Lecture 8 - There was no week 7 (svatek)

(Reminder of last time.)

- $\Omega$ a signature
- $X$ a set, $T_\Omega(X)$ is set defined ind. by
  - if $x \in X$ then $x \in T_\Omega(X)$
  - if $s \in \Omega_n$ & $t_1, \ldots, t_n \in T_\Omega(X)$ then $s(t_1, \ldots, t_n) \in T_\Omega(X)$.
- $T_\Omega(X)$ is an $\Omega$-alg w/ substitution of terms:
  if $s \in \Omega_n$, $t_1, \ldots, t_n \in T_\Omega(X) \mapsto s(t_1, \ldots, t_n) \in T_\Omega(X)$
- It is the free $\Omega$-algebra on $X$:

$$\begin{tikzcd} & T_\Omega X \arrow[dr, dashed, "\exists! \bar{v}"] & \\ X \arrow[ur, "\eta_X"] \arrow[rr, "v"'] & & A \end{tikzcd}$$

$x \mapsto x$

$\exists! \bar{v}$ an $\Omega$-alg hom st $\bar{v} \circ \eta_X = v$.

we have $X \xrightarrow{v} A$, $A$ an $\Omega$-alg, $v$ an $X$-tuple of elements of $A$

- $\bar{v}(x) = v(x)$
- $\bar{v}(s(t_1, \ldots, t_n)) = s^A(\bar{v}(t_1), \ldots, \bar{v}(t_n))$.
- What $\bar{v}$ does is substitutes the $X$-tuple $v$ for the variables in $X$.

Eg. $\Omega = \{\cdot, e\}$ for magmas

$X = \{x, y, z\} \xrightarrow{v = (a, b, c)} A$

$T_\Omega(X)$

$\Psi$

$(x.y).z, \quad x.(y.z) \xmapsto{\bar{v}} (a.b).c, \quad a.(b.c)$

## Equations

**Def$^n$)** Let $\Omega$ be signature, $X$ a set. By an **$\Omega$-equation** in variables $X$, we mean a pair $(s,t) \in T_\Omega(X)^2$.

**Remark)** Often informally write an $\Omega$-equation $(s,t)$ as "$s = t$".

**Example**

$\Omega = (e, \cdot)$ sig. For magmas, the following are equations:

- $x.e = x$
- $e.x = x$
- $(x.y).z = x.(y.z)$

$\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\}$ in variables $\{x, y, z\}$

**Def$^n$)** Let $(s,t) \in T_\Omega(X)^2$ be an equation.

- An **X-tuple** $v : X \longrightarrow A$ of elements of an $\Omega$-algebra **A** **satisfies** **the equation $s = t$** if $\bar{v}(s) = \bar{v}(t)$ where $\bar{v} : T_\Omega(X) \longrightarrow A$ is the map described above.
- The **$\Omega$-algebra A** sat. the equation $s = t$ if **each** X-tuple $v$ of $A$ sat. the equation (We sometimes write $A \models s = t$.)

## Example

$\Omega = (\cdot, e)$

$(x.y).z \,,\; x.(y.z) \in T_\Omega(\{x,y,z\})$.

- Then $\{x,y,z\} \xrightarrow{(a,b,c)} A$ sat the eq$^n$ $\iff$ $(a.b).c = a.(b.c) \in A$
- $A \models (x.y).z = x.(y.z)$ iff $A$ is associative.

**Def$^n$)** Let $\Omega$ be a sig., & $E$ a set of equations. By an $(\Omega, E)$-algebra we mean an $\Omega$-algebra $A$ such that $A \models s = t$ for each equation $(s,t) \in E$.

## Examples

- For $\Omega = (\cdot, e)$ &

$$E = \left\{ \begin{array}{c} (x.y).z = x.(y.z) \\ x.e = x \\ e.x = e \end{array} \right\}$$

an $(\Omega, E)$-alg is a monoid,
- If take

$$\Omega' = (\cdot, e, (-)^{-1}) \text{ unary op.}$$

& $E' = E \cup \{ x.x^{-1} = e \,,\; x^{-1}.x = e \}$

then $(\Omega, E)$-algebra is a group.

