Properness

Theorem. Let $\mathcal{M}$ be a model category.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail] & B \arrow[d, tail] \\ X \arrow[r, "g"'] & Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

If all objects are cofibrant then

f w.e. $\Rightarrow$ g w.e.

Definition We say that $\mathcal{M}$ is left proper if the same holds for all objects A, B, X, Y (not necessarilly cofibrant), i.e. w.e.'s are preserved by pushouts along cofibrations. Dually, $\mathcal{M}$ is right proper if we.'s are preserved by pullbacks along fibrations.

Example. If all objects of $\mathcal{M}$ are cofibrant $\Rightarrow$ $\mathcal{M}$ left proper

e.g. $\mathcal{M} = \mathrm{Cat}$, sSet ; serious example: Top, Ch.

Dually $\mathcal{M} = \mathrm{Cat}$, Top, Ch right proper for free; sSet serious example.

$\mathcal{M}$ left proper $\Rightarrow$ $[\mathcal{A}^{op}, \mathcal{M}]$ left proper (pushouts and w.e. pointwise, cof $\Rightarrow$ ptwise cof)

Proof. Clearly holds if f is a trivial cofibration $\xRightarrow[???]{\text{Brown}}$ holds also for w.e.'s between cofibrant objects.

not that simple

$$f_* : A/\mathcal{M} \rightleftarrows B/\mathcal{M} : f^*$$

$$\begin{tikzcd} & A \arrow[dl, "f^*y"'] \arrow[dr, "f^*z"] & \\ Y \arrow[rr, "g"'] & & Z \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dl, "y"'] \arrow[dr, "z"] & \\ Y \arrow[rr, "g"'] & & Z \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dd, bend left, "f^*y"] \\ B \arrow[d, "y"'] \\ Y \end{tikzcd} \longleftarrow\!\shortmid \begin{tikzcd} B \arrow[d, "y"] \\ Y \end{tikzcd}$$

$\longleftarrow$ preserves & reflects fibrations and weak equivalences (they are determined via $\mathrm{cod} : A/\mathcal{M} \to \mathcal{M}$ and f^* commutes with them. $(A \to X) \mapsto X$)

$$\begin{tikzcd} A \arrow[d, "x"'] \\ X \end{tikzcd} \longmapsto \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "x"'] & B \arrow[d, "f_* x"] \\ X \arrow[r] & Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

$\Rightarrow$ derived unit on cofibrant objects

$$\eta' : 1 \xrightarrow{\eta} f^* f_* \xrightarrow[\sim]{f^* r f_*} f^* R f_*$$

$\uparrow$ serves equally well

Summary:

when f is a trivial cofibration then $\mathbb{L}f_* : \mathrm{Ho}\, A/\mathcal{M} \rightleftarrows \mathrm{Ho}\, B/\mathcal{M} : \mathbb{R}f^*$ is an equivalence (η' w.e. $\Rightarrow \varepsilon'$ w.e.)

2-out-of-3: $\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[rr, bend right, "h"'] & \cdot \arrow[r, "g"] & \cdot \end{tikzcd}$ induces

$$\begin{tikzcd} & \mathrm{Ho}(B/\mathcal{M}) \arrow[dl, shift left] \arrow[dr, shift left] & \\ \mathrm{Ho}(A/\mathcal{M}) \arrow[ur, shift left] \arrow[rr, shift left] & & \mathrm{Ho}(C/\mathcal{M}) \arrow[ul, shift left] \arrow[ll, shift left] \end{tikzcd}$$

two equivalences $\Rightarrow$ so is third

cofibrant i.e. x cofibration

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "x"'] \arrow[dr, bend left, "f^* f_* x"] & B \arrow[d, "f_* x"] \\ X \arrow[r, "\eta"'] & Y \end{tikzcd}$$

$\Rightarrow$ if η' is a w.e. then so is ε' by the triangle identity:

$$\begin{tikzcd} \mathbb{R}f^* X \arrow[rr, "1"] \arrow[dr, "\eta' \mathbb{R}f^*"', "\cong"] & & \mathbb{R}f^* X \\ & \mathbb{R}f^* \mathbb{L}f_* \mathbb{R}f^* X \arrow[ur, "\mathbb{R}f^* \varepsilon'"'] & \end{tikzcd}$$

$\mathbb{R}f^* \varepsilon'$ iso

$\Downarrow$

ε' iso

Now we can finally apply Brown's lemma 🙂

ε' iso

Now we can finally apply Brown's lemma:

$$\begin{tikzcd} & \mathrm{Cyl} f \arrow[dr, "p"] & \\ A \arrow[ur, "i"] \arrow[rr, "f"'] & & B \arrow[ul, dashed, "j"'] \end{tikzcd}$$

triv. cof.   triv. cof.

w.e. between cof. obj's

$$\begin{tikzcd} & \mathrm{Cyl} f \arrow[dr, "p"] & \\ B \arrow[ur, "j"] \arrow[rr, "1"'] & & B \end{tikzcd}$$

$i_* \dashv i^*$ and $j_* \dashv j^*$ Q.e.

$\Downarrow$

$f_* \dashv f^*$ Q.e. $\Leftarrow$ $p_* \dashv p^*$ Q.e. $\square$

Remark. A different and still conceptual proof: let $A = \{ \overset{0}{\cdot} \longrightarrow \overset{1}{\cdot} \longleftarrow \overset{2}{\cdot} \}$ and equip $[A^{op}, \mathcal{M}]$ with the so-called Reedy model structure (see later) in which a span $M_0 \longleftarrow M_1 \longrightarrow M_2$ is cofibrant iff all M_0, M_1, M_2 are cofibrant and the map $M_1 \longrightarrow M_2$ is a cofibration (unlike in the proj. model str. where both maps need to be cofibrations) we still get a Quillen adjunction

$$\mathrm{colim} : [A^{op}, \mathcal{M}] \rightleftarrows \mathcal{M} : \Delta$$

and thus colim preserves w.e. between cofibrant objects. Apply to

$$\begin{tikzcd} A \arrow[r, "f", "\sim"'] & B \\ A \arrow[u, "1"] \arrow[r, "1"] \arrow[d, tail] & A \arrow[u, "f"'] \arrow[d, tail] \\ X \arrow[r, "1"] & X \end{tikzcd} \quad \overset{\mathrm{colim}}{\rightsquigarrow} \quad X \xrightarrow[\sim]{g} Y$$

Quillen bifunctors, monoidal and enriched model categories

Definition. A left adjoint bifunctor is a bifunctor

$$\frac{A \otimes B \to C}{A \to \mathrm{Hom}(B,C)}$$

$$F : \mathcal{M} \times \mathcal{N} \to \mathcal{P}$$

$-\otimes B \dashv \mathrm{Hom}(B,-)$

that, for each $N \in \mathcal{N}$, yields a left adjoint functor $F(-,N) : \mathcal{M} \to \mathcal{P}$
and, for each $M \in \mathcal{M}$, yields a left adjoint functor $F(M,-) : \mathcal{N} \to \mathcal{P}$

⤳ get $G : \mathcal{N}^{op} \times \mathcal{P} \to \mathcal{M}$ s.t. $\mathcal{P}(F(M,N),P) \cong \mathcal{M}(M, G(N,P))$
and $H : \mathcal{M}^{op} \times \mathcal{P} \to \mathcal{N}$ s.t. $\mathcal{P}(F(M,N),P) \cong \mathcal{N}(N, H(M,P))$

We may call (F,G,H) an adjunction of two variables.

