Examples

$\mathcal{M} = Ch$, say non-negatively graded chain complexes of (right) R-modules

$\cdots \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$

$\mathcal{W}$ = quasi-isomorphisms

$\mathcal{F}$ = maps that are surjective in positive dimensions $Ch_f = Ch$

Define for $n > 0$: $D^n = (\cdots \to 0 \to \overset{n}{R} \xrightarrow{1} \overset{n-1}{R} \to 0 \to \cdots)$

$Ch(D^n, C) \cong C_n$ so that

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & C \arrow[d, "f"] \\ D^n \arrow[r] \arrow[ur, dashed] & D \end{tikzcd} \iff \begin{tikzcd} C_n \arrow[d, "surj."] \\ D_n \end{tikzcd} \iff f \in \mathcal{F}$$

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \overset{n}{R} \arrow[r, "1"] \arrow[d] & \overset{n-1}{R} \arrow[r] \arrow[d] & 0 \arrow[d] \\ C_{n+1} \arrow[r] & C_n \arrow[r] & C_{n-1} \arrow[r] & C_{n-2} \end{tikzcd}$$

$\mathcal{W} \cap \mathcal{F}$ = surjective quasi-iso's; $f: C \to D$ induces

= surjective + kernel acyclic

zero homology

$$\begin{tikzcd} C_1 \arrow[r] \arrow[d] & C_0 \arrow[r] \arrow[d, "surj"] & H_0 C \arrow[r] \arrow[d, "\cong"] & 0 \\ D_1 \arrow[r] & D_0 \arrow[r] & H_0 D \arrow[r] & 0 \end{tikzcd}$$

Define for $n \geq 0$: $S^{n-1} = (\cdots \to 0 \to \overset{n}{0} \to \overset{n-1}{R} \to 0 \to \cdots) \subseteq D^n$

$Ch(S^{n-1}, K) \cong Z_{n-1} K$ $(n-1)$-cycles of K

$$\begin{tikzcd} S^{n-1} \arrow[r, "z"] \arrow[d, hook] & K \\ D^n \arrow[ur, dashed, "c"'] & \end{tikzcd} \iff H_{n-1} K = 0$$

$\partial z = 0$

$z = \partial c$

$$\begin{tikzcd} Ch(D^n, K) \cong K_n \arrow[d, "i^*"] \arrow[d, "\partial", shift left=8ex] \\ Ch(S^{n-1}, K) \cong Z_{n-1} K \end{tikzcd}$$

$$\begin{tikzcd} S^{n-2} \arrow[r] \arrow[d, "0"] & D^{n-1} \arrow[d, "pr"] & S^{n-1} \arrow[r] \arrow[d, "1"] & D^n \arrow[d, "1"] \\ 0 \arrow[r] & S^{n-1} & S^{n-1} \arrow[r] & D^n \end{tikzcd}$$

Remark. $n = 0$

$S^{-1} \to D^0$ $\qquad Z_{-1} C = 0$

$= \quad 0 \to S^0$ $\qquad C_0 = Z_0 C$

$\Rightarrow \quad \mathcal{I}^\square \subseteq \mathcal{J}^\square$ = surj's in pos. dim's

$\rightsquigarrow \quad \mathcal{I}^\square \subseteq \mathcal{W} \cap \mathcal{J}^\square$ since

$$\begin{tikzcd} S^{n-1} \arrow[r, "z"] \arrow[d, hook] & C \arrow[d, "f"] \\ D^n \arrow[r, "0"] \arrow[ur, dashed] & D \end{tikzcd} \equiv \begin{tikzcd} S^{n-1} \arrow[r, "z"] \arrow[d, hook] & \ker f \\ D^n \arrow[ur, dashed] & \end{tikzcd} \iff H_{n-1} \ker f = 0$$

For the opposite direction: let $f \in \mathcal{W} \cap \mathcal{J}^\square$, i.e. f is a surj. quasi-iso

$$\begin{tikzcd} S^{n-1} \arrow[r, "z"] \arrow[d, hook] & C \arrow[d, "f"] \\ D^n \arrow[r, "d"] \arrow[ur, dashed] & D \end{tikzcd} - \begin{tikzcd} S^{n-1} \arrow[r, "\partial c"] \arrow[d, hook] & C \arrow[d, "f"] \\ D^n \arrow[r, "d"] \arrow[ur, "c"] & D \end{tikzcd} = \begin{tikzcd} S^{n-1} \arrow[r, "z - \partial c"] \arrow[d, hook] & C \arrow[d, "f"] \\ D^n \arrow[r, "0"] \arrow[ur, dashed] & D \end{tikzcd}$$

has a solution since $\ker f$ is acyclic

by surjectivity, c exists

Summary. $\mathcal{I} = \{ S^{n-1} \hookrightarrow D^n \mid n \geq 0 \}$, $\mathcal{J} = \{ 0 \to D^n \mid n > 0 \}$

gives the model category structure on $\mathcal{M} = Ch$ with

$\mathcal{W}$ = q-iso's, $\mathcal{F}$ = surj's in pos dim's

$\mathcal{C} = {}^\square(\mathcal{I}^\square)$ = retracts of relative $\mathcal{I}$-cell complexes

