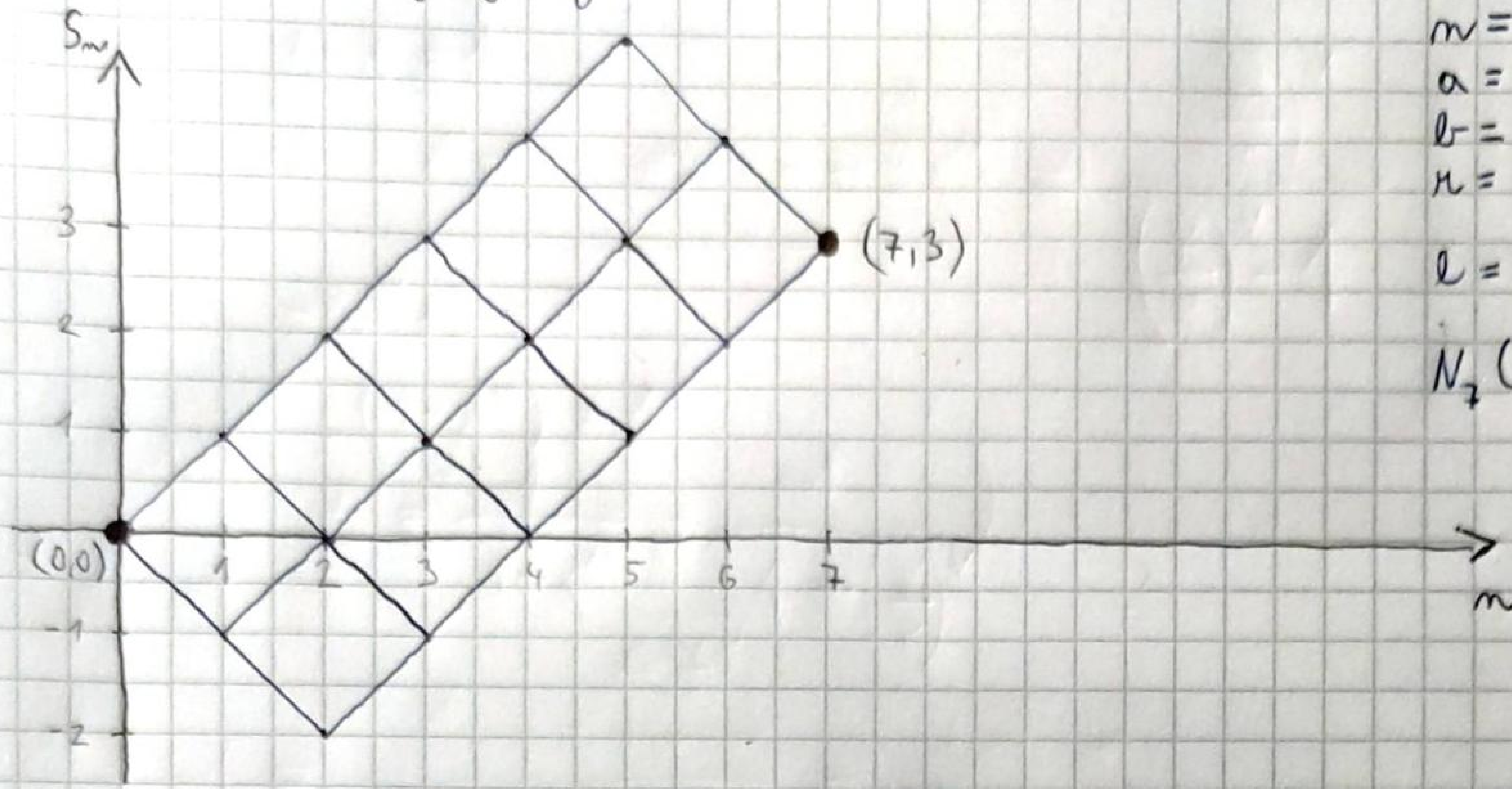
PR 3.1.

