Properness

Theorem. Let M be a model category.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & \ulcorner & \downarrow \\ X & \xrightarrow{g} & Y \end{array}$$

If all objects are cofibrant then
f w.e. $\Rightarrow$ g w.e.

Definition. We say that M is left proper if the same holds for all objects A, B, X, Y (not necessarily cofibrant), i.e. w.e.'s are preserved by pushouts along cofibrations. Dually, M is right proper if we.'s are preserved by pullbacks along fibrations.

Example. If all objects of M are cofibrant $\Rightarrow$ M left proper.
e.g. $M = Cat,\ sSet$; serious example: Top, Ch.
Dually $M = Cat, Top, Ch$ right proper for free; $sSet$ serious example.
M left proper $\Rightarrow [A^{op}, M]$ left proper (pushouts and w.e. pointwise, cof $\Rightarrow$ ptwise cof)

Proof. Clearly holds if f is a trivial cofibration $\xrightarrow[???]{Brown}$ holds also for w.e.'s between cofibrant objects. not that simple

$$f_* : A/M \rightleftarrows B/M : f^*$$

$$\left(\begin{array}{c} A \\ f \downarrow \\ B \\ y \downarrow \\ Y \end{array}\right) f_* y \quad \longleftarrow\!\!\mid \quad \begin{array}{c} B \\ \downarrow y \\ Y \end{array} \quad \longleftarrow$$

preserves & reflects fibrations and weak equivalences (they are determined via $cod : A/M \to M$ and f^* commutes with them. $(A \to x) \mapsto X$)

$$\begin{array}{ccc} A \\ x \downarrow \\ X \end{array} \quad \longmapsto \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ x\downarrow & \ulcorner & \downarrow f_* x \\ X & \longrightarrow & Y \end{array}$$

$\Rightarrow$ derived unit on cofibrant objects
$\eta' : 1 \xrightarrow{\eta} f^* f_* \xrightarrow[\sim]{f^* r f_*} f^* R f_*$
$\uparrow$ serves equally well

Summary:
when f is a trivial cofibration
then $Lf_* : Ho\ A/M \rightleftarrows Ho\ B/M : Rf^*$
is an equivalence (η' w.e. $\Rightarrow \varepsilon'$ w.e.)
2-out-of-3: $\overset{f}{\to}\overset{g}{\to}\ \underset{h}{\searrow}$ induces

$$Ho(A/M) \rightleftarrows Ho(B/M) \rightleftarrows Ho(C/M)$$

two equivalences $\Rightarrow$ so is third

Now we can finally apply Brown's lemma

cofibrant
i.e. $\Rightarrow$
x cofibration

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ x\downarrow & \overrightarrow{f^* f_* x} & \downarrow f_* x \\ X & \xrightarrow{\eta} & Y \end{array}$$

$\Rightarrow$ if η' is a w.e. then so is ε'
by the triangle identity:

$$\begin{array}{ccc} Rf^* X & \xrightarrow{\quad 1 \quad} & Rf^* X \\ \eta' Rf^* \downarrow \cong & & \nearrow Rf^* \varepsilon' \quad iso \\ Rf^* Lf_* Rf^* X & & \varepsilon'\ iso \end{array}$$

Now we can finally apply Brown's lemma $\overset{?}{\underset{\circ}{}}$
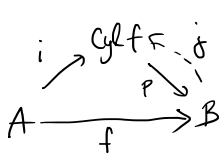
$i_* \dashv i^*$ and $j_* \dashv j^*$ Q.e.

$f_* \dashv f^*$ Q.e. $\quad p_* \dashv p^*$ Q.e. $\square$

Remark. A different and still conceptual proof: let $A = \{0 \longrightarrow 1 \longleftarrow 2\}$ and equip $[A^{op}, M]$ with the so-called Reedy model structure (see later) in which a span $M_0 \longleftarrow M_1 \longrightarrow M_2$ is cofibrant iff all M_0, M_1, M_2 are cofibrant and the map $M_1 \longrightarrow M_2$ is a cofibration (unlike in the proj. model str. where both maps need to be cofibrations)

We still get a Quillen adjunction

$$\text{colim} : [A^{op}, M] \rightleftarrows M : \Delta$$

and thus colim preserves w.e. between cofibrant objects. Apply to

$$
\begin{array}{ccc}
A & \xrightarrow{\;f\;}_{\sim} & B \\
\uparrow & & \uparrow f \\
\Delta & \xrightarrow{\;1\;} & A \\
\downarrow & & \downarrow \\
X & \xrightarrow{\;1\;} & X
\end{array}
\qquad \xrightarrow{\;\text{colim}\;} \quad X \xrightarrow{\;g\;}_{\sim} Y
$$

