Bousfield localization

$u: X \to Y$ an f-local equiv iff $L_f u: L_f X \to L_f Y$ weak equivalence

Theorem (Whitehead)

A map $u: X \to Y$ between f-local spaces is an f-local equiv iff it is a weak equivalence.

Proof.

$$\begin{tikzcd} \tilde{X} \arrow[r, "r", "\sim"'] \arrow[d, "\tilde{u}"'] & X \arrow[d, "u"] \\ \tilde{Y} \arrow[r, "\sim", "s"'] \arrow[ur, "v"] & Y \end{tikzcd}$$

$$\tilde{u}^*: \mathrm{Map}(\tilde{Y}, X) \xrightarrow{\sim} \mathrm{Map}(\tilde{X}, X) \ni r$$

$$\tilde{u}^*: \mathrm{Map}(\tilde{Y}, Y) \xrightarrow{\sim} \mathrm{Map}(\tilde{X}, Y)$$

$s, uv \in \mathrm{Map}(\tilde{Y}, Y)$, $\quad s\tilde{u},\ uv\tilde{u} \simeq ur$ $\quad \square$

cellular model categories: assume that M is cofibrantly generated with a set I of generating cofibrations that are effective mono's (= equalizers). Then any relative I-cell complex

$$X = X_0 \to X_1 \to \cdots \to X_\lambda = Y \quad \text{with cells} \quad \begin{tikzcd} \sum_{t \in T_\alpha} A_t \arrow[r] \arrow[d] & X_\alpha \arrow[d] \\ \sum_{t \in T_\alpha} B_t \arrow[r] & X_{\alpha+1} \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

admits a reasonable notion of a subcomplex – namely a collection of subsets $\bar{T}_\alpha \subseteq T_\alpha$ that give via subpushouts

$$\begin{tikzcd} X = \bar{X}_0 \arrow[r] \arrow[d, "="] & \bar{X}_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & \bar{X}_\lambda = \bar{Y} \arrow[d] \\ X = X_0 \arrow[r] & X_1 \arrow[r] & \cdots \arrow[r] & X_\lambda = Y \end{tikzcd}$$

these then exist uniquelly and are effective monos

IF they exist = condition on $\bar{T}_\alpha$'s

We call $|\sum_\alpha T_\alpha|$ the **size** of the complex Y (relative to X) = number of cells

– **compactness** of A: $\operatorname{colim} M(A, X_i) \xrightarrow{\cong} M(A, \operatorname{colim} X_i)$
(κ-compactness)

for X_i the collection of all subcomplexes of a complex $X = \operatorname{colim} X_i$ of size $< \kappa$

Hirschhorn $\leq \kappa$

Since we assume I to consist of effective monos, this is always injective, i.e. compactness means

$$\begin{tikzcd} & X_i \arrow[r] \arrow[d] & X_j \arrow[dl] \\ A \arrow[r] \arrow[ur, dashed, "\exists"] & X & \end{tikzcd} \Longrightarrow \quad A \text{ is small}$$

since the cells of X_i lie in some step of the transfinite composition

Definition M is **cellular** if it is lie in some step of the transfinite composition

Definition. $\mathcal{M}$ is cellular if it is

- cofibrantly generated by sets I, J
- domains and codomains of I are compact $\Rightarrow$ all cofibrant objs compact
- domains and codomains of J are ~~small w.r.t. cofibrations~~ compact
- cofibrations are effective monos

Example. If $\mathcal{M}$ is locally presentable, 2 and 3 are automatic.
($\mathcal{M}$ combinatorial)

Important ingredient:

- Let X be a set of relative cell complexes with compact domains and $\underset{\text{cell complex}}{f: X \longrightarrow Y}$ a map. By applying the SOA factor it as $f: X \xrightarrow{i} \hat{X} \xrightarrow{p} Y$. Then size $\hat{X}$ = size X provided that it is big enough.
  - bounded set of squares
    $$\begin{tikzcd} A_i \arrow[r] \arrow[d] & X_\alpha \arrow[d] \\ B_i \arrow[r] & Y \end{tikzcd}$$
    $$\underbrace{A_i \dashrightarrow (X_\alpha)_\beta}_{\text{set of such up to iso}} \longrightarrow X_\alpha$$
  - the size of X_α does not increase

- Application:
  - X cell complex $\Rightarrow$ $\mathrm{Cyl}\,X$ cell complex of equal size
    (SOA w.r.t. I applied to $X+X \longrightarrow X$)
  - $f: A \longrightarrow B$ inclusion of cell complexes
    SOA w.r.t. $\overline{\Lambda f} = J \cup \{L^n B +_{L^n A} A^n \longrightarrow B^n\}$ applied to $X \longrightarrow *$ gives the f-localization $X \longrightarrow L_f X$ of equal size

Properties of the f-localization:

- it respects intersection of subcomplexes: $L_f(\cap X_i) = \cap L_f X_i$
  (follows from the way that SOA works)
- for each subcomplex $Z \subseteq L_f X$ there is a minimal subcomplex $W \subseteq X$ s.t. $Z \subseteq L_f W$ and size W = size Z (above some μ ...)
- it respects directed unions of subcomplexes $\mathrm{colim}\, L_f X_i = L_f\, \mathrm{colim}\, X_i$

• it respects directed unions of subcomplexes ...