**Def$^n$)** For $(\Omega, E)$ as above, we define
$$(\Omega,E)\text{-Alg} \hookrightarrow \Omega\text{-Alg}$$
as the Full subcategory of $(\Omega,E)$-algebras.

This means:

- obs: are $(\Omega,E)$-algebras
- morphisms: $\Omega$-alg. homomorphisms.

- The inclusion $(\Omega,E)\text{-Alg} \xhookrightarrow{i} \Omega\text{-Alg}$ is then a fully faithful functor.
- We obtain a composite forgetful functor to Set, as depicted below

$$\begin{tikzcd}
(\Omega,E)\text{-Alg} \arrow[rr, hook, "i"] \arrow[dr, "U"'] & {} \arrow[d, phantom, "="] & \Omega\text{-Alg} \arrow[dl, "U_\Omega"] \\
& \text{Set} &
\end{tikzcd}$$

- later, we will see that $i$ & $U$ have left adjoints.

## Examples

- When I spoke of "algebraic categories" earlier in course, the precise meaning is cat. of the form
$$(\Omega, E)\text{-Alg}.$$
This framework captures all

of the examples we have been talking about –
Vect, Grp, Ring, Mon, G-Set...

Goal now: study good properties of categories of the form $\Omega$-Alg & $(\Omega, E)$-Alg.

- Firstly (today) we look at $\Omega$-Alg – the case of $(\Omega, E)$-Alg follows easily from $\Omega$-Alg.
- In particular, will study limits, kernels and quotients.

Firstly, subalgebras, homomorphic images & image factorisation.

Def) - Let $A \in \Omega$-Alg. A subalgebra $B \hookrightarrow A$ of $A$ is a subset $B$ of $A$ such that – if $s \in \Omega_n$ & $b_1, \ldots, b_n \in B$ then $s^A(b_1, \ldots, b_n) \in B$.

- In particular, $B$ is then an $\Omega$-alg & the incl. $B \hookrightarrow A$ a injective homomorphism.

Def$^n$) Let $A \in \Omega$-Alg. A homomorphic image of $A$ is a surjective homomorphism $f : A \twoheadrightarrow B$.

means surjective $\twoheadrightarrow$

- Let $f : A \longrightarrow B \in \Omega$-Alg.

- Then let $\operatorname{im} f = \{ b \in B : \exists a \in A \text{ with } fa = b \}$
- Then $\operatorname{im} f \hookrightarrow B$ is a subalgebra of $B$: indeed, if $s \in \Omega_n$, $b_1 = fa_1, \ldots, b_n = fa_n$, then $s(b_1, \ldots, b_n) = s(fa_1, \ldots, fa_n)$
$$= fs(a_1, \ldots, a_n) \in \operatorname{im} f.$$
- In particular, we obtain a factorisation

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "e"'] \arrow[ddr, "q"'] & & B \text{ in } \Omega\text{-Alg} \\ & \operatorname{im} f \arrow[ur, hook, "i"'] & \\ & fa & \end{tikzcd}$$

homomomorphic image

subalgebra inclusion

- Later, we will look at the first isomorphism theorem which explains how to view $\operatorname{im} f$ as a quotient.

## Limits of $\Omega$-algebras

**Proposition**

$\Omega$-Alg has (infinite) products and equalisers & $U : \Omega\text{-Alg} \longrightarrow \text{Set}$ preserves them.

(Remark: these generate all limits - not proved in course.)

**Proof**

- Consider a set $I$ and family $(A_i)_{i \in I}$

of $\Omega$-algebras. (i.e. $A: I \longrightarrow \Omega\text{-Alg}$)

- Their product as sets is the direct product

$$\prod_{i\in I} A_i = \{ \bar{a} = (a_i)_{i\in I} : a_i \in A_i \} \xrightarrow{p_i} A_i$$
$$(a_i)_{i\in I} \longmapsto a_i$$

- We want to show that $\prod_{i\in I} A_i$ has the structure of $\Omega$-algebra such that each $p_i$ is a homomorphism:

This says given $s \in \Omega_n$ and $\bar{a}^1, \dots, \bar{a}^n$ we have $s(\bar{a}^1, \dots, \bar{a}^n)_i = s^{A_i}(\bar{a}^1_i, \dots, \bar{a}^n_i) \in A_i$.

