D_1(\omega \wedge D_2 \eta)}_{(-1)^{r_2 k} D_1 \omega \wedge D_2 \eta + (-1)^{r_2 k}(-1)^{k r_1}\, \omega \wedge D_1 D_2 \eta}$$

$$= D_1 D_2 \omega \wedge \eta + (-1)^{(r_2+k)r_1} D_2 \omega \wedge D_1 \eta + (-1)^{r_2 k} D_1 \omega \wedge D_2 \eta$$

$$+ (-1)^{k(r_1+r_2)} \omega \wedge D_1 D_2 \eta$$

$$- (-1)^{r_1 r_2} \Big(D_2 D_1 \omega \wedge \eta + (-1)^{(r_1+k)r_2} D_1 \omega \wedge D_2 \eta$$

$$+ (-1)^{r_1 k} D_2 \omega \wedge D_1 \eta + (-1)^{k(r_1+r_2)} \omega \wedge D_2 D_1 \eta \Big).$$

$$= [D_1, D_2](\omega) \wedge \eta + (-1)^{(r_1+r_2)k} \omega \wedge [D_1, D_2](\eta)$$

(b) D graded deriv. Suppose $\omega \in \Omega^k(M)$ s.t. $\omega|_U = 0$, $U \subseteq M$ open subset.

$D(\omega)|_U = 0$?

Fix $x \in U$. Suppose $f \in C^\infty(M, \mathbb{R})$ with $f(x) = 1$ and $\mathrm{Supp}(f) \subseteq U$.

$\Longrightarrow \quad f\omega = 0$

$0 = D(f\omega) = D(f) \wedge \omega + f \wedge D\omega$

$\Longrightarrow \quad 0 = (\underbrace{D(f) \wedge \omega}_{=0})(x) + \underbrace{f(x)}_{=1} D\omega(x) = D\omega(x)$

$\Longrightarrow$ $D\omega = 0$ on all of U.

(c) $D, \tilde{D}$ are graded deriv. $\omega \in \Omega^k(M)$

(U, u) a coord. $\omega|_U = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} du^{i_1} \wedge \dots \wedge du^{i_k}$.

By (b)

$$D(\omega)|_U = D(\omega|_U) = D\Big(\sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} du^{i_1} \wedge \dots \wedge du^{i_k}\Big)$$

$$= \sum_{i_1 < \dots < i_k} \underline{D\omega_{i_1 \dots i_k} \wedge du^{i_1} \wedge \dots} \wedge du^{i_k} + \omega_{i_1 \dots i_k} \underline{D(du^{i_1} \wedge \dots \wedge du^{i_k})}$$

$$D(du^{i_1} \wedge \dots \wedge du^{i_k}) = D(du^{i_1}) \wedge du^{i_2} \wedge \dots \wedge du^{i_k}$$
$$+ (-1)^r du^{i_1} \wedge D(du^{i_2} \wedge \dots \wedge du^{i_k})$$
$$\longrightarrow \dots$$

is determined by $D(du^i)$

$\Longrightarrow$ $D\omega|_U$ is uniquely determined by $D(f)$ and $D(df)$ for $f \in C^\infty(M,\mathbb{R})$.

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a) $i_\xi : \Omega^k(M) \to \Omega^{k-1}(M)$ a graded deriv. of degree -1.

$i_\xi\omega \in \Omega^{k-1}(M)$ ✓

$i_\xi : \Omega(M) \to \Omega(M)$ is linear ✓

$i_\xi(\omega \wedge \eta) = \underline{\underline{i_\xi\omega}} \wedge \eta + (-1)^k \underline{\omega \wedge i_\xi \eta}$ $\qquad \omega \in \Omega^k(M)$.

~~k=0~~

Per induction, $k = 0$ $\quad i_\xi(f\mu) = f\, i_\xi \mu$

$k = 1 \qquad \xi = \xi_0$

$$i_\xi(\omega\wedge\mu)(\xi_1,\dots,\xi_m) = \omega\wedge\mu(\xi_0,\xi_1,\dots,\xi_m)$$

$$= \sum_{i=0}^{m}(-1)^{i+1}\omega(\xi_i)\,\mu(\xi_0,\dots,\hat{\xi}_i,\dots,\xi_m)$$

$$= \underline{\omega(\xi_0)\,\mu(\xi_1,\dots,\xi_m)} - \sum_{i=1}^{m}(-1)^{i}\,\omega(\xi_i)\, i_\xi\mu(\xi_1,\dots,\hat{\xi}_i,\dots,\xi_m)$$

$$= \underline{\underline{i_\xi}}\omega\wedge\mu(\xi_1,\dots,\xi_m) - (\omega\wedge i_\xi\mu)(\xi_1,\dots,\xi_m).$$

Assume now it holds for $k \geq 2$ and let $\omega \in \Omega^{k}(M)$.
$\omega = \omega'\wedge\omega'' \qquad \omega' \in \Omega^1(M)$, $\omega'' \in \Omega^{k-1}(M)$.

→ Induction step is now easy.

5. $\mathcal{L}_q$, l_3 , d .

$\uparrow$ $\uparrow$ $\uparrow$