Theorem. (Bousfield Cardinality Argument)

Every inclusion of cell complexes that is an f-equivalence can be expressed as a relative cell complex w.r.t. small inclusions of cell complexes that are f-equivalences.

Proof.

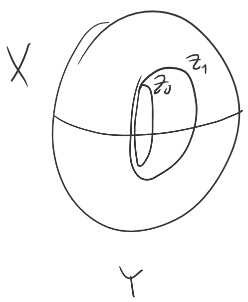
$$\begin{tikzcd} X \arrow[dr, hook, "\sim_f"] \arrow[ddd, "\sim_f"'] & \\ & X_1 \arrow[d, "\sim_f"] \\ & X_2 \arrow[dl, dotted] \\ Y & \end{tikzcd}$$

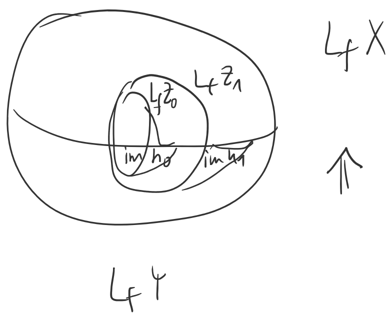
– by maximality principle, enough to extend

Whitehead theorem

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\sim_f"'] & L_f X \arrow[d, "\sim"] \\ Y \arrow[r] & L_f Y \end{tikzcd} \Longrightarrow$$

there exists a deformation $h: \mathrm{Cyl}\, L_f Y \to L_f Y$ constant on $\mathrm{Cyl}\, L_f X$





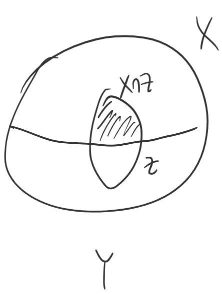
$h_0 = h|_{\mathrm{Cyl}\, L_f Z_0}$

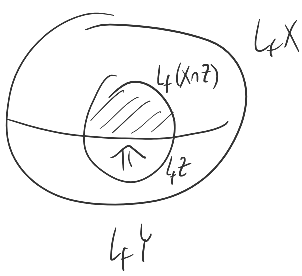
$h_1 = h|_{\mathrm{Cyl}\, L_f Z_1}$

$\vdots$

$$\begin{array}{c} Z_0 \\ \cap \\ Z_1 \\ \cap \\ \vdots \\ \cap \\ Z \end{array} \qquad \begin{tikzcd} \mathrm{Cyl} L_f Z_0 \arrow[dr] & L_f Z_0 \\ \cap & \cap \\ \mathrm{Cyl} L_f Z_1 \arrow[dr] & L_f Z_1 \\ \cap & \cap \\ \vdots & \vdots \end{tikzcd}$$

$\mathrm{Cyl}\, L_f Z \to L_f Z$ deformation onto $L_f(X \cap Z)$





$$\text{small}\left\{\begin{tikzcd} X\cap Z \arrow[r] \arrow[d, "\sim_f"'] & X \arrow[d, "\sim_f"] \\ Z \arrow[r] & X\cup Z \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}\right\} \text{ extension } \qquad \begin{tikzcd} L_f(X\cap Z) \arrow[d, "\sim"'] \\ L_f Z \end{tikzcd}$$

$\square$

Theorem. (Left Bousfield localization)

$\mathcal{M}$ left proper cellular, f inclusion of cell complexes

$\Rightarrow$ f-local model structure exists and is left proper cellular

Proof. • recognizing f-local fibrations:

$$\mathcal{F}_f = (\mathcal{W}_f \cap \mathcal{I})^{\square} \overset{\text{left proper}}{=} \mathcal{J}^{\square} \cap (\mathcal{W}_f \cap \mathcal{I})_{cell}^{\square} \overset{BCA}{=} \mathcal{J}^{\square} \cap (\mathcal{J}_f^{o})^{\square} = \mathcal{J}_f^{\square}$$

$$\Rightarrow \mathcal{W}_f \cap \mathcal{I} \overset{?\,\square}{=} {}^{\square}((\mathcal{W}_f \cap \mathcal{I})^{\square}) = {}^{\square}(\mathcal{J}_f^{\square})$$

• recognizing f-local trivial fibrations $\overset{?}{=}$ trivial fibrations $\mathcal{I}^{\square}$