- attaching a cell:

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d, hook] & C \arrow[d] \\ D^n \arrow[r] & D \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \vdots \arrow[d, "\partial"] \\ C_{n+1} \arrow[d, "\partial"] \\ C_n \oplus R \end{tikzcd} \qquad \begin{tikzcd} & \vdots \arrow[d, "\partial"] & \\ & C_{n+1} \arrow[d, "\partial"] & \\ & C_n \oplus R \arrow[d, "\partial"] \arrow[dl, "f"] & \end{tikzcd}$$

arbitrary

- attaching more cells

- attaching more cells
  = more copies of R
  possibly in varying dimensions

$$\begin{tikzcd} C_n \oplus R \arrow[d, "\partial"'] \arrow[d, shift left=6, "1"] \\ C_{n-1} \oplus R \arrow[d, "\partial"'] \arrow[u, phantom, "f\text{-glue}"] \\ \vdots \end{tikzcd} \qquad \begin{tikzcd} C_n \arrow[d, "\partial"'] & R \arrow[dl, "f"] \\ C_{n-1} \arrow[d, "\partial"'] \\ \vdots \end{tikzcd}$$

arbitrary with $\partial \circ f = 0$

- transfinite composition
  = injective map such that
  the cokernel consists of free modules

  $\Rightarrow$ easy, $\Leftarrow$ uses that our chain complexes are non-negatively graded - build like a CW-cx

- retracts
  = injective map such that the cokernel consists of proj. modules

$^\square(\mathcal{J}^\square)$ simpler: relative $\mathcal{J}$-cell complexes are inclusions $C \longrightarrow C \oplus \bigoplus_\alpha D^{n_\alpha} \longrightarrow \bigoplus_\alpha D^{n_\alpha}$ with cokernel composed of free modules and contractible $\Rightarrow$ retracts will be inclusions with cokernel composed of proj. modules and still contractible (= projective in Ch)

$$\begin{tikzcd} \bigoplus 0 \arrow[r] \arrow[d] & C \arrow[d] \\ \bigoplus_\alpha D^{n_\alpha} \arrow[r] & C \oplus \bigoplus D^{n_\alpha} \end{tikzcd}$$

Remains: $\mathcal{J}$-cell $\subseteq \mathcal{W}$, but this is clear from the description.

Variations.

- Unbounded chain complexes, $\mathcal{F}$ = surj's, $\mathcal{I}$ & $\mathcal{J}$ similar, but cofibrations not so nice
- Bounded below chain complexes, $\mathcal{F}$ = surj's, $\mathcal{I}$ & $\mathcal{J}$ similar, nice description? of cofibrations, but only finitely bicomplete.

Homotopy. The path object of D is always

$$\mathrm{Tot}\left(D^0 \oplus D^0 \xrightarrow[{[-1,1]}]{} D^{-1}\right) = \mathrm{Hom}\big(\underbrace{R^1 \xrightarrow{(-1,1)} R^0 \oplus R}_{I},\ D\big) = \mathrm{Path}\, D$$

$$\begin{tikzcd} R[0] \arrow[r, "i_0", "\sim"'] \arrow[dr, "1"'] & I \arrow[d, "\sim"] & R[0] \arrow[l, "i_1"', "\sim"] \arrow[dl, "1"] \\ & R[0] & \end{tikzcd} \quad \text{yields} \quad \begin{tikzcd} D \arrow[r, leftarrow] \arrow[dr, leftarrow, "1"'] & \mathrm{Path}\, D \arrow[d, leftarrow, "\sim"] & D \arrow[l, leftarrow] \arrow[dl, leftarrow, "1"] \\ & D & \end{tikzcd}$$

and right htpy w.r.t. this path object is the usual chain homotopy. Since Ch_c = cx's of proj's, $Ch_f = Ch$, Whitehead theorem says that q-iso between cxs of proj's is a htpy equiv.