Quillen bifunctors, monoidal and enriched model categories

Definition. A left adjoint bifunctor is a bifunctor

$$F: M \times N \longrightarrow P$$

that, for each $n \in N$, yields a left adjoint functor $F(-, n): M \to P$
and, for each $m \in M$, yields a left adjoint functor $F(m, -): N \to P$

$\rightsquigarrow$ get $\quad G: N^{op} \times P \to M \quad$ s.t. $\quad P(F(m,n), p) \cong M(m, G(n,p))$

and $\quad H: M^{op} \times P \to N \quad$ s.t. $\quad P(F(m,n), p) \cong N(n, H(m,p))$

We may call (F, G, H) an adjunction of two variables.

Definition. A left Quillen bifunctor is a left adjoint bifunctor such that for each cofibration $i: A \rightarrowtail B$ in M and $j: K \rightarrowtail L$ in N, the map $\quad F_{\lrcorner}(i,j)) \overset{\text{def}}{=} F(A,L) +_{F(A,K)} F(B,K) \longrightarrow F(B,L) \quad$ as in

is a cofibration that is trivial if at least one of i, j is.

We may write this as:
$$F_{\lrcorner}(\mathcal{C}, \mathcal{C}) \subseteq \mathcal{C}$$
$$F_{\lrcorner}(W \cap \mathcal{C}, \mathcal{C}) \subseteq W \cap \mathcal{C}$$
$$F_{\lrcorner}(\mathcal{C}, W \cap \mathcal{C}) \subseteq W \cap \mathcal{C}$$

Remark. If M, N are cofibrantly generated, it is enough to check this for the generating cofibrations and trivial cofibrations. This also applied to the case of Quillen functors (easier).

Remark. This easily generalizes to $\underline{n}$ variables giving as special cases:

$n=1$: left Quillen functor
$n=2$: left Quillen bifunctor

Lemma. F is left Quillen $\iff$ G is right Quillen, i.e.
$$G_{\ulcorner}(\mathcal{C}, \mathcal{F}) \subseteq \mathcal{F}$$
$$G_{\ulcorner}(W \cap \mathcal{C}, \mathcal{F}) \subseteq W \cap \mathcal{F}$$
$$G_{\ulcorner}(\mathcal{C}, W \cap \mathcal{F}) \subseteq W \cap \mathcal{F}$$

here I decided to denote fibrations in N^{op} by $\mathcal{C}$ since they are cofibrations in N

Proof. This boils down to $\quad F_{\lrcorner}(i,j) \boxempty p \iff i \boxempty G_{\ulcorner}(j,p)$ that must be checked and is tedious but completely elementary. $\quad\square$

Definition. A monoidal model category is a category with

lecture 6 - stránka 3

- a closed monoidal structure, i.e. $\otimes$ is a left biadjoint with right adjoints $\hom_r$ and $\hom_\ell$

- a model structure $\left[\!\!\begin{array}{l}\text{symmetric for us} \implies \hom_r = \hom_\ell = \{,\}\end{array}\right.$

such that

- the unit of the monoidal structure S is either cofibrant or, more generally, the cofibrant replacement $S^c \xrightarrow{\sim} S$ tensors with all cofibrant objects to w.e.'s

$$S^c \otimes X \xrightarrow{\sim} S \otimes X \cong X$$
$$X \otimes S^c \xrightarrow{\sim} X \otimes S \cong X$$

this does not depend on the choice of the cofibrant replacement in view of

- the tensor product bifunctor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is left Quillen. $\Leftarrow$

Examples. All $\underbrace{\text{Cat}, \text{Ch}, \text{Top}, \text{sSet}}$ are monoidal model categories.

(with cofibrant unit)

$$\left.\begin{array}{l} i_n : S^{n-1} \to D^n \\ j_0 : \cdot \to I \\ j^n = j_0 \otimes_{\lrcorner} i_n \end{array}\right\} \begin{array}{l} i_n \otimes_{\lrcorner} i_m \text{ is a cofibration} \\ \left.\begin{array}{l} j_n \otimes_{\lrcorner} i_m \\ i_n \otimes_{\lrcorner} j_m \end{array}\right\} \text{ are trivial cofibrations} \end{array}$$

A functorial cylinder object for cofibrant objects: pick a cylinder object for the unit $S \in \mathcal{V}_c$: $S + S \rightarrowtail \text{Cyl } S \xrightarrow{\sim} S$ and tensor it with $A \in \mathcal{V}_c$ to obtain $\qquad$ a w.e. between cofibrant objects

$$\begin{array}{ccccc} (S+S) \otimes A & \rightarrowtail & \text{Cyl } S \otimes A & \xrightarrow{\sim} & S \otimes A \\ \| & & \|\text{ def} & & \| \\ A + A & \rightarrowtail & \text{Cyl } A & \xrightarrow{\sim} & A \end{array}$$

Dually a path object is $\{\text{Cyl } S, X\}$.