Definition. A left Quillen bifunctor is a left adjoint bifunctor such that for each cofibration $i : A \rightarrowtail B$ in $\mathcal{M}$ and $j : K \rightarrowtail L$ in $\mathcal{N}$, the map $F_\lrcorner(i,j) : F(A,L) +_{F(A,K)} F(B,K) \longrightarrow F(B,L)$ as in

pushout corner map

$$\begin{tikzcd}
F(A,K) \arrow[r, "{F(i,1)}"] \arrow[d, "{F(1,j)}"'] & F(B,K) \arrow[d] \arrow[ddr, bend left, "{F(1,j)}"] & \\
F(A,L) \arrow[r] \arrow[drr, "{F(i,1)}"'] & \bullet \arrow[ul, phantom, "\ulcorner"] \arrow[dr, "{F_\lrcorner(i,j)}"] & \\
 & & F(B,L)
\end{tikzcd}$$

$N \xrightarrow{f} N'$

$$\begin{tikzcd}
G(N,P) & F(G(N,P),N) \arrow[r, "\varepsilon_N"] & P \\
G(N',P) \arrow[u, dashed, "\exists!"'] & F(G(N',P),N) \arrow[u, dashed, "\exists!"'] \arrow[r, "{F(1,f)}"'] & F(G(N',P),N') \arrow[u, "\varepsilon_{N'}"']
\end{tikzcd}$$

mate $F \Rightarrow F'$ $G' \Rightarrow G$

is a cofibration that is trivial if at least one of i,j is.

We may write this as:

$$F_\lrcorner(\mathcal{C},\mathcal{C}) \subseteq \mathcal{C}$$
$$F_\lrcorner(\mathcal{W}\cap\mathcal{C}, \mathcal{C}) \subseteq \mathcal{W}\cap\mathcal{C}$$
$$F_\lrcorner(\mathcal{C}, \mathcal{W}\cap\mathcal{C}) \subseteq \mathcal{W}\cap\mathcal{C}$$

sSet: $A \subseteq B$ $K \subseteq L$ $F = \times$

$(A \times L) +_{A \times K} (B \times K) \longrightarrow B \times L$

$\times$ is left Q. bifunctor

Remark. If $\mathcal{M}, \mathcal{N}$ are cofibrantly generated, it is enough to check this for the generating cofibrations and trivial cofibrations. This also applied to the case of Quillen functors (easier).

Remark. This easily generalizes to $\underline{n}$ variables giving as special cases:

$n=1$: left Quillen functor
$n=2$: left Quillen bifunctor

Lemma. F is left Quillen $\Longleftrightarrow$ G is right Quillen, i.e.

$$G_r(\mathcal{C},\mathcal{F}) \subseteq \mathcal{F}$$
$$G_r(\mathcal{W}\cap\mathcal{C},\mathcal{F}) \subseteq \mathcal{W}\cap\mathcal{F}$$
$$G_r(\mathcal{C},\mathcal{W}\cap\mathcal{F}) \subseteq \mathcal{W}\cap\mathcal{F}$$

here I decided to denote fibrations in $\mathcal{N}^{op}$ by $\mathcal{C}$ since they are cofibrations in $\mathcal{N}$

Proof. This boils down to $F_\lrcorner(i,j) \boxslash p \Longleftrightarrow i \boxslash G_r(j,p)$ that must be checked and is tedious but completely elementary. □

Definition. A monoidal model category is a category with a monoidal structure, i.e. $\otimes$ is a left biadjoint

$$\frac{\dfrac{A \otimes B \to C}{A \to \mathrm{hom}_r(B,C)}}{B \to \mathrm{hom}_\ell(A,C)}$$

$$\frac{\frac{A\otimes B}{A \to \hom_r(B,C}}{B \to \hom_\ell(A,C}$$

Definition. A monoidal model category is a category with

- a closed monoidal structure, i.e. $\otimes$ is a left biadjoint with right adjoints $\hom_r$ and $\hom_\ell$

  symmetric for us $\to$ $\hom_r = \hom_\ell = \{\ ,\ \}$

- a model structure

such that

- the unit of the monoidal structure S is either cofibrant or, more generally, the cofibrant replacement $S^c \xrightarrow{\sim} S$ tensors with all cofibrant objects to w.e.'s

$$S^c \otimes X \xrightarrow{\sim} S\otimes X \cong X$$
$$X\otimes S^c \xrightarrow{\sim} X\otimes S \cong X$$

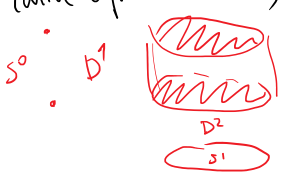
this does not depend on the choice of the cofibrant replacement in view of

- the tensor product bifunctor $\otimes : \mathcal{V}\times\mathcal{V} \to \mathcal{V}$ is left Quillen.

Examples. All Cat, Ch, $\underbrace{\text{Top}}$, sSet are monoidal model categories.

cat of compactly gen. (weak) Hausdorff spaces

(with cofibrant unit)



$i_n : S^{n-1} \to D^n$

$j_0 : \bullet \to I$

$j_n = j_0 \otimes_\lrcorner i_n$

i_{n+m} v Top

$i_n \otimes_\lrcorner i_m$ is a cofibration

$j_n \otimes_\lrcorner i_m$, $i_n \otimes_\lrcorner j_m$ are trivial cofibrations

$$\begin{tikzcd}
m & \mathbb{R} \arrow[d, "(-1)^{m-1}"'] & \mathbb{R} \arrow[l, "1"'] \arrow[d, "(-1)^n"] \\
m-1 & \mathbb{R} & \mathbb{R} \arrow[l, "1"] \\
 & n-1 & n
\end{tikzcd}$$

A functorial cylinder object for cofibrant objects: pick a cylinder object for the unit $S\in\mathcal{V}_c$: $S+S \rightarrowtail \mathrm{Cyl}\,S \xrightarrow{\sim} S$ and tensor it with $A\in\mathcal{V}_c$ to obtain

↑ a w.e. between cofibrant objects

$$\begin{tikzcd}
(S+S)\otimes A \arrow[r, tail] \arrow[d, equal] & \mathrm{Cyl}\,S\otimes A \arrow[r, "\sim"] \arrow[d, equal, "\text{def}"] & S\otimes A \arrow[d, equal] \\
A+A \arrow[r, tail] & \mathrm{Cyl}\,A \arrow[r, "\sim"] & A
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbb{R} \arrow[r] \arrow[d, "(-1,1)"] & 0 \arrow[d] \\
\mathbb{R}\oplus\mathbb{R} \arrow[r, tail] & \mathbb{R}\oplus\mathbb{R} \arrow[r, "\Delta"'] & \mathbb{R}
\end{tikzcd}$$

Dually a path object is $\{\mathrm{Cyl}\,S, X\}$.

Remark. tensor-hom adjunction

$$\frac{A\otimes B \to C}{A \to \mathrm{Hom}(B,C)}$$ sets of maps $\mathcal{V}(B,C) \in \mathrm{Set}$

better: objects of maps $\mathrm{Hom}(A\otimes B, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B,C))$ ← works since

$$\frac{\frac{\frac{X \to \mathrm{Hom}(A\otimes B, C)}{X\otimes A\otimes B \to C}}{X\otimes A \to \mathrm{Hom}(B,C)}}{X \to \mathrm{Hom}(A, \mathrm{Hom}(B,C))}$$

If $\mathcal{M}$ is enriched over $\mathcal{V}$, this can be generalized:

$$\mathcal{M}(K\otimes M, N) \cong \mathcal{M}(M, \{K,N\})$$ ← $\{-,-\} : \mathcal{V}^{op}\times\mathcal{M} \to \mathcal{M}$

$$\mathcal{M}(K\otimes M, N) \cong \mathcal{V}(K, \mathcal{M}(M,N))$$

$\mathcal{M}(M,\{K,N\}) \not\cong \mathcal{V}(K,\mathcal{M}(M,N))$ ← two different Hom's !