Derived functors. Let $F\colon \mathrm{Mod}\text{-}R \to \mathrm{Mod}\text{-}S$ be a right exact functor. It induces a functor $F\colon Ch_R \to Ch_S$ that clearly preserves weak equivalences between cofibrant objects (since by Whitehead theorem these are just homotopy equivalences) and we thus obtain a total left derived functor $\mathbb{L}F(C) = F(C^c)$. In particular, for $C = A[0]$, we have $C^c = P \xrightarrow{\sim} A[0]$ — a projective resolution, and $\mathbb{L}F(A[0]) = FP$, whose homology is $H_n\, \mathbb{L}F(A[0]) = L_n F(A)$.

$\rightsquigarrow$ this gives the left derived functors in a compact way

– as an object of $Ho(Ch_S) \longrightarrow$ [illegible]

$\mathcal{M} = \mathrm{Top}$

$\mathcal{W}$ = weak homotopy equivalences

$\mathcal{F}$ = Serre fibrations $= \{ 0 \times D^n \hookrightarrow I \times D^n \}^{\boxslash}$ ($\mathcal{J}$)

$\mathcal{W} \cap \mathcal{F}$ = Serre fibrations with weakly contractible fibres

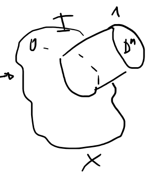
$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & F \\ D^n \arrow[ur, dashed] & \end{tikzcd} \qquad \text{again equivalent to (but more complicated – the problem is homotoped into a fibre)} \qquad \begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & X \arrow[d] \\ D^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

$\rightsquigarrow \mathcal{I} = \{ S^{n-1} \hookrightarrow D^n \mid n \geq 0 \}$, $\mathcal{W} \cap \mathcal{F} = {}^{\boxslash}(\mathcal{I}^{\boxslash})$

! domains of $\mathcal{I}$ & $\mathcal{J}$ are not small but SOA still works

Remains: ${}^{\boxslash}(\mathcal{J}^{\boxslash}) \subseteq \mathcal{W}$

"$\subseteq$": a relative $\mathcal{J}$-cell complex is obtained by attaching →



$\rightsquigarrow$ get a deformation retraction

$\Rightarrow$ the inclusion is w.h.e. $\square$

Derived functors. The functor $A/\mathrm{Top} \to \mathrm{Top}$ gives a total left
$(A \to X) \longmapsto X/A$

derived functor $(A \xrightarrow{f} X) \longmapsto (A \to \mathrm{Cyl}\, A +_f X) \longmapsto (\mathrm{Cyl}\, A +_f X)/A = \underbrace{\mathrm{Cone}\, A +_f X}_{\text{mapping cone}}$

↑ not cofibrant repl. in this model structure but in a different one (sufficient) (would be cofibrant if A, X were cofibrant)

X/A "cofibre" $\qquad \mathrm{Cone}\, A +_f X$ "homotopy cofibre"

$\mathcal{M} = \mathrm{sSet} = [\Delta^{op}, \mathrm{Set}]$

$\mathcal{W}$ = w.h.e. complicated, many equivalent formulations,
e.g. maps $f : K \to L$ inducing iso $f^* : [L, X] \to [K, X]$ $\forall X$ Kan cx. (fibrant obj.)

$\mathcal{F} \overset{\text{def}}{=}$ Kan fibrations $= \{ \Lambda_k \Delta^n \hookrightarrow \Delta^n \mid n \geq 1, k \in \{0, \ldots, n\} \}^{\boxslash}$
$= \mathcal{J}^{\boxslash}$; ${}^{\boxslash}(\mathcal{J}^{\boxslash})$ = anodyne extensions $= \mathcal{W} \cap \mathcal{C}$

$\mathcal{W} \cap \mathcal{F} = \{ \partial \Delta^n \hookrightarrow \Delta^n \mid n \geq 0 \}^{\boxslash} = \mathcal{I}^{\boxslash}$; ${}^{\boxslash}(\mathcal{I}^{\boxslash})$ = $\mathcal{I}$-cell = monos $= \mathcal{C}$

There is an adjunction
$|\cdot| : \mathrm{sSet} \rightleftarrows \mathrm{Top} : S$ ($\perp$)
that clearly takes $|\mathcal{I}_{sSet}| = \mathcal{I}_{Top}$, $|\mathcal{J}_{sSet}| = \mathcal{J}_{Top}$ and preserves

colimits $\Rightarrow$ Quillen adjunction

Theorem. This is a Quillen equivalence, so that

$$|\cdot| : Ho(sSet) \underset{\leftarrow}{\xrightarrow{\simeq}} Ho(Top) : S$$

L Top fibrant
L no need to derive, everything in sSet cofibrant

In particular, $X \in Top \Rightarrow |SX| \xrightarrow{\sim} X$ a functorial CW-replacement
$K \in sSet \Rightarrow K \to S|K|$ a functorial Kan-replament