"$\Leftarrow$": $\mathcal{I}^{\square} = \mathcal{W} \cap \mathcal{I}^{\square} \subseteq \mathcal{W}_f \cap \mathcal{J}_f^{\square}$

"$\Rightarrow$": uses retract argument (part of a more technical recognition theorem for cofibrantly generated model cat's)

$$\begin{tikzcd} & \cdot \arrow[dr, "p", "\sim"'] & \\ \cdot \arrow[ur, "i"] \arrow[rr, "u"'] & & \cdot \end{tikzcd}$$

$u \in \mathcal{W}_f \cap \mathcal{J}_f^{\square}$

$u \in \mathcal{W}_f$, $p \in \mathcal{W} \subseteq \mathcal{W}_f$ $\Rightarrow$ $i \in \mathcal{W}_f \cap \mathcal{I} = {}^{\square}(\mathcal{J}_f^{\square})$

$\Rightarrow$ u retract of $p \in \mathcal{W} \cap \mathcal{I}^{\square}$

• $\mathcal{J}_f$-cell $\subseteq \mathcal{W}_f$

• left proper easy

• cellular clear

$\square$

Fibrant objects lie in $\mathcal{J}_f^{\triangledown} \subseteq \Lambda f^{\triangledown}$ $\Rightarrow$ they are f-local.

In the opposite direction:

$$\begin{tikzcd} K \arrow[r] \arrow[d] & X & \text{f-local} \arrow[l] \\ L \arrow[ur, dashed, "\exists \text{ up to htpy that can be removed}"'] & & \end{tikzcd} \qquad \mathcal{J}_f^{o} \ni (K \to L)$$

$$\begin{tikzcd} \mathrm{Map}(K,X) \\ \mathrm{Map}(L,X) \arrow[u, "\sim"'] \end{tikzcd}$$

Theorem. Fibrant objects are exactly the f-local objects.

We will also describe fibrations between fibrant objects.

Proposition.
$$\begin{tikzcd} X \arrow[rr, "u", "\sim"'] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$
p is an f-local fibration
$\iff$ q is an f-local fibration

Proof. We start with the easier "$\Leftarrow$".

$$\mathcal{J}_f \ni \begin{tikzcd} K \arrow[r] \arrow[d] & X \arrow[r, "u", "\sim"'] \arrow[d] & Y \arrow[dl] \\ L \arrow[r] \arrow[ur, dashed] & Z & \end{tikzcd}$$

in $\mathcal{M}$ has a simple interpretation
in $K/\mathcal{M}/Z$:
$$\begin{tikzcd} & X \arrow[d, "\sim"', "u"] \\ L \arrow[ur, dashed] & Y \end{tikzcd}$$
$$\begin{tikzcd} \mathrm{Map}(L,X) \arrow[d, "\sim"] & \text{non-empty} \\ \mathrm{Map}(L,Y) & \text{non-empty} \arrow[u, Rightarrow] \end{tikzcd}$$

The direction "$\Rightarrow$" reduces to u being a trivial fibration (Brown)

K cell complex $\Rightarrow$
$$\begin{tikzcd} & X \arrow[d, "\sim"', "u"] \\ K \arrow[ur, dashed, "\exists"] \arrow[r] \arrow[d, tail] & Y \arrow[d] \\ L \arrow[r] & Z \end{tikzcd}$$
$\rightsquigarrow$ again can be turned to $K/\mathcal{M}/Z$.

$\square$

Theorem. A map $p: X \to Y$ between f-local objects is an f-local fibration $\iff$ it is a fibration.

Proof.
$$\begin{tikzcd} X \arrow[rr, tail, "\sim_f", "u"'] \arrow[dr, "p"'] & & Z \arrow[dl, two heads, "q"] \\ & Y & \end{tikzcd}$$
$\Rightarrow$ Z also f-local
$\overset{\text{Whitehead}}{\Rightarrow}$ u a weak equivalence
$\Rightarrow$ p also f-local fibration. $\square$

Reminder

20. ledna 2021 15:41

$f: A \to B$ inclusion of I-cell complexes

$\Lambda f = \{ L^n B +_{L^n A} A^n \to B^n \}$

$\overline{\Lambda f} = J \cup \Lambda f$

$\overline{\Lambda f}^{\square} =$ f-local objects

f-local equivalences: $u: X \to Y$ s.t. $\tilde{u}: \tilde{X} \rightarrowtail \tilde{Y}$ induces w.e. $\tilde{u}^*: Map(\tilde{Y}, W) \to Map(\tilde{X}, W)$ $\forall W$ f-local

Important: $\overline{\Lambda f} \subseteq I \cap W_f$ and the latter is closed under coproducts, pushouts, transfinite compositions, retracts.

$\Rightarrow$ SOA gives f-localization $X \to L_f X \to *$
relative $\overline{\Lambda f}$-cell complex