$\overset{\mathcal{V}}{}$ $\overset{\mathcal{M}}{}$ $\overset{\mathcal{M}}{}$ ← $\mathcal{M}(-,-) : \mathcal{M}^{op}\times\mathcal{M} \to \mathcal{V}$

Associativity:

$$\mathcal{M}(K\otimes(L\otimes M), N) \cong \mathcal{V}(K, \mathcal{M}(L\otimes M, N)) \cong \mathcal{V}(K, \mathcal{V}(L, \mathcal{M}(M,N)))$$
$$\cong \mathcal{V}(K\otimes L, \mathcal{M}(M,N)) \cong \mathcal{M}((K\otimes L)\otimes M, N)$$
$$\Rightarrow K\otimes(L\otimes M) \cong (K\otimes L)\otimes M$$

Unit $S\otimes M \cong M$

Definition. A $\mathcal{V}$-category $\mathcal{M}$ is said to be tensored if

$$\mathcal{V}(K, \mathcal{M}(M,N)) \cong \mathcal{M}(K \otimes M, N) \quad \text{naturally}$$

This gives a functor $\otimes : \mathcal{V}_0 \times \mathcal{M}_0 \to \mathcal{M}_0$ (in fact a $\mathcal{V}$-functor $\mathcal{V} \otimes \mathcal{M} \to \mathcal{M}$) that makes $\mathcal{M}_0$ into a "module" over $\mathcal{V}_0$. Dually $\mathcal{M}$ is cotensored if $\mathcal{V}(K, \mathcal{M}(M,N)) \cong \mathcal{M}(M, \{K,N\})$ naturally

This gives a functor $\{\ ,\ \} : \mathcal{V}_0^{op} \times \mathcal{M}_0 \to \mathcal{M}_0$

Together with the hom-functor $\mathcal{M}(-,-) : \mathcal{M}_0 \times \mathcal{M}_0 \to \mathcal{V}_0$ these yield an adjunction of two variables.

If $S^c \otimes A \xrightarrow{\sim} S \otimes A \cong A$ for $A \in \mathcal{M}_c$ and this adjunction is Quillen, we say that $\mathcal{M}$ is a model $\mathcal{V}$-category.

Again $\mathrm{Cyl}\, S \otimes A$ is a cylinder object for $A \in \mathcal{M}_c$
$\{\mathrm{Cyl}\, S, X\}$ is a path object for $X \in \mathcal{M}_f$.

Left adjoint $\mathcal{V}$-functors preserve these cylinder objects (they are colimits),
right adjoint $\mathcal{V}$-functors preserve these path objects (they are limits).

Remark. A small $\mathcal{V}$-cat st. $\mathcal{A}(A,B) \in \mathcal{V}_c$, $\mathcal{M}$ cof.-gen. model $\mathcal{V}$-cat $\Rightarrow$ $\Rightarrow$ $[\mathcal{A}, \mathcal{V}]$ is a cof. gen. model. $\mathcal{V}$-cat

Deriving bifunctors.

$F : \mathcal{M} \times \mathcal{N} \to \mathcal{P}$ left Quillen bifunctor

$\Rightarrow F : \mathcal{M}_c \times \mathcal{N}_c \to \mathcal{P}_c$ preserves weak equivalences :

$$\begin{tikzcd}
0 \arrow[d, tail] & 0 \arrow[d, tail] & {} \arrow[r, squiggly] & {} \\
M & N & &
\end{tikzcd}
\qquad
\begin{tikzcd}
F(0,0) \arrow[rr] \arrow[dd] & & F(M,0) \arrow[dl] \arrow[dd] \\
& \cdot \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, tail] & \\
F(0,N) \arrow[ur] \arrow[rr] & & F(M,N)
\end{tikzcd}
\qquad F_\lrcorner(M,N) : 0 \rightarrowtail F(M,N)$$

$$\begin{tikzcd}
M' \arrow[d, tail, "i"', "\sim"] & 0 \arrow[d, tail] & {} \arrow[r, squiggly] & {} \\
M & N & &
\end{tikzcd}
\qquad
\begin{tikzcd}
F(M',0) \arrow[rr] \arrow[dd] & & F(M,0) \arrow[dl] \arrow[dd] \\
& \cdot \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, tail, "\sim"] & \\
F(M',N) \arrow[ur] \arrow[rr] & & F(M,N)
\end{tikzcd}
\qquad F_\lrcorner(i,N) : F(M',N) \overset{\sim}{\underset{F(i,N)}{\rightarrowtail}} F(M,N)$$

now apply Brown's lemma

Thus we get

$$\begin{tikzcd}
\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \times \mathrm{Ho}(\mathcal{N}) \arrow[r] & \mathrm{Ho}(\mathcal{P}) \\
\mathcal{M} \times \mathcal{N} \arrow[u, "\gamma \times \gamma"] \arrow[r, "F(Q \times Q)"'] & \mathcal{P} \arrow[u, "\gamma"']
\end{tikzcd}$$

induced by

$$\mathbb{L}F(M,N) = F(QM, QN)$$
$$= F(M^c, N^c)$$

In particular, for a monoidal model category $\mathcal{V}$, we get

$$\otimes^{\mathbb{L}} : \mathrm{Ho}(\mathcal{V}) \times \mathrm{Ho}(\mathcal{V}) \longrightarrow \mathrm{Ho}(\mathcal{V})$$

making the homotopy category into a monoidal closed category

$$\begin{array}{ccc}
(K \otimes^{\mathbb{L}} L) \otimes^{\mathbb{L}} M & \overset{?}{\cong} & K \otimes^{\mathbb{L}} (L \otimes^{\mathbb{L}} M) \\
\| & & \| \\
Q(QK \otimes QL) \otimes QM & & QK \otimes Q(QL \otimes QM)
\end{array}$$

unit again S

the "weird" axiom : $X \in \mathcal{V}_c$

$$\begin{tikzcd}
Q(QK \otimes QL) \otimes QM \arrow[d, "q \otimes 1"', "\cong"] & QK \otimes Q(QL \otimes QM) \arrow[d, "1 \otimes q"', "\cong"] \\
(QK \otimes QL) \otimes QM \arrow[r, phantom, "\cong"] & QK \otimes (QL \otimes QM)
\end{tikzcd}$$

the "weird" axiom : $X \in \mathcal{V}_c$

$$\begin{tikzcd}
QS \otimes X \arrow[r] & S \otimes X \cong X \\
QS \otimes QX \arrow[u, "\cong", "1 \otimes q"'] \arrow[ur, "\cong", "q \otimes q"'] & \\
S \otimes^L X \arrow[u, equal] \arrow[uur, "\text{unit } \underline{\underline{\text{iso}}}"'] &
\end{tikzcd}$$

More generally, if $\mathcal{M}$ is a model $\mathcal{V}$-category, then $Ho(\mathcal{M})$ is

- a category enriched in $Ho(\mathcal{V})$
- tensored and cotensored

} need to derive the Hom's $\mathcal{M}^R(-,-)$, $\{-,-\}^R$

$\otimes^L : Ho(\mathcal{V}) \times Ho(\mathcal{M}) \to Ho(\mathcal{M})$

Weighted colimits

tensor adjunction (or any other)

$\mathrm{Hom}_R(K, \mathrm{Hom}(M,N)) \cong \mathrm{Hom}(K \otimes_R M, N)$
$[R^{op}, Ab]$ Ab Ab

$$\mathcal{V}(K, \mathcal{M}(M,N)) \cong \mathcal{M}(K \otimes M, N)$$

, e.g. abelian groups and their ⊗

how about $\otimes_R$?