Transfer of the model structure

$F : \mathcal{N} \rightleftarrows \mathcal{M} : G$ and assume that $\mathcal{N}$ has a model structure

Define $\mathcal{W}, \mathcal{F}$ in $\mathcal{M}$ as $G^{-1}\mathcal{W}$, $G^{-1}\mathcal{F}$. Then

$$G^{-1}\mathcal{W} \cap G^{-1}\mathcal{F} = G^{-1}(\mathcal{W} \cap \mathcal{F}) = (F\mathcal{I})^{\boxslash}$$
$$G^{-1}\mathcal{F} = (F\mathcal{J})^{\boxslash}$$

$\Rightarrow$ we get a model category provided that

- $F\mathcal{I}, F\mathcal{J}$ have small domains
- $F\mathcal{J}\text{-cell} \subseteq G^{-1}\mathcal{W}$

Theorem. Suppose that $\mathcal{M}$ possesses a functorial fibrant replacement for which a path object exists. Then the assumption $F\mathcal{J}\text{-cell} \subseteq G^{-1}\mathcal{W}$ is satisfied.

automatic in the enriched context
($\text{Path}\, M = \{Cyl\, S, M\}$
cylinder on the monoidal unit = "interval")

Proof. $F\mathcal{J}\text{-cell} \boxslash G^{-1}\mathcal{F}$

$$\begin{tikzcd} M \arrow[r, "i_M"] \arrow[d, "k"'] & M^f \arrow[d] \\ N \arrow[r] \arrow[ur, dashed, "\ell"] & 1 \end{tikzcd} \quad F\mathcal{J}\text{-cell} \ni k \qquad \Rightarrow \qquad \begin{tikzcd} M \arrow[r, "i_M"] \arrow[d, "k"'] \arrow[dr, phantom, "\text{comm}"] & M^f \arrow[d, "k^f"] \\ N \arrow[r, "i_N"'] \arrow[ur, "\ell"] & N^f \end{tikzcd}$$

functorial $M \xrightarrow{\sim} M^f$
i.e. M^f functor, $\downarrow$ nat. transf.

similarly:

$$\begin{tikzcd} M \arrow[r, "j i_N k"] \arrow[d, "k"'] & \text{Path}\, N^f \arrow[d, "q"] \\ N \arrow[r, "{(i_N, k^f \ell)}"'] \arrow[ur, dashed] & N^f \times N^f \end{tikzcd} \quad \Rightarrow \quad i_N \sim k^f \ell \quad \text{so that the lower triangle above commutes up to homotopy}$$

Now apply G that preserves path objects and right homotopies:

$$\underset{\mathcal{M}}{\begin{tikzcd} M \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{comm}"] & M^f \arrow[d] \\ N \arrow[r] \arrow[ur, "\text{htpy}"] & N^f \end{tikzcd}} \rightsquigarrow \underset{\mathcal{N}}{\begin{tikzcd} GM \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\text{comm}"] & GM^f \arrow[d] \\ GN \arrow[r, "\sim"'] \arrow[ur, "\text{htpy}"] & GN^f \end{tikzcd}} \overset{\gamma}{\rightsquigarrow} \underset{Ho(\mathcal{N})}{\begin{tikzcd} GM \arrow[r, "\cong"] \arrow[d] & GM^f \arrow[d] \\ GN \arrow[r, "\cong"'] \arrow[ur] & GN^f \end{tikzcd}}$$

$$\Downarrow$$

$$M \xrightarrow{\sim} N \quad \Leftarrow \quad GM \xrightarrow{\sim} GN \quad \Leftarrow \quad GM \xrightarrow{\cong} GN \qquad \square$$

Examples.