In other words, we are forced to equip $\prod A_i$ with component-wise $\Omega$-algebra structure.

- Given $\Omega$-alg. $B$ & homs $(f_i : B \longrightarrow A_i)_{i\in I}$ we have a unique function

$$f : B \longrightarrow \prod_{i\in I} A_i \text{ such that } p_i \circ f = f_i;$$

namely $(fb)_i = f_i(b)$.

- Must check $f$ is a homomorphism:

$$fs(b^1, \dots, b^n) = s(fb^1, \dots, fb^n) \in \prod_{i\in I} A_i$$

Check components at $i \in I$

$$f_i s(b^1, \dots, b^n) = \underbrace{s((fb^1)_i, \dots, (fb^n)_i)}_{\text{pointwise } \Omega\text{-alg. structure}}$$

$$\underset{f \text{ a hom.}}{\parallel} \qquad s(f_i b^1, \dots, f_i b^n)$$

- Given $A \underset{g}{\overset{f}{\rightrightarrows}} B$ their equaliser is $E = \{ x \in A : fx = gx \} \overset{i}{\hookrightarrow} A$ in Set.

In fact, $E$ is a subalgebra of $A$:
if $s \in \Omega_n$ & $x_1, \dots, x_n$ st $fx_i = gx_i$ then
$f s(x_1, \dots, x_n) \underset{f \text{ a hom}}{=} s(fx_1, \dots, fx_n) \underset{\text{assump.}}{=} s(gx_1, \dots, gx_n) \underset{g \text{ hom}}{=} g s(x_1, \dots, x_n)$.

In partic., $i : E \hookrightarrow A$ is a homomorphism and easy to check univ. prop. of the equaliser. $\square$

What about colimits?
- Key sort - quotients by congruences.

Congruences generalise
- e.rels for sets
- normal subgroups of groups
- 2 sided ideals for comm rings

Def) Let $A$ be an $\Omega$-algebra. An equivalence relation $E \subseteq A^2$ is called a congruence if $E$ is a subalgebra of $A$.

- In elementary terms, a cong. is an e-rel ($(x,x) \in E$, $(x,y) \in E \Rightarrow (y,x) \in E$, $(x,y) \in E, (y,z) \in E \Rightarrow (x,z) \in E$) such that
$s \in \Omega_n$, $(x_1, y_1) \in E, \dots, (x_n, y_n) \in E$,
$(s(x_1, \dots, x_n), s(y_1, \dots, y_n)) \in E$.
- I will write $xEy$ to mean $(x,y) \in E$.

- If $E \xhookrightarrow{i} A^2$ is a congruence, can form diagram $E \xhookrightarrow{i} A^2 \overset{p_1}{\underset{p_2}{\rightrightarrows}} A$ in $\Omega$-Alg

& so obtain

$$E \overset{d \text{ domain}}{\underset{c \text{ codomain}}{\rightrightarrows}} A : (x,y) \in E \overset{d}{\underset{c}{\rightrightarrows}} \begin{matrix} x \\ y \end{matrix}.$$

a pair of $\Omega$-algebra homomorphisms.

- We are interested in their coequaliser $A \xrightarrow{p} A/E$ in $\Omega$-Alg defined as follows:

- elements of $A/E$ are equiv. classes $[a]$ with $p(a) = [a] = \{x : xEa\}$.

- Observe $p$ is surjective. Therefore if $p$ is to be a homomorphism we are forced to define

$$s^{A/E}([a_1], \ldots, [a_n]) = [s^A(a_1, \ldots, a_n)].$$

- Is this well defined?