$\bullet \circlearrowleft_R \longmapsto M$
$R \xrightarrow{\text{ring homo}} \mathrm{End}(M)$

R-module = $R \longrightarrow Ab$ left

$R = \bullet \circlearrowleft_R$ $R^{op} \longrightarrow Ab$ right

Hom_R = hom in $[R^{op}, Ab]$?

$\uparrow$ $\uparrow$
replace by diagrams

weight $W : \mathcal{A}^{op} \longrightarrow \mathcal{V}$ $D : \mathcal{A} \longrightarrow \mathcal{M}$ diagram

$$[\mathcal{A}^{op}, \mathcal{V}](W, \underbrace{\mathcal{M}(D-, N)}_{\mathcal{A}^{op} \xrightarrow{D^{op}} \mathcal{M}^{op} \xrightarrow{\mathcal{M}(-,N)} \mathcal{V}}) \cong \mathcal{M}(W \otimes_{\mathcal{A}} D, N)$$

Definition. The weighted colimit $W \otimes_{\mathcal{A}} D$ ... the colimit of D weighted by W is the object representing $[\mathcal{A}^{op}, \mathcal{V}](W, \mathcal{M}(D, -))$ as above.

Examples.
- The tensor product of R-modules
- The tensors ... take $\mathcal{A} = \bullet \circlearrowleft_S$ were $S \in \mathcal{V}$ is the unit

$W_* = K$
$W \otimes_{\mathcal{A}} M = K \otimes M$

- The geometric realization

tensor over Set: $K.P = \coprod_K P$

$$Set(K, Top(P, X)) \cong Top(K \cdot P, X)$$

$$sSet(K_\bullet, \underbrace{Top(|\Delta^\bullet|, X)}_{SX}) \cong Top(\underbrace{K_\bullet \cdot_\Delta |\Delta^\bullet|}_{|K_\bullet|}, X)$$

make Δ - indexed

$$\begin{tikzcd} \Delta^{op} \arrow[r, "K_\bullet"] & Set & \text{weight} \\ \Delta \arrow[r, bend left, "\text{\textquotedblleft Yoneda\textquotedblright}"] \arrow[r, "|\Delta^\bullet|"'] & Top & \text{diagram} \end{tikzcd}$$

$n \longmapsto |\Delta^n|$

- The ordinary colimits: $\mathcal{V} = Set$ $W = \Delta *$

$$\underbrace{[\mathcal{A}^{op}, Set](\Delta *, \mathcal{M}(D, N))}_{\mathrm{cone}(D,N)} \cong \mathcal{M}(\Delta * \otimes_{\mathcal{A}} D, N)$$

$$\begin{tikzcd} \Delta * (A) : & DA \arrow[r] \arrow[d] & N \arrow[d, equal] \\ & DB \arrow[r] & N \end{tikzcd}$$

The general weighted colimits can be translated to ordinary colimits

$$W \otimes_{\mathcal{A}} D = \operatorname*{colim}_{El\, W} DP \qquad El\, W \xrightarrow{P} \mathcal{A} \xrightarrow{D} \mathcal{M}$$

"take DA multiple times - once for each element of WA"

$\bullet$ $\mathcal{A}$ $W_\bullet = K$

$\#K \times \left\{ \begin{matrix} \bullet \\ \vdots \\ \bullet \end{matrix} \right.$ $El\, W$

Another point of view:

$\Delta_n = \Delta(n, -)$
$\Delta^n = \Delta(-, n)$

- $[\mathcal{A}^{op}, \mathcal{V}](W, \mathcal{M}(D, N)) \cong \mathcal{M}(W \otimes_{\mathcal{A}} D, N)$
  take $W = \mathcal{A}(-, A) = \mathcal{A}^A$ representable functor
  $\Rightarrow$ get on the left $\mathcal{M}(D, N)(A) = \mathcal{M}(DA, N)$ by Yoneda
  $\mathcal{M}(D-, N)(A) =$
  $\Rightarrow \mathcal{A}^A \otimes_{\mathcal{A}} D = DA$
- clearly $W \otimes_{\mathcal{A}} D$ preserves colimits (ordinary or weighted) in the W-variable $\Rightarrow$ for $\mathcal{V} = Set$ any weight = presheaf is a colimit of representables $\Rightarrow$ weighted colimits can be replaced by colimits (over the cat of elements)

$\longrightarrow$ $\mathcal{V} = Ab$, $\mathcal{A} = \begin{tikzcd} a \arrow[r, "\mathbb{Z}"] \arrow[loop above, "\mathbb{Z} \cdot 1"] \arrow[r, bend right] & b \arrow[loop above, "\mathbb{Z} \cdot 1_b"] \end{tikzcd} \Rightarrow [\mathcal{A}, \mathcal{M}]$ = arrows of $\mathcal{M}$

$\longrightarrow$ $\mathcal{V} = Ab$, $\mathcal{A} = \begin{tikzcd} a \arrow[loop above, "\mathbb{Z}\cdot 1"] \arrow[r, "\mathbb{Z}"] & b \arrow[loop above, "\mathbb{Z}\cdot 1_b"] \end{tikzcd}$ $\Rightarrow [\mathcal{A}, \mathcal{M}]$ = arrows of $\mathcal{M}$

$\mathcal{A} \longrightarrow [\mathcal{A}^{op}, Ab]$

$$\begin{tikzcd} a \arrow[d, "f"] & \mathcal{A}^a \arrow[d, "in"] \\ b & \mathcal{A}^b \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{A}^a \arrow[d, "in", shift right] \arrow[d, "0", shift left] \\ \mathcal{A}^b \arrow[d] \\ \mathcal{A}^b/\mathcal{A}^a \end{tikzcd}$$

colimit

$\mathcal{A}^a = \mathbb{Z} \leftarrow 0$

$\mathcal{A}^b = \mathbb{Z} \xleftarrow{1} \mathbb{Z}$

$\mathcal{A}^b/\mathcal{A}^a = 0 \leftarrow \mathbb{Z}$

$\rightsquigarrow$

$\mathcal{A}^a \otimes_{\mathcal{A}} D = Da$

$\mathcal{A}^b \otimes_{\mathcal{A}} D = Db$

$\mathcal{A}^b/\mathcal{A}^a \otimes_{\mathcal{A}} D = \mathrm{coker}(Da \to Db)$

coeq's: $\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \end{tikzcd}$

$\mathcal{A}^a = \{1\} \leftleftarrows \emptyset$

$f_* \downarrow \downarrow g_*$

$\mathcal{A}^b = \{f, g\} \leftleftarrows \{1\}$

$\mathrm{coeq} = * \leftleftarrows *$

pushouts: $\begin{tikzcd} a \arrow[r] \arrow[d] & b \\ c \end{tikzcd}$ $\Delta * = \mathrm{colim}(\mathcal{A}^a \rightrightarrows \mathcal{A}^b + \mathcal{A}^c)$

$$\begin{tikzcd} * & * \arrow[l] \\ * \arrow[u] \end{tikzcd} = \mathrm{colim}\left(\begin{tikzcd} * & \emptyset \arrow[l] \\ \emptyset \arrow[u] \end{tikzcd} \rightrightarrows \left(\begin{tikzcd} * & * \arrow[l] \\ \emptyset \arrow[u] \end{tikzcd} + \begin{tikzcd} * & \emptyset \arrow[l] \\ * \arrow[u] \end{tikzcd} \right) \right)$$