Remark. tensor-hom adjunction

$$\frac{A \otimes B \to C}{A \to \text{Hom}(B, C)} \qquad \text{sets of maps}$$

better: objects of maps $\qquad \frac{\text{Hom}(A \otimes B, C)}{\text{Hom}(A, \text{Hom}(B, C))}$ $\quad\leftarrow$ works since

$$\frac{X \to \text{Hom}(A \otimes B, C)}{X \otimes A \otimes B \to C}$$
$$\frac{X \otimes A \to \text{Hom}(B, C)}{X \to \text{Hom}(A, \text{Hom}(B, C))}$$

If M is enriched over $\mathcal{V}$, this can be generalized:

$$M(K \otimes M, N) \cong M(M, \{K, N\})$$
$$\frac{}{\mathcal{V}(K, M(M, N))} \quad\leftarrow \text{ two different Hom's } \overset{?}{\circ}$$

Associativity:
$$M(K \otimes (L \otimes M), N) \cong \mathcal{V}(K, M(L \otimes M, N)) \cong \mathcal{V}(K, \mathcal{V}(L, M(M, N)))$$
$$\cong \mathcal{V}(K \otimes L, M(M, N)) \cong M((K \otimes L) \otimes M, N)$$
$$\implies K \otimes (L \otimes M) \cong (K \otimes L) \otimes M$$

Unit $S \otimes M \cong M$

Definition. A V-category M is said to be tensored it

$$V(K, M(M,N)) \cong M(K \otimes M, N) \qquad \text{naturally}$$

This gives a functor $\otimes : V_0 \times M_0 \to M_0$ (in fact a V-functor $V \otimes M \to M$) that makes M_0 into a "module" over V_0. Dually M is cotensored if $V(K, M(M,N)) \cong M(M, \{K,N\})$ naturally

This gives a functor $\{ , \} : V_0^{op} \times M_0 \to M_0$

Together with the hom-functor $M(-,-): M_0 \times M_0 \to V_0$

these yield an adjunction of two variables.

If $A \otimes S^c \xrightarrow{\sim} A \otimes S = A$ for $A \in M_c$ and this adjunction is Quillen, we say that M is a model V-category.

Again $Cyl \, S \otimes A$ is a cylinder object for $A \in M_c$
$\{Cyl S, X\}$ is a path object for $X \in M_f$.

Left adjoint V-functors preserve these cylinder objects (they are colimits),
right adjoint V-functors preserve these path objects (they are limits).

Deriving bifunctors.

$F : M \times N \to P \qquad$ left Quillen bifunctor

$\Rightarrow F : M_c \times N_c \to P_c \qquad$ preserves weak equivalences:

$\begin{array}{cc} 0 & 0 \\ \downarrow & \downarrow \\ M & N \end{array} \rightsquigarrow$

$$\begin{array}{ccc} F(0,0) & \longrightarrow & F(M,0) \\ \downarrow & & \downarrow \\ F(0,N) & \longrightarrow & F(M,N) \end{array}$$

$F_\lrcorner(M,N): 0 \rightarrowtail F(M,N)$

$\begin{array}{cc} M' & 0 \\ i \downarrow \sim & \downarrow \\ M & N \end{array} \rightsquigarrow$

$$\begin{array}{ccc} F(M',0) & \longrightarrow & F(M,0) \\ \downarrow & & \downarrow \\ F(M',N) & \longrightarrow & F(M,N) \end{array}$$

$F_\lrcorner(i,N): F(M',N) \rightarrowtail F(M,N)$

$F(i,N)$

now apply Brown's lemma

Thus we get $\mathbb{L}F : Ho(M) \times Ho(N) \to Ho(P)$

$$\begin{array}{ccc} & & \mathbb{L}F \\ Ho(M) \times Ho(N) & \longrightarrow & Ho(P) \\ \gamma \times \gamma \uparrow & & \uparrow \gamma \\ M \times N & \longrightarrow & P \\ & F(Q \times Q) & \end{array}$$

induced by $M \times N \xrightarrow{F(Q \times Q)} P$

In particular, for a monoidal model category V, we get

$$\otimes^{\mathbb{L}} : Ho(V) \times Ho(V) \longrightarrow Ho(V)$$

making the homotopy category into a monoidal closed category

$$(K \otimes^{\mathbb{L}} L) \otimes^{\mathbb{L}} M \overset{?}{\cong} K \otimes^{\mathbb{L}} (L \otimes^{\mathbb{L}} M)$$