Suppose $[b_1] = [a_1], \ldots, [b_n] = [a_n]$

Then $b_1 E a_1, \ldots, b_n E a_n$ so as $E$ a congruence we have

$$s(b_1, \ldots, b_n) \; E \; s(a_1, \ldots, a_n)$$

so $[s(b_1, \ldots, b_n)] = [s(a_1, \ldots, a_n)]$ as required.

- In particular, $A/E$ is a $\Omega$-algebra & $p : A \longrightarrow A/E$ a surjective homomorphism.

<u>Proposition</u>

$$E \underset{c}{\overset{d}{\rightrightarrows}} A \xrightarrow{p} A/E$$ is a coequaliser in $\Omega$-Alg.

<u>Proof</u>

- Firstly if $(x,y) \in E$ then
  $$pd(x,y) = [x] = [y] = pc(x,y)$$
  so $pd = pc$.
- Given $A \xrightarrow{f} B$ with $fd = fc$.
  This means precisely that if $(x,y) \in E$ then $fx = fy$.
  Therefore $[x] = [y] \implies fx = fy$.
- Therefore we can extend $f$ along $p$ by

$$\begin{tikzcd} A \arrow[r, two heads, "p"] \arrow[rr, bend right, "f"'] & A/E \arrow[r, "\bar{f}"] & B \end{tikzcd}$$

  where $\bar{f}[a] = fa$
- Clearly $\bar{f}$ is a homomorphism, since $f$ is.
- Since $p$ is surjective, $\bar{f}$ is only map extending $f$ along $p$.

Def) Let $f : A \to B \in \Omega$-Alg.
The kernel of $f$ is the <u>congruence</u>
$$K_f = \{(x,y) : fx = fy\} \hookrightarrow A^2.$$

- Easy to see this is a congruence: check it!

Categorically, $K_f$ is the pullback

$$
\begin{tikzcd}
(x,y) \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & x \\
& K_f \arrow[r, "d"] \arrow[d, "c"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] & \\
& A \arrow[r, "f"'] & B & \\
y & & &
\end{tikzcd}
$$

so, in particular, we have $fd = fc$.

Therefore, we get a unique factorisation of $f$ through the coequaliser

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "p"'] & & B \\
& A/K_f \arrow[ur, "\bar{f}"'] &
\end{tikzcd}
\qquad
a \mapsto [a] \mapsto fa
$$

## First isomorphism theorem

- Given $f: A \to B \in \Omega\text{-Alg}$, we have an isomorphism
$$t : A/K_f \longrightarrow \operatorname{im} f$$
commuting with the factorisations of $f$: that is,

the diag:

$$\begin{tikzcd} & A/K_f \arrow[dd, "s" ', "t"] \arrow[dr, "\bar{f}"] & \\ A \arrow[ur, "p"] \arrow[dr, "e" '] & & B \\ & \operatorname{im} f \arrow[ur, "i" '] & \end{tikzcd}$$

$\boxed{\text{Proof}}$ - Define $t[a] = fa$.

Clearly makes diagram commute. It is a homom. as $\bar{f}$ is.

- For surj., if $b \in \operatorname{im} f$, then $b = fa$ so $t[a] = fa = b$.
- For inj, suppose $t[a] = t[b]$. That is, $fa = fb$. Then $(a,b) \in K_f$ so $[a] = [b]$.

$\square$

<u>Corollary</u>

If $f : A \twoheadrightarrow B$ is <u>surjective</u>, then <u>$A/K_f \cong B$</u>. In particular,

$$K_f \underset{c}{\overset{d}{\rightrightarrows}} A \xrightarrow{f} B$$ is a <u>coequaliser</u>

<u>Proof</u>

In this case $\operatorname{im} f = B$

so $A/K_f \xrightarrow[\sim]{t} B$

&

$$\begin{tikzcd} & A/K_f \arrow[d, "t"] \\ A \arrow[ur, "p"] \arrow[r, "f"'] & B \end{tikzcd}$$

so

& then use that coequalisers are invariant up to iso, so as $p$ coequaliser map, $f = t \circ p$ is also coequaliser.
$\phantom{f = }$ iso