- The conical colimits "=" the ordinary colimits in the underlying category $\mathcal{M}_0$: $\cdot S : Set \rightleftarrows \mathcal{V}_0 : \mathcal{V}_0(S, -)$ tensor-hom adjunction

$\mathbb{Z} : Set \rightleftarrows Ab : Ab_0(\mathbb{Z}, -)$

free ab. gp

$\mathcal{A}$ an ordinary category

$\to \mathcal{A}\cdot S$ a $\mathcal{V}$-category that admits a canonical weight $\Delta * \cdot S = \Delta S$

$$[\mathcal{A}^{op}, Set](\Delta *, \mathcal{M}_0(D, N)) \cong \mathcal{M}_0(\Delta * \cdot_{\mathcal{A}} D, N)$$

$$\cong$$

$$[\mathcal{A}^{op}, \mathcal{V}_0](\Delta S, \mathcal{M}(D, N))$$

$$\cong$$

$$[(\mathcal{A}\cdot S)^{op}, \mathcal{V}]_0(\Delta S, \mathcal{M}(D, N))$$

this is almost saying that

$\Delta * \cdot_{\mathcal{A}} D = \Delta S \otimes_{\mathcal{A}\cdot S} D$

up to the index 0

Lemma. If $\mathcal{M}$ has cotensors, this works in the enriched sense

Proof. We need $[(\mathcal{A}\cdot S)^{op}, \mathcal{V}](\Delta S, \mathcal{M}(D, N)) \cong \mathcal{M}(\Delta * \cdot_{\mathcal{A}} D, N)$

$\overset{\text{Yoneda}}{\Longleftrightarrow}$ $\mathcal{V}_0(K, [(\mathcal{A}\cdot S)^{op}, \mathcal{V}](\Delta S, \mathcal{M}(D, N))) \cong \mathcal{V}_0(K, \mathcal{M}(\Delta * \cdot_{\mathcal{A}} D, N))$

$K = S \otimes K$

$$\mathcal{V}_0(S, [(\mathcal{A}\cdot S)^{op}, \mathcal{V}](\Delta S, \mathcal{M}(D, N^K))) \qquad \mathcal{V}_0(S, \mathcal{M}(\Delta * \cdot_{\mathcal{A}} D, N^K))$$

$$[(\mathcal{A}\cdot S)^{op}, \mathcal{V}]_0(\Delta S, \mathcal{M}(D, N^K)) \qquad \mathcal{M}_0(\Delta * \cdot_{\mathcal{A}} D, N^K)$$

$\square$

Summary. ordinary colimits $\overset{\text{often}}{=}$ conical colimits $\subseteq$ weighted colimits $\supseteq$ tensors

Construction. $W \otimes_{\mathcal{A}} D$ can be constructed as the coequalizer of

$$\sum_{B,C} WC \otimes \mathcal{A}(B,C) \otimes DB \underset{1 \otimes act}{\overset{act \otimes 1}{\rightrightarrows}} \sum_{A} WA \otimes DA$$

like $M \otimes_R N$

the coend

Explanation: $[\mathcal{A}^{op}, \mathcal{V}](W, \mathcal{M}(D, N))$ is the equalizer of

$[\mathcal{A}^{op}, \mathcal{V}](F, G) = \int_{\mathcal{A}} \underbrace{\mathcal{V}(F, G)}_{\mathcal{A}^{op} \times \mathcal{A} \to \mathcal{V}}$

$$\prod_{A} \mathcal{V}(WA, \mathcal{M}(DA, N)) \rightrightarrows \prod_{B,C} \{\mathcal{A}(B,C), \mathcal{V}(WC, \mathcal{M}(DB, N))\}$$

$$\begin{tikzcd} WB \arrow[r] & \sim B \\ WC \arrow[u] \arrow[r] \arrow[ur] & \sim C \arrow[u] \end{tikzcd}$$

$$\cong \qquad\qquad \cong$$

$$\prod \mathcal{M}(WA \otimes DA, N) \qquad \prod \mathcal{M}(WC \otimes \mathcal{A}(B,C) \otimes DB, N)$$

A $\qquad\qquad\qquad\qquad$ $B\,C$

$$\begin{tikzcd} \prod_A \mathcal{M}(WA \otimes DA, N) \arrow[u, phantom, "\cong" rotate=90] & \prod_{B,C} \mathcal{M}(WC \otimes \mathcal{A}(B,C) \otimes DB, N) \arrow[u, phantom, "\cong" rotate=90] \end{tikzcd}$$

obtained from the paraller pair from the construction upon applying $\mathcal{M}(-,N)$ that turns colimits into limits. □

Example. $\mathcal{M}$ a simplicial category; then the geometric realization is a functor $|-| : s\mathcal{M} \to \mathcal{M}$ given by

$|X| = \Delta^{\bullet} \otimes_{\Delta} X$ $\quad$ $\Delta^{\bullet} : \Delta \to sSet$ $\quad$ $X : \Delta^{op} \to \mathcal{M}$ $\quad$ is weight diag $\quad$ the colimit weighted by $\Delta^{\bullet} : \Delta \to sSet$

$\quad$ " Yoneda embedding

We will see that $\Delta^{\bullet}$ is Reedy cofibrant.

The $- \otimes_{\Delta} - : [\Delta, sSet] \times [\Delta^{op}, \mathcal{M}] \to \mathcal{M}$ is left Quillen w.r.t. Reedy structures $\Rightarrow$ $|-|$ homotopy invariant on Reedy cofibrant diag's

E.g. for $\mathcal{M} = sSet$ $\Rightarrow$ $|X| = \operatorname{diag} X$

$$|X| = \int^n \Delta^n \otimes \underbrace{X_n}_{\int^k \Delta^k \otimes X_{nk}}$$

$$= \int^{n,k} (\Delta^n \times \Delta^k) \otimes X_{nk}$$

$$= \int^{n,k} \Delta \times \Delta(\operatorname{diag} -, (n,k)) \otimes X_{nk}$$

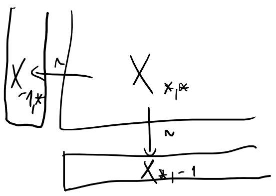
$$= X \circ \operatorname{diag}$$

$$\begin{tikzcd} \Delta^{op} \arrow[r, "\operatorname{diag}"] \arrow[rr, bend right, "X \circ \operatorname{diag}"'] & \Delta^{op} \times \Delta^{op} \arrow[r, "X"] & Set \end{tikzcd}$$

Any X is Reedy cofibrant

If all augmentations $X_n \xrightarrow{\sim} X_{-1}$ are w.e.