$$\parallel \qquad\qquad\qquad \parallel$$

$$Q(QK \otimes QL) \otimes QM \qquad QK \otimes Q(QL \otimes QM)$$

unit again S

the "weird" axiom: $X \in V_c$

$$QS \otimes X \to S \otimes X \cong X$$

lecture 6 - stránka 5

$$\| $$
$$Q(QK \otimes QL) \otimes QM \qquad\qquad QK \otimes Q(QL \otimes QM)$$

$$q\otimes 1 \Big\downarrow \cong \qquad\qquad 1\otimes q \Big\downarrow \cong$$

$$(QK \otimes QL) \otimes QM \quad \cong \quad QK \otimes (QL \otimes QM)$$

the "weird" axiom : $X \in V_c$

$$QS \otimes X \longrightarrow S \otimes X \cong X$$

$$\cong \uparrow 1\otimes q \qquad \nearrow \overset{\cong}{\underset{q\otimes q}{\nearrow}}$$

$$QS \otimes QX$$

$$\| \qquad\qquad \nearrow \text{ unit iso } \underset{=}{}$$

$$S \otimes^{L} X$$

More generally, if M is a model V-category, then $Ho(M)$ is
- a category enriched in $Ho(V)$ } need to derive the Hom's
- tensored and cotensored } $M^{R}(-,-)$, $\{-,-\}^{R}$

Reedy model categories, framings

Motivation. Any model category M is in some weak sense enriched over sSet and, as a result, $Ho(M)$ will be enriched over $Ho(sSet)$.

Start with an honest sSet-model category = simplicial model category

— since $K \in sSet$ is a colimit $K = \operatorname{colim}_{(n, \Delta^n \to K)} \Delta^n$

we have $K \otimes M = \operatorname{colim}(\Delta^n \otimes M)$ and it is enough to give $\Delta^n \otimes M$
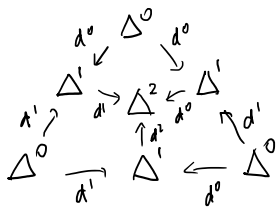
— clearly $\Delta^0 \otimes M \cong M$, since Δ^0 is the monoidal unit

— what is the essential property of $\Delta^1 \otimes M$?
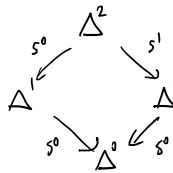 it is a cylinder? ... at least for $M \in M_c$:

$$\Delta^0 + \Delta^0 \xrightarrow{[d^0, d^1]} \Delta^1 \xrightarrow[\sim]{s^0} \Delta^0 \quad \rightsquigarrow \quad M + M \rightarrowtail \Delta^1 \otimes M \xrightarrow{\sim} M$$

— what about $\Delta^2 \otimes M$?

) more compactly $\partial\Delta^2 \to \Delta^2$
$\| $
$L_2 \dot\Delta$

$\rightsquigarrow L_2 \dot\Delta \otimes M \rightarrowtail \dot\Delta^2 \otimes M \xrightarrow{\sim} M_2 \dot\Delta \otimes M$

) more compactly $\Delta^2 \xrightarrow{\sim} M_2 \dot\Delta$ $\qquad \coprod_0^{\nabla} \dot\Delta : \Delta \longrightarrow sSet$
a cosimplicial object in sSet

— need some calculus of such diagrams $\Delta \to sSet$ $\qquad \rightsquigarrow \dot\Delta \otimes M : \Delta \to M$
$\to$ Reedy categories, Reedy model structures $\Delta \to M$ $\qquad$ a cosimplicial object in M

A Reedy category has two kinds of maps — direct and inverse
$\qquad\qquad\qquad$ (like d^i and s^i in Δ)

Definition. A direct category is a category A together with a functor
$\deg : A \to \lambda$ (ordinal) that satisfies $f : A \to B \rightsquigarrow f = 1 \Leftrightarrow \deg A = \deg B$

Now we denote by $A_a \in [A, Set]$ the representable $A(a, -)$

Dually $A^a \in [A^{op}, Set]$ denotes $A(-, a)$ and we define

• $\partial A^a \subseteq A^a$ a subfunctor obtained by removing 1
• $i_a : \partial A^a \hookrightarrow A^a$ the inclusion

We say that $f : X \to Y$ is a (trivial) cofibration if

$\qquad i_a \otimes_{\Delta, f} f$ is a (trivial) cofibration