- $F : sSet \rightleftarrows sAb : G$ satisfies the assumption since any simplicial group is fibrant (as a simplicial set, i.e. its G-image is fibrant) $\Rightarrow$ can take $M^f = M$.
- more generally for any variety of algebras $\mathcal{C}$

  $F : sSet \rightleftarrows s\mathcal{C} : G$

  there is a fibrant replacement Ex^∞ on $sSet$ that preserves finite limits (it is a filtered colimit of right adjoints) so that it gives a functor

  $Ex^\infty : s\mathcal{C} \longrightarrow s\mathcal{C}$

  and gives the desired functorial fibrant replacement on $s\mathcal{C}$

- given $\mathcal{M}$ cofibrantly generated and $\mathcal{A}$ a small category

  $F : [\text{ob}\,\mathcal{A}^{op}, \mathcal{M}] \rightleftarrows [\mathcal{A}^{op}, \mathcal{M}] : G$

  $\cong \prod_{\text{ob}\,\mathcal{A}} \mathcal{M}$ $\leftarrow$ a model category with all $\mathcal{W}, \mathcal{C}, \mathcal{F}$ objectwise

  i.e. $\mathcal{I} = \{ i_A \mid A \in \mathcal{A}, i \in \mathcal{I} \}$

  $\llcorner$ product of $\cdot\ i : K \to L$ at object a
  $\cdot\ 1 : 0 \to 0$ at object $\neq a$

  $\Rightarrow F i_A = i \cdot \mathcal{A}(-, A) : K \cdot \mathcal{A}(-, A) \longrightarrow L \cdot \mathcal{A}(-, A)$

with components: $K \cdot \mathcal{A}(B,A) \longrightarrow L \cdot \mathcal{A}(B,A)$

$$\sum_{B\to A} K \xrightarrow{\Sigma i} \sum_{B\to A} L$$

The assumption is satisfied – SOA gives a functorial fibrant replacement. (Or simply $F\mathcal{J}$-cell has component at B a relative cell complexes generated from $\sum_{B\to A} K \xrightarrow{\Sigma j} \sum_{B\to A} L$ for various $A \in \mathcal{A}$, $j \in \mathcal{J}$ $\Rightarrow$ it lies in $\mathcal{J}$-cell $\subseteq \mathcal{W}$)

We get the so called projective model structure on $[\mathcal{A}^{op}, \mathcal{M}]$.

The adjunction:

$$\mathrm{colim}: [\mathcal{A}^{op}, \mathcal{M}] \rightleftarrows \mathcal{M} : \Delta$$

is a Quillen adjunction – Δ preserves fibrations and weak equivalences.

$$\mathbb{L}\,\mathrm{colim}: \mathrm{Ho}\,[\mathcal{A}^{op}, \mathcal{M}] \rightleftarrows \mathrm{Ho}\,\mathcal{M} : \mathbb{R}\Delta$$

$\llcorner$ essentially Δ

$\mathrm{Colim} \circ Q : [\mathcal{A}^{op}, \mathcal{M}] \longrightarrow \mathcal{M}$ the "homotopy colimit"; concrete models using concrete cofibrant replacements

Remark. Colim does not preserve general weak equiv's, only w.e. between cofibrant objects (i.e. cofib. repl. is necessary):

$$\mathcal{A}\;\begin{pmatrix} 1 \\ \downarrow \\ 0 \\ \uparrow \\ 2 \end{pmatrix} \qquad \begin{tikzcd} D^n \arrow[r, "\sim"] & * \\ S^{n-1} \arrow[u] \arrow[d] \arrow[r, "="] & S^{n-1} \arrow[u] \arrow[d] \\ D^n \arrow[r, "\sim"] & * \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} S^n \arrow[r, "\not\sim"] & * \end{tikzcd}$$

cofibrant $\qquad$ not cofibrant

the homotopy colimit of the second diag.

$\Downarrow$

$$\begin{tikzcd} 0 = 0 \cdot \mathcal{A}(-,0) \arrow[r, "\text{gen cof.}"] & S^{n-1} \cdot \mathcal{A}(-,0) \arrow[r] & D \\ & S^{n-1} \cdot \mathcal{A}(-,1) \arrow[u] \arrow[r, "\text{gen cof.}"] & D^n \cdot \mathcal{A}(-,1) \arrow[u] \\ & + & + \\ & S^{n-1} \cdot \mathcal{A}(-,2) \arrow[r, "\text{gen cof.}"'] & D^n \cdot \mathcal{A}(-,2) \end{tikzcd}$$

Remark. Generally, cofibrations are natural transformations

$$\begin{tikzcd} X \arrow[r, tail] & X' \\ A \arrow[u] \arrow[d] \arrow[r, tail] & A' \arrow[u] \arrow[d] \\ Y \arrow[r, tail] & Y' \end{tikzcd}$$

(all components and the maps from the pushouts are cofibrations

$\Rightarrow$ cofibrant diagrams = all objects cofibrant and both maps cofibrations

Properness

Theorem. Let $\mathcal{M}$ be a model category.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail] & B \arrow[d, tail] \\ X \arrow[r, "g"'] & Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

If all objects are cofibrant then f w.e. $\Rightarrow$ g w.e.