$\Rightarrow$ $X \xrightarrow{\sim} \operatorname{cst} X_{-1}$ $\Rightarrow$ $|X| \xrightarrow{\sim} |\operatorname{cst} X_{-1}| = X_{-1}$



"balancing"

Further interesting topics. Restriction and extension of scalars

$F_! : [\mathcal{A}, \mathcal{M}] \rightleftarrows [\mathcal{B}, \mathcal{M}] : F^*$ $\qquad$ $F : \mathcal{A} \to \mathcal{B}$

$\operatorname{Lan}_F \nearrow$ $\qquad$ $F^* \;=\; \mathcal{B}(F,-) \otimes_{\mathcal{B}} -$

$\mathcal{B}(-,F) \otimes_{\mathcal{A}} -$ $\qquad\qquad$ ${}_R S_S \otimes_S -$

${}_S S_R \otimes_R -$

Reedy model categories, framings

$\mathcal{M}(M,N)$ $\overset{?}{\downarrow}(\neq)$

Motivation. Any model category $\mathcal{M}$ is in some weak sense enriched over sSet and, as a result, $Ho(\mathcal{M})$ will be enriched over $Ho(sSet)$.

Start with an honest sSet-model category = simplicial model category

- since $K \in sSet$ is a colimit $K = \underset{(n,\, \Delta^n \to K)}{\mathrm{colim}}\, \Delta^n$

  we have $K \otimes M = \mathrm{colim}\, (\Delta^n \otimes M)$ and it is enough to give $\Delta^n \otimes M$
- clearly $\Delta^0 \otimes M \cong M$, since Δ^0 is the monoidal unit
- what is the essential property of $\Delta^1 \otimes M$?

  it is a cylinder ! ... at least for $M \in \mathcal{M}_c$:

$$\Delta^0 + \Delta^0 \overset{[d^0,d^1]}{\rightarrowtail} \Delta^1 \underset{\sim}{\xrightarrow{s^0}} \Delta^0 \quad \rightsquigarrow \quad M + M \rightarrowtail \Delta^1 \otimes M \xrightarrow{\sim} M$$

- what about $\Delta^2 \otimes M$?

$$\begin{tikzcd}
 & & \Delta^0 \arrow[dl] \arrow[dr, "d^0"] & & \\
 & \Delta^1 \arrow[r, "d^1"'] & \Delta^2 & \Delta^1 \arrow[l, "d^0"] & \\
\Delta^0 \arrow[ur] \arrow[rr, "d^1"'] & & \Delta^1 \arrow[u, "d^2"] & & \Delta^0 \arrow[ll, "d^0"] \arrow[ul, "d^1"]
\end{tikzcd}$$

, more compactly $\partial\Delta^2 \to \Delta^2$, $\partial\Delta^2 = L_2\Delta^\bullet$

$\rightsquigarrow L_2\Delta^\bullet \otimes M \rightarrowtail \Delta^2 \otimes M \xrightarrow{\sim} M_2\Delta^\bullet \otimes M$

$$\begin{tikzcd}
 & \Delta^2 \arrow[dl, "s^0"'] \arrow[dr, "s^1"] & \\
\Delta^1 \arrow[dr, "s^0"'] & & \Delta^1 \arrow[dl, "s^0"] \\
 & \Delta^0 &
\end{tikzcd}$$

, more compactly $\Delta^2 \xrightarrow{\sim} M_2\Delta^\bullet$

$\forall$ $\Delta^\bullet : \Delta \longrightarrow sSet$ a cosimplicial object in sSet

$\rightsquigarrow \Delta^\bullet \otimes M : \Delta \to \mathcal{M}$ a cosimplicial object in $\mathcal{M}$

- need some calculus of such diagrams $\Delta \to sSet$, $\Delta \to \mathcal{M}$

  $\to$ Reedy categories, Reedy model structures

A Reedy category has two kinds of maps – direct and inverse (like d^i and s^i in Δ)

Definition A direct category is a category $\mathcal{A}$ together with a functor $\deg : \mathcal{A} \longrightarrow \lambda$ that satisfies $f : A \to B \rightsquigarrow f = 1 \Leftrightarrow \deg A = \deg B$

λ ordinal

An inverse category is a dual notion, i.e. a category $\mathcal{A}$ together with a functor $\deg : \mathcal{A}^{op} \longrightarrow \lambda$ satisfying $f = 1 \Leftrightarrow \deg A = \deg B$.

A Reedy category is a category $\mathcal{A}$ together with two subcategories $\mathcal{A}^+$, $\mathcal{A}_-$ and a function $\deg : \mathrm{ob}\, \mathcal{A} \to \lambda$ that

- makes $\mathcal{A}^+$ into a direct category
- makes $\mathcal{A}_-$ into an inverse category
- any morphism has a unique decomposition:

$$\sum_{A \in \mathcal{A}} \mathcal{A}^+(A,C) \times \mathcal{A}_-(B,A) \xrightarrow[\cong]{\circ} \mathcal{A}(B,C)$$

as a composition of an inverse and a direct morphism

The Yoneda lemma gives $X \cong \mathcal{A}(-,-) \cdot_{\mathcal{A}} X$ and we will describe a way of building X by decomposing $\mathcal{A}(-,-)$. The axioms actually give

$$\mathcal{A}(-,-) = \sum_{A \in \mathcal{A}} \mathcal{A}^+(A,-) \times \mathcal{A}_-(-,A) = \sum_{A \in \mathcal{A}} \mathcal{A}^+_A \times \mathcal{A}^A_-$$

(but only of functors $\mathcal{A}_-^{op} \times \mathcal{A}^+ \to Set$!)

we factor it into

but in order to understand $\mathcal{A}(-,-)$ we factor it into

$\mathrm{sk}_n \mathcal{A}(-,-)$ ... maps that factor through some A of degree $\leq n$

and clearly we have

$$\mathcal{A}(-,-) = \operatorname*{colim}_{n<\infty} \mathrm{sk}_n \mathcal{A}(-,-)$$

so that it remains to study the difference between $\mathrm{sk}_n \mathcal{A}$ and $\mathrm{sk}_{n-1} \mathcal{A}$ or, better for n limit, $\mathrm{sk}_{<n} \mathcal{A} = \operatorname*{colim}_{i<n} \mathrm{sk}_i \mathcal{A}$.

Quite clearly, we have a pushout square

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathrm{sk}_{<n} \mathcal{A} \arrow[d] \\
\sum\limits_{A\in\mathcal{A}_n} \mathcal{A}^+_A \times \mathcal{A}^A_- \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \mathrm{sk}_n \mathcal{A}
\end{tikzcd}$$

with A ranging over all objects of degree n

However, it will be crucial to express this in terms of representable functors on $\mathcal{A}$, rather than on $\mathcal{A}^+$, $\mathcal{A}_-$.

Examples.

- any ordinal is a direct category
- $$\begin{tikzcd} & \bullet \\ \bullet \arrow[ur, "+"] \arrow[dr, "+"'] & \\ & \bullet \end{tikzcd}$$ is a direct category
- $\bullet \xleftarrow{-} \bullet \xrightarrow{+} \bullet$ is a Reedy category
- Δ^+ is a direct category (non-empty finite ordinals + monos)
- Δ_- is an inverse category (non-empty finite ordinals + epis)
- Δ is a Reedy category

Notation. We denote

- $\mathcal{A}_A = \mathcal{A}(A,-) \in [\mathcal{A}, \mathrm{Set}]$ the covariant representable
- $\mathcal{A}^A = \mathcal{A}(-,A) \in [\mathcal{A}^{op}, \mathrm{Set}]$ the contravariant representable ... think Δ^n

We further define two subfunctors

- $i_A : \partial \mathcal{A}_A \subseteq \mathcal{A}_A$ of maps that factor through an object of lower degree.
- $i^A : \partial \mathcal{A}^A \subseteq \mathcal{A}^A$ of maps that factor through an object of lower degree.