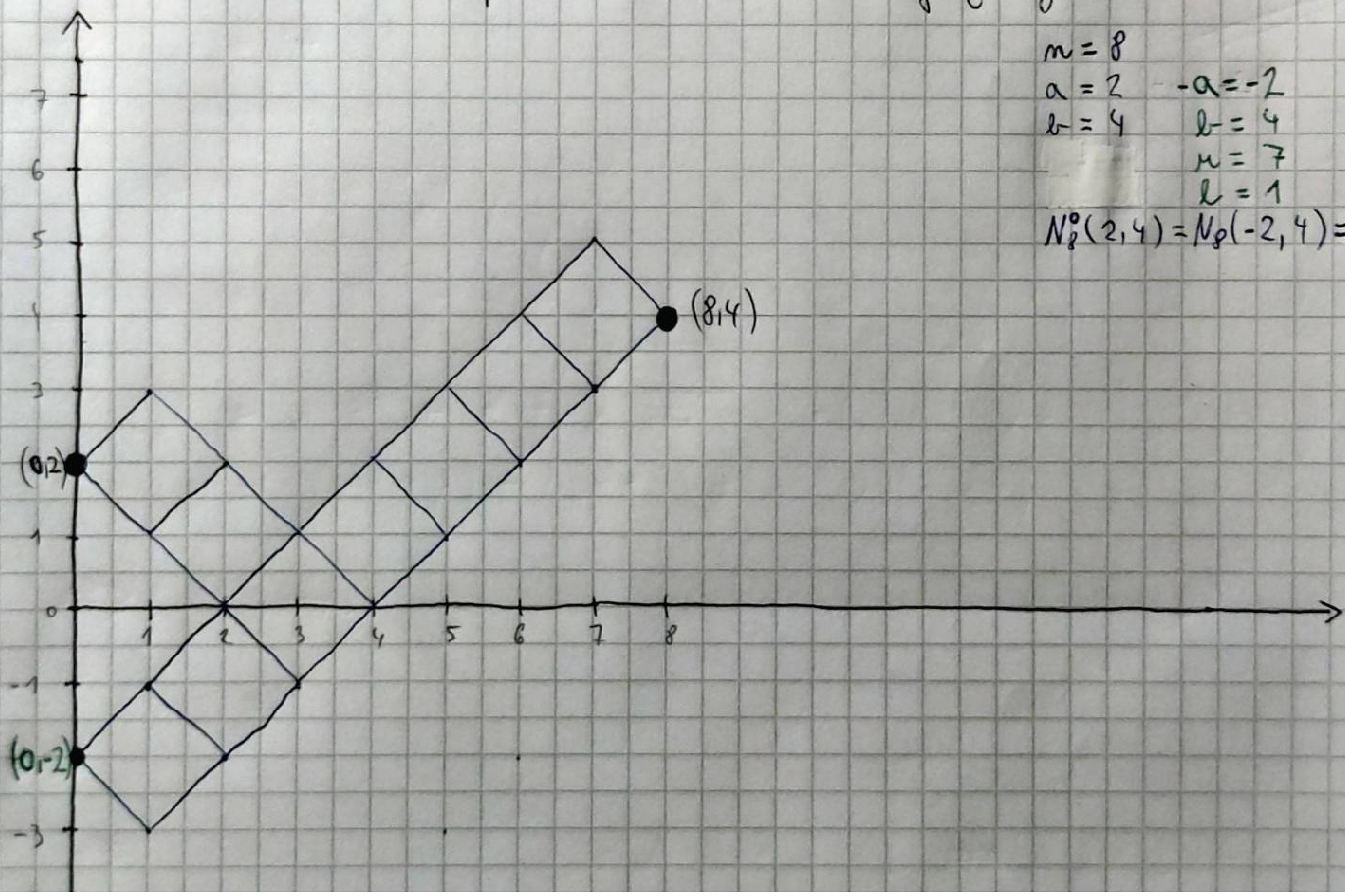
Nájdite počet ciest z bodu (0,0) do bodu (7,3) a znázornite graficky.