Definition. We say that $\mathcal{M}$ is left proper if the same holds for all objects A, B, X, Y (not necessarill cofibrant), i.e. w.e.'s are preserved by pushouts along cofibrations. Dually, $\mathcal{M}$ is right proper if we.'s are preserved by pullbacks along fibrations.

Example. If all objects of $\mathcal{M}$ are cofibrant $\Rightarrow$ $\mathcal{M}$ left proper
e.g. $\mathcal{M} = \mathrm{Cat}, \mathrm{sSet}$; serious example: $\mathrm{Top}, \mathrm{Ch}$.
Dually $\mathcal{M} = \mathrm{Cat}, \mathrm{Top}, \mathrm{Ch}$ right proper for free; sSet serious example.
$\mathcal{M}$ left proper $\Rightarrow$ $[\mathcal{A}^{op}, \mathcal{M}]$ left proper (pushouts and w.e. pointwise, cof $\Rightarrow$ ptwise cof)

Proof. Clearly holds if f is a trivial cofibration $\overset{\text{Brown}}{\underset{???}{\Longrightarrow}}$ holds also for w.e.'s between cofibrant objects.
not that simple

$$f_* : A/\mathcal{M} \rightleftarrows B/\mathcal{M} : f^*$$

$$\begin{tikzcd} A \arrow[d, "f"] \arrow[dd, bend left, "f^*y"] \\ B \arrow[d, "y"] \\ Y \end{tikzcd} \longmapsfrom \begin{tikzcd} B \arrow[d, "y"] \\ Y \end{tikzcd}$$

$\leftarrow$ preserves & reflects fibrations and weak equivalences
(they are determined via $\mathrm{cod} : A/\mathcal{M} \to \mathcal{M}$, $(A \to X) \mapsto X$
and f^* commutes with them.)

$$\begin{tikzcd} A \arrow[d, "x"'] \\ X \end{tikzcd} \longmapsto \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "x"'] & B \arrow[d, "f_* x"] \\ X \arrow[r] & Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

$\Rightarrow$ derived unit on cofibrant objects
$$\eta' : 1 \xrightarrow{\eta} f^* f_* \underset{\sim}{\xrightarrow{f^* r f_*}} f^* R f_*$$
$\uparrow$ serves equally well

Summary:
when f is a trivial cofibration
then $\mathbb{L}f_* : \mathrm{Ho}\, A/\mathcal{M} \rightleftarrows \mathrm{Ho}\, B/\mathcal{M} : \mathbb{R}f^*$
is an equivalence (η' w.e. $\Rightarrow \varepsilon'$ w.e.)
2-out-of-3:
$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[rr, bend right, "h"'] & \cdot \arrow[r, "g"] & \cdot \end{tikzcd}$$
induces

$$\begin{tikzcd} & \mathrm{Ho}(B/\mathcal{M}) \arrow[dl, shift left] \arrow[dr, shift left] & \\ \mathrm{Ho}(A/\mathcal{M}) \arrow[ur, shift left] \arrow[rr, shift left] & & \mathrm{Ho}(C/\mathcal{M}) \arrow[ul, shift left] \arrow[ll, shift left] \end{tikzcd}$$

two equivalences $\Rightarrow$ so is third

cofibrant i.e. x cofibration

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "x"'] & B \arrow[d, "f^* f_* x"', shift right] \arrow[d, "f_* x", shift left] \\ X \arrow[r, "\eta"'] & Y \end{tikzcd}$$

$\Rightarrow$ if η' is a w.e. then so is ε'
by the triangle identity:

$$\begin{tikzcd} \mathbb{R}f^* X \arrow[rr, "1"] \arrow[dr, "\eta' \mathbb{R}f^*"', "\cong"] & & \mathbb{R}f^* X \\ & \mathbb{R}f^* \mathbb{L}f_* \mathbb{R}f^* X \arrow[ur] & \end{tikzcd}$$

$\mathbb{R}f^* \varepsilon'$ iso
$\Downarrow$
ε' iso

Now we can finally apply Brown's lemma ⸮

Now we can finally apply Brown's lemma:

$$\begin{tikzcd} & Cyl\,f \arrow[dr, "p"] & \\ A \arrow[ur, "i"] \arrow[rr, "f"] & & B \arrow[ul, dashed, "j"'] \end{tikzcd} \longleftarrow \begin{tikzcd} & Cyl\,f \arrow[dr, "p"] & \\ B \arrow[ur, "j"] \arrow[rr, "1"] & & B \end{tikzcd}$$

ε' iso

$i_* \dashv i^*$ and $j_* \dashv j^*$ Q.e.