In the decomposition

$$\begin{aligned}
\mathcal{A}_A = \mathcal{A}(A,-) &= \sum \mathcal{A}^+(B,-) \times \mathcal{A}_-(A,B) \\
\partial \mathcal{A}_A \qquad\qquad &= \sum \mathcal{A}^+(B,-) \times \partial \mathcal{A}_-(A,B)
\end{aligned}$$

↳ only 1 is excluded

This means that there is a pushout (coproduct)

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \partial \mathcal{A}_A \arrow[d, "i_A"] \\
\mathcal{A}^+_A \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \mathcal{A}_A
\end{tikzcd}$$

Dually, we get

$$\begin{aligned}
\mathcal{A}^A = \mathcal{A}(-,A) &= \sum \mathcal{A}^+(B,A) \times \mathcal{A}_-(-,B) \\
\partial \mathcal{A}^A \qquad\qquad &= \sum \partial \mathcal{A}^+(B,A) \times \mathcal{A}_-(-,B)
\end{aligned}$$

and a pushout (coproduct)

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \partial \mathcal{A}^A \arrow[d, "i^A"] \\
\mathcal{A}^A_- \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \mathcal{A}^A
\end{tikzcd}$$

Putting these together yields that

0×0 $\qquad$ each square pushout $\Longrightarrow$ $\qquad$ $\partial \mathcal{A}_A \times \partial \mathcal{A}^A$

$$\begin{tikzcd} & 0\times 0 \arrow[dl] \arrow[dr] & \\ 0\times\mathcal{A}^A_- \arrow[r] \arrow[dr] & \cdot \arrow[d] \arrow[u, phantom, "\ulcorner"] & \mathcal{A}^+_A\times 0 \arrow[l] \arrow[dl] \\ & \mathcal{A}^+_A\times\mathcal{A}^A_- & \end{tikzcd}$$

zero

gives as a pushout $\Longrightarrow$

$$\begin{tikzcd} \partial\mathcal{A}_A\times\mathcal{A}^A \arrow[r] \arrow[dr] & \cdot \arrow[d, "i_A\times\lrcorner\, i^A"] \arrow[u, phantom, "\ulcorner"] & \mathcal{A}_A\times\partial\mathcal{A}^A \arrow[l] \arrow[dl] \\ & \mathcal{A}_A\times\mathcal{A}^A & \end{tikzcd}$$

Now we get a diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \sum\limits_A \partial\mathcal{A}_A\times\mathcal{A}^A \underset{\partial\mathcal{A}_A\cdot\partial\mathcal{A}^A}{+} \mathcal{A}_A\times\partial\mathcal{A}^A \arrow[r] \arrow[d, "\sum\limits_A i_A\times\lrcorner\, i^A"] & \mathrm{sk}_{<n}\mathcal{A}(-,-) \arrow[d] \\ \sum\limits_A \mathcal{A}^+_A\times\mathcal{A}^A_- \arrow[r] & \sum\limits_A \mathcal{A}_A\times\mathcal{A}^A \arrow[r] \arrow[u, phantom, "\ulcorner"] & \mathrm{sk}_n\mathcal{A}(-,-) \end{tikzcd}$$

sums range over $A\in\mathcal{A}$ with $\deg A = n$

in which the outer square is a pushout $\Rightarrow$ so is the one on the right but now in $[\mathcal{A}^{op}\times\mathcal{A}, Set]$?

Upon applying $-\cdot_{\mathcal{A}} X$, we denote:

$$\begin{tikzcd} \partial\mathcal{A}^A\cdot_{\mathcal{A}} X \arrow[r, "i^A\cdot_{\mathcal{A}} X"] \arrow[d, equal, "\text{def}"] & \mathcal{A}^A\cdot_{\mathcal{A}} X \arrow[d, equal] \\ L_A X \arrow[r, "\partial_A X"] & X_A \end{tikzcd}$$

<u>Example</u>. $\mathcal{A}=\Delta^{op}$, $\mathcal{M}=Set$
$\Rightarrow L_n X\subseteq X_n$ the subset of deg. simplices (needs a bit of work).

<u>Theorem</u>. For any $X\in[\mathcal{A},\mathcal{M}]$ we get $0=\mathrm{sk}_{-1}X$, $X=\mathrm{colim}\,\mathrm{sk}_n X$ and

$$\begin{tikzcd} \sum\limits_A \partial\mathcal{A}_A\cdot X_A \underset{\partial\mathcal{A}_A\cdot L_AX}{+} \mathcal{A}_A\cdot L_AX \arrow[r] \arrow[d, "i_A\cdot\lrcorner\,\partial_A X"] & \mathrm{sk}_{<n}X \arrow[d] \\ \sum \mathcal{A}_A\cdot X_A \arrow[r] & \mathrm{sk}_n X \arrow[u, phantom, "\ulcorner"] \end{tikzcd}$$

<u>Important special case</u>.

When $\mathcal{A}$ is direct, we have $\partial\mathcal{A}_A=0$ and, consequently,

$$\begin{tikzcd} \sum\limits_A \mathcal{A}_A\cdot L_AX \arrow[r] \arrow[d] & \mathrm{sk}_{<n}X \arrow[d] \\ \sum\limits_{\mathcal{A}} \mathcal{A}_A\cdot X_A \arrow[r] & \mathrm{sk}_n X \arrow[u, phantom, "\ulcorner"] \end{tikzcd}$$

so that: $\forall A\in\mathcal{A}$: $\partial_A X: L_AX\rightarrowtail X_A$ cofibration

$\Rightarrow$ X is cofibrant in the projective model structure

More generally, for $f: X\to Y$ we denote $\partial_A f = i^A\cdot_{\mathcal{A}}\lrcorner\, f$, i.e. the pushout corner map in

$$\begin{tikzcd} L_AX \arrow[r] \arrow[d, "\partial_A X"'] & L_AY \arrow[d] \arrow[dd, bend left, "\partial_A Y"] \\ X_A \arrow[r] \arrow[dr] & \cdot \arrow[d, "\partial_A f"] \arrow[u, phantom, "\ulcorner"] \\ & Y_A \end{tikzcd}$$

$$\partial_A f = \begin{tikzcd} \partial\mathcal{A}^A \arrow[d, "i^A"] \\ \mathcal{A}^A \end{tikzcd} \cdot_{\mathcal{A}}\lrcorner \begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd}$$

<u>Theorem</u>. For any map $f: X\to Y$ of $[\mathcal{A},\mathcal{M}]$ we get $X=\mathrm{sk}^X_{-1}Y$, $Y=\mathrm{colim}_{n<\infty}\,\mathrm{sk}^X_n Y$ and

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\sum i_A\cdot_{\mathcal{A}}\lrcorner\,\partial_A f"'] & \mathrm{sk}^X_{<n}Y \arrow[d] \\ \cdot \arrow[r] & \mathrm{sk}^X_n Y \arrow[u, phantom, "\ulcorner"] \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[d, dashed] \\ \mathrm{sk}_n\mathcal{A}(-,-) \arrow[d, dashed] \\ \mathcal{A}(-,-) \end{tikzcd} \cdot_{\mathcal{A}}\lrcorner \begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd} = \begin{tikzcd} X \arrow[d, dashed] \\ \mathrm{sk}^X_n Y = \mathrm{sk}_n Y \underset{\mathrm{sk}_n X}{+} X \arrow[d, dashed] \\ Y \end{tikzcd}$$