$n = 7$
$a = 0$
$b = 3$
$r = \frac{n+b-a}{2} = 5$
$l = 2$
$N_7(0,3) = \binom{7}{5} = 21$

PR 3.2.

Nájdite počet ciest z bodu (0,2) do bodu (8,4), ktoré navštívia počiatok a znázornite graficky



$n = 8$
$a = 2$ $\quad -a = -2$
$b = 4$ $\quad b = 4$
$r = 7$
$l = 1$
$N_8^0(2,4) = N_8(-2,4) = \binom{8}{7} = 8$

PR 3.3. Nájdi generujúce funkcie postupností

- $\sum |a_n| < \infty \Rightarrow \sum a_n < \infty$
- Nech $\{a_n\}$ je nerastúca postupnosť s kladnými čl.

$$\sum_{1}^{\infty} (-1)^{n-1} a_n < \infty \Leftrightarrow \lim_{n \to \infty} a_n = 0$$

a) $\{1, -1, 1, -1, \dots\} = \{a_n\}_{n=0}^{\infty}$

$$1 - s + s^2 - s^3 + s^4 - \dots = \sum_{n=0}^{\infty} (-1)^n \cdot s^n \overset{|s|<1}{\underset{\substack{a_0 = 1 \\ q = -s}}{=}} \frac{1}{1+s}$$

$$\Rightarrow G_a(s) = \frac{1}{1+s}$$

$$\to a_n = \frac{G_a^{(n)}(0)}{n!} = \begin{cases} n=0: & 1 \\ n=1: & \frac{-1(1+0)^{-2}}{1} = -1 \\ n=2: & \frac{2(1+0)^{-3}}{2} = 1 \\ & \dots \end{cases}$$

b) $\{1, 2, 3, 4, 5, \dots\} = \{b_n\}_{n=0}^{\infty}$

$$1 + 2s + 3s^2 + 4s^3 + 5s^4 + \dots = \sum_{n=0}^{\infty} (n+1) \cdot s^n = \frac{1}{(1-s)^2}$$

- $\sum_{n=0}^{\infty} (n+1) s^n =$

$$R = \frac{1}{\lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right|} = \frac{1}{\underbrace{\lim_{n \to \infty} \frac{n+1}{n}}_{\to 1}} = 1$$

$\to$ Interval konvergencie je $(-1, 1)$

- $\underline{s = -1}$: $\sum_{0}^{\infty} (n+1)(-1)^n \to$ diverguje
- $\underline{s = 1}$: $\sum_{0}^{\infty} (n+1) \to$ diverguje

$\Rightarrow$ obor konvergencie je $(-1, 1)$

$$= \sum_{n=0}^{\infty} (s^{n+1})' \underset{\text{na } (-1,1)}{\overset{\text{rovnomerne}}{\underline{\underline{\text{konverguje}}}}} \left(\sum_{n=0}^{\infty} s^{n+1} \right)' \underset{\text{rad}}{\overset{\text{geometrický}}{=}}$$

$$\underset{q = s}{\overset{b_0 = s}{=}} \left(\frac{s}{1-s} \right)' = \frac{1}{(1-s)^2}$$

$$\Rightarrow G_b(s) = \frac{1}{(1-s)^2}$$

$$\to b_n = \frac{G_b^{(n)}(0)}{n!} = \begin{cases} n=0: & 1 \\ n=1: & \frac{2(1-0)^{-3}}{1} = 2 \\ n=2: & \frac{6(1-0)^{-4}}{2} = 3 \\ & \dots \end{cases}$$

c) $\left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \dots\right\} \longrightarrow \left\{c_n\right\}_{n=0}^{\infty}$

$$1-\frac{s}{2}+\frac{s^2}{3}-\frac{s^3}{4}+\frac{s^4}{5}-\dots = \sum_{n=0}^{\infty}(-1)^n\frac{s^n}{n+1} = \frac{\ln(1+s)}{s}$$