$$\Downarrow$$

$f_* \dashv f^*$ Q.e. $\quad p_* \dashv p^*$ Q.e. $\square$

Remark. A different and still conceptual proof: let $\mathcal{A} = \{ 0 \longrightarrow 1 \longleftarrow 2 \}$ and equip $[\mathcal{A}^{op}, \mathcal{M}]$ with the so-called Reedy model structure (see later) in which a span $M_0 \longleftarrow M_1 \longrightarrow M_2$ is cofibrant iff all M_0, M_1, M_2 are cofibrant and the map $M_1 \longrightarrow M_2$ is a cofibration (unlike in the proj. model str. where both maps need to be cofibrations)

We still get a Quillen adjunction

$$\operatorname{colim} : [\mathcal{A}^{op}, \mathcal{M}] \rightleftarrows \mathcal{M} : \Delta$$

and thus colim preserves w.e. between cofibrant objects. Apply to

$$\begin{tikzcd} A \arrow[r, "f", "\sim"'] & B \\ A \arrow[u, "1"] \arrow[r, "1"] \arrow[d, tail] & A \arrow[u, "f"'] \arrow[d, tail] \\ X \arrow[r, "1"] & X \end{tikzcd} \quad \overset{\text{colim}}{\rightsquigarrow} \quad X \xrightarrow[\sim]{g} Y$$

Quillen bifunctors, monoidal and enriched model categories

Definition. A left adjoint bifunctor is a bifunctor

$$F : \mathcal{M} \times \mathcal{N} \to \mathcal{P}$$

that, for each $n \in \mathcal{N}$, yields a left adjoint functor $F(-,n) : \mathcal{M} \to \mathcal{P}$
and, for each $m \in \mathcal{M}$, yields a left adjoint functor $F(m,-) : \mathcal{N} \to \mathcal{P}$

$\rightsquigarrow$ get $G : \mathcal{N}^{op} \times \mathcal{P} \to \mathcal{M}$ s.t. $\mathcal{P}(F(m,n), p) \cong \mathcal{M}(m, G(n,p))$
and $H : \mathcal{M}^{op} \times \mathcal{P} \to \mathcal{N}$ s.t. $\mathcal{P}(F(m,n), p) \cong \mathcal{N}(n, H(m,p))$

We may call (F, G, H) an adjunction of two variables.

Definition. A left Quillen bifunctor is a left adjoint bifunctor such that for each cofibration $i : A \rightarrowtail B$ in $\mathcal{M}$ and $j : K \rightarrowtail L$ in $\mathcal{N}$, the map $F_\lrcorner(i,j) \overset{\text{def}}{=} F(A,L) +_{F(A,K)} F(B,K) \longrightarrow F(B,L)$ as in

$$\begin{tikzcd}
F(A,K) \arrow[r] \arrow[d] & F(B,K) \arrow[d] \arrow[dd, bend left] \\
F(A,L) \arrow[r] \arrow[drr, bend right] & F_\lrcorner(i,j) \arrow[dr] \\
& & F(B,L)
\end{tikzcd}$$

is a cofibration that is trivial if at least one of i, j is.

We may write this as: $F_\lrcorner(\mathcal{C}, \mathcal{C}) \subseteq \mathcal{C}$
$F_\lrcorner(\mathcal{W} \cap \mathcal{C}, \mathcal{C}) \subseteq \mathcal{W} \cap \mathcal{C}$
$F_\lrcorner(\mathcal{C}, \mathcal{W} \cap \mathcal{C}) \subseteq \mathcal{W} \cap \mathcal{C}$

Remark. If $\mathcal{M}, \mathcal{N}$ are cofibrantly generated, it is enough to check this for the generating cofibrations and trivial cofibrations.
This also applied to the case of Quillen functors (easier).

Remark. This easily generalizes to $\underline{n}$ variables giving as special cases:
$n = 1$: left Quillen functor
$n = 2$: left Quillen bifunctor

Lemma. F is left Quillen $\Leftrightarrow$ G is right Quillen, i.e.