<u>Theorem</u>. For a map $f: X\to Y$ of $[\mathcal{A},\mathcal{M}]$ TFAE

- $\forall A$: $\partial_A f$ is a (trivial) cofibration
- $f\in$ cell$\{i_A\cdot i \mid A\in\mathcal{A},\ i\in\mathcal{M}$ a (trivial) cofibration$\}$

$$\begin{tikzcd} \partial\mathcal{A}_A\cdot L \underset{\partial\mathcal{A}_A\cdot K}{+} \mathcal{A}_A\cdot K \arrow[d] \\ \mathcal{A}_A\cdot L \end{tikzcd}$$

- $\forall A$: $\partial_A f$ is a (trivial) cofibration
- $f \in \operatorname{cell}\{ i_A \cdot_\lrcorner i \mid A \in \mathcal{A},\ i \in \mathcal{M}$ a (trivial) cofibration$\}$

$$\begin{tikzcd} \partial\mathcal{A}_A \cdot K \arrow[d] \\ \mathcal{A}_A \cdot L \end{tikzcd}$$

These maps are called Reedy (trivial) cofibrations.

Proof. The implication "$\Downarrow$" is the previous theorem.
For the implication "$\Uparrow$" observe that ∂_A commutes with all cellular constructions so that it is enough to study $\partial_A(i_B \cdot_\lrcorner i)$:

$$\begin{array}{c} \partial\mathcal{A}^A \\ \downarrow i^A \\ \mathcal{A}^A \end{array} \cdot_{\mathcal{A}\lrcorner} \left(\begin{array}{c} \partial\mathcal{A}_B \\ \downarrow i_B \\ \mathcal{A}_B \end{array} \cdot_\lrcorner \begin{array}{c} K \\ \downarrow i \\ L \end{array} \right) = \underbrace{\left(\begin{array}{c} \partial\mathcal{A}^A \\ \downarrow i^A \\ \mathcal{A}^A \end{array} \times_{\mathcal{A}\lrcorner} \begin{array}{c} \partial\mathcal{A}_B \\ \downarrow i_B \\ \mathcal{A}_B \end{array} \right)}_{\text{either iso for } A \neq B} \cdot_\lrcorner \begin{array}{c} K \\ \downarrow i \\ L \end{array}$$

$\mathcal{A}_{\neq 1}(B,A) \hookrightarrow \mathcal{A}(B,A)$ for $A = B$

$\Downarrow$

pushout of $0 \to 1$

$\underbrace{\qquad\qquad}_{\text{pushout of } i}$ ∎

Dually, we denote

$$\begin{tikzcd} M_A X = \{\partial\mathcal{A}_A, X\}^{\mathcal{A}} \\ X_A = \{\mathcal{A}_A, X\}^{\mathcal{A}} \arrow[u, "\delta_A X"', "\{i_A, X\}^{\mathcal{A}}"] \end{tikzcd}$$

and more generally for $f\colon X \to Y$:

$\delta_A f = \{i_A, f\}^{\mathcal{A}}_\ulcorner$ = pullback corner map in

$$\begin{tikzcd} X_A \arrow[r] \arrow[d] & M_A X \arrow[d] \\ Y_A \arrow[r] & M_A Y \end{tikzcd}$$

Theorem. TFAE

- $\forall A \in \mathcal{A}$: $\delta_A f$ is a (trivial) fibration
- $f \in \operatorname{cocell}\{ \{i^A, p\}_\ulcorner \mid A \in \mathcal{A},\ p \in \mathcal{M}$ a (trivial) fibration$\}$

$\rightsquigarrow$ Reedy (trivial) fibrations.

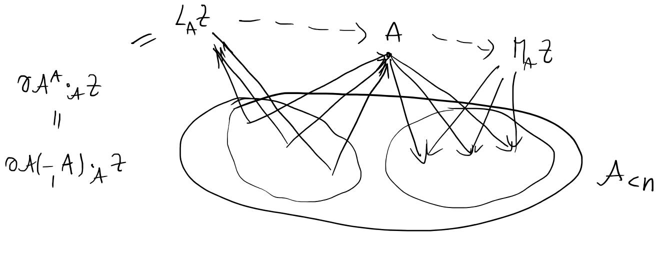
Theorem. There is a model structure on $[\mathcal{A}, \mathcal{M}]$, called the Reedy model structure with $\mathcal{C}$ = Reedy cofibrations, $\mathcal{F}$ = Reedy fibrations, $\mathcal{W}$ = pointwise weak equivalences.

Proof. We need to show that $\mathcal{W} \cap \mathcal{C}$ are exactly the Reedy trivial cofibrations. Easily $\underset{\text{trivial cofibration}}{\overset{\uparrow}{i_A \cdot_\lrcorner j}} \boxslash \mathcal{F} \Leftarrow j \boxslash \underbrace{\{i_A, \mathcal{F}\}_{\mathcal{A}\lrcorner}}_{\text{fibrations by definition}}$.

inductively

$$\begin{tikzcd} L_A X \arrow[r, tail, "\sim"] \arrow[d] & L_A Y \arrow[d] \arrow[dr] & \\ X_A \arrow[r, tail, "\sim"] & \bullet \arrow[r] & Y_A \end{tikzcd}$$

The factorizations are produced inductively using the following idea:



to give an extension of Z from $\mathcal{A}_{<n}$ to $\mathcal{A}_{\leq n}$ we need $\forall A$ of degree n to factor

$$\begin{tikzcd} L_A Z \arrow[r] \arrow[rr, bend right, "\text{obtained from } Z|_{\mathcal{A}_{<n}}"'] & Z_A \arrow[r] & M_A Z \end{tikzcd}$$

$$\begin{tikzcd} L_A \arrow[rr, bend right, "\text{obtained from } Z|_{A<n}"'] & & {} \end{tikzcd}$$

Now
$$\begin{tikzcd}
L_A X \arrow[rr] \arrow[d] & & L_A Z \arrow[rr] \arrow[dl] & & L_A Y \arrow[d] \\
X_A \arrow[r] & \bullet \arrow[r, tail] \arrow[ul, phantom, "\ulcorner"] & Z_A \arrow[r, two heads, "\sim"] & \bullet \arrow[r] \arrow[dl] \arrow[dr, phantom, "\lrcorner"] & Y_A \arrow[d] \\
M_A X \arrow[rr] \arrow[u, leftarrow] & & M_A Z \arrow[rr] & & M_A Y
\end{tikzcd}$$

$\square$

Application. • Properness in $\mathcal{M}_c / \mathcal{M}_f$ differently.

• Given $A \in \mathcal{M}_c$ consider $\mathrm{cst}\, A \in [\Delta, \mathcal{M}]$ and consider its Reedy cofibrant replacement $A^* \in [\Delta, \mathcal{M}]$

$$0 \rightarrowtail A^* \xrightarrow{\sim} \mathrm{cst}\, A$$

– we may achieve that $A^*_0 = A$ and then A^* is called a cosimplicial frame on A. Any two are related by a zig-zag of cosimplicial frames. Thus for $X \in \mathcal{M}_f$

$$\mathcal{M}(A^*, X) \in sSet$$

Dually $\mathcal{M}(A, X_*) \in sSet$ via a simplicial frame on X.