$\cdot s$ $(s \neq 0)$

$\cdot \frac{1}{s}$ $(s \neq 0)$

$$s-\frac{s^2}{2}+\frac{s^3}{3}-\frac{s^4}{4}+\frac{s^5}{5}-\dots = \sum_{n=0}^{\infty}(-1)^n\frac{s^{n+1}}{n+1} = \ln(1+s)$$

$\cdot \sum_{n=0}^{\infty}(-1)^n\frac{s^{n+1}}{n+1} =$

$R = 1$

$\rightarrow$ interval konvergencie: $(-1,1)_{\infty}$

$\cdot$ $s=-1$: $\sum_0^{\infty}(-1)^n\frac{(-1)^{n+1}}{n+1} = \sum_0^{\infty}\frac{1}{n+1} = \infty$ ... Harmonický rad

$\cdot$ $s=1$: $\sum_0^{\infty}\frac{(-1)^n}{n+1} < \infty$ ... Leibnizov rad

$\rightarrow$ obor konvergencie: $(-1,1]$

$$= \sum_{n=0}^{\infty}(-1)^n\int_0^s t^n\,dt \overset{\text{rovnomerná}}{\underset{\text{konvergencia na }(-1,1]}{=}} \int_0^s \sum_{n=0}^{\infty}(-1)^n t^n\,dt \overset{\text{geometrický}}{\underset{\text{rad}}{=}} \int_0^s \frac{1}{1+t}\,dt =$$

$\boxed{a_0 = 1,\ q = -t}$

$$= \Big[\ln(1+t)\Big]_0^s = \ln(1+s)$$

$$\Longrightarrow G_c(s) = \frac{\ln(1+s)}{s}$$

PR 3.4. Pomocou generujúcej funkcie nájdite strednú hodnotu a rozptyl náhodnej veličiny s ...

a) Geometrickým rozdelením

$X \sim G_e(p)$

$$G_X(s) = \frac{p}{1-s+s\cdot p}$$

$$\boxed{\begin{aligned} E(X) &= G_X'(1) \\ Var(X) &= G_X''(1) + G_X'(1) - \left[G_X'(1)\right]^2 \end{aligned}}$$

$$G_X'(s) = \left[\frac{p}{1-s+sp}\right]' = \frac{-p(-1+p)}{(1-s+sp)^2} = \frac{p(1-p)}{(1-s+sp)^2}$$

- $E(X) = G_X'(1) = \frac{p(1-p)}{p^2} = \frac{1-p}{p}$ //

$$G_X''(s) = \frac{-p(1-p)\cdot 2\cancel{(1-s+sp)}\cdot(-1+p)}{(1-s+sp)^{\cancel{4}3}} = \frac{2\cdot p\cdot(1-p)^2}{(1-s+sp)^3}$$

$$G_X''(1) = \frac{2(1-p)^2}{p^2}$$

- $Var(X) = G_X''(1) + G_X'(1) - \left[G_X'(1)\right]^2 = \frac{2(1-p)^2}{p^2} + \frac{(1-p)p}{p^2} - \frac{(1-p)^2}{p^2} =$

$$= \frac{(1-p)(2-2p+p-1+p)}{p^2} = \frac{1-p}{p^2}$$ //

b) Poissonovo rozdelenie

$X \sim Poisson(\lambda)$

$G_X(s) = e^{\lambda(s-1)}$

$G'_X(s) = \lambda e^{\lambda(s-1)}$

- $E(X) = G'_X(1) = \lambda$

$G''_X(s) = \lambda^2 e^{\lambda(s-1)}$

$G''_X(1) = \lambda^2$

- $Var(X) = G''_X(1) + G'_X(1) - [G'_X(1)]^2 = \cancel{\lambda^2} + \lambda - \cancel{\lambda^2} = \lambda$