$G_\ulcorner(\mathcal{C}, \mathcal{F}) \subseteq \mathcal{F}$
$G_\ulcorner(\mathcal{W} \cap \mathcal{C}, \mathcal{F}) \subseteq \mathcal{W} \cap \mathcal{F}$
$G_\ulcorner(\mathcal{C}, \mathcal{W} \cap \mathcal{F}) \subseteq \mathcal{W} \cap \mathcal{F}$

here I decided to denote fibrations in $\mathcal{N}^{op}$ by $\mathcal{C}$ since they are cofibrations in $\mathcal{N}$

Proof. This boils down to $F_\lrcorner(i,j) \boxslash p \Leftrightarrow i \boxslash G_\ulcorner(j,p)$ that must be checked and is tedious but completely elementary. $\square$

Definition. A monoidal model category is a category with

- a closed monoidal structure, i.e. $\otimes$ is a left biadjoint with right adjoints $\hom_r$ and $\hom_\ell$
- a model structure

symmetric for us $\Rightarrow$ $\hom_r = \hom_\ell = \{\ ,\ \}$

such that
- the unit of the monoidal structure S is either cofibrant or, more generally, the cofibrant replacement $S^c \xrightarrow{\sim} S$ tensors with all cofibrant objects to w.e.'s
$$S^c \otimes X \xrightarrow{\sim} S \otimes X \cong X$$
$$X \otimes S^c \xrightarrow{\sim} X \otimes S \cong X$$
this does not depend on the choice of the cofibrant replacement in view of
- the tensor product bifunctor $\otimes : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ is left Quillen.

Examples. All $\underbrace{\mathrm{Cat}, \mathrm{Ch}, \mathrm{Top}, \mathrm{sSet}}$ are monoidal model categories.
(with cofibrant unit)

$$\left.\begin{array}{l} i_n : S^{n-1} \longrightarrow D^n \\ j_0 : \cdot \longrightarrow I \\ j_n = j_0 \otimes_\lrcorner i_n \end{array}\right\} \quad \begin{array}{l} i_n \otimes_\lrcorner i_m \text{ is a cofibration} \\ \left.\begin{array}{l} j_n \otimes_\lrcorner i_m \\ i_n \otimes_\lrcorner j_m \end{array}\right\} \text{ are trivial cofibrations} \end{array}$$

$$\begin{tikzcd} m & R & R \arrow[l, "1"'] \arrow[d, "(-1)^n"] \\ m-1 & R \arrow[from=u, "(-1)^{m+1}"'] & R \arrow[l, "1"] \\ & n-1 & n \end{tikzcd}$$

A functorial cylinder object for cofibrant objects: pick a cylinder object for the unit $S \in \mathcal{V}_c$: $S+S \rightarrowtail \mathrm{Cyl}\, S \xrightarrow{\sim} S$ and tensor it with $A \in \mathcal{V}_c$ to obtain
a w.e. between cofibrant objects

$$\begin{array}{ccccc} A \otimes (S+S) & \rightarrowtail & A \otimes \mathrm{Cyl}\, S & \xrightarrow{\sim} & A \otimes S \\ \| & & \|\text{def} & & \| \\ A + A & \rightarrowtail & \mathrm{Cyl}\, A & \xrightarrow{\sim} & A \end{array}$$

Dually a path object is $\{\mathrm{Cyl}\, S, X\}$.

Definition. A $\mathcal{V}$-category $\mathcal{M}$ is said to be tensored if
$$\mathcal{V}(V, \mathcal{M}(M,N)) \cong \mathcal{M}(M \otimes V, N) \quad \text{naturally}$$
This gives a functor $\otimes : \mathcal{M}_0 \times \mathcal{V}_0 \longrightarrow \mathcal{M}_0$ (in fact $\mathcal{M} \otimes \mathcal{V} \to \mathcal{M}$) that makes $\mathcal{M}_0$ into a "module" over $\mathcal{V}_0$. Dually $\mathcal{M}$ is cotensored if $\mathcal{V}(V, \mathcal{M}(M,N)) \cong \mathcal{M}(M, \{V, N\})$ naturally
This gives a functor $\{\ ,\ \} : \mathcal{V}_0^{op} \times \mathcal{M}_0 \longrightarrow \mathcal{M}_0$
Together with the hom-functor $\mathcal{M}(-,-) : \mathcal{M}_0 \times \mathcal{M}_0 \longrightarrow \mathcal{V}_0$ these yield an adjunction of two variables.
If $A \otimes S^c \xrightarrow{\sim} A \otimes S = A$ for $A \in \mathcal{M}_c$ and this adjunction is Quillen, we say that $\mathcal{M}$ is a model $\mathcal{V}$-category.

Again $A \otimes \mathrm{Cyl}\, S$ is a cylinder object for $A \in \mathcal{M}_c$
$\{\mathrm{Cyl}\, S, X\}$ is a path object for $X \in \mathcal{M}_f$.

Left adjoint $\mathcal{V}$-functors preserve these cylinder objects (they are colimits),
right adjoint $\mathcal{V}$-functors preserve these path objects (they are limits).

Reedy model categories, framings