2.1 $f(x,y) = \operatorname{arctg}(x^2+y^2)$

diferenciál v bodě $[1,-1]$

Diferenciál:

$$\underline{df}(1,-1) : \mathbb{R}^2 \to \mathbb{R}$$

lineární zob.

$$df(1,-1)(v) = d_v f(1,-1)$$

směrová derivace ve směru vektoru v

$$f(x,y) = \operatorname{arctg}(x^2+y^2)$$

$$f_x(x,y) = \frac{2x}{1+(x^2+y^2)^2} \qquad f_x(1,-1) = \frac{2}{5}$$

$$f_y(x,y) = \frac{2y}{1+(x^2+y^2)^2} \qquad f_y(1,-1) = \frac{-2}{5}$$

$$df(1,-1)\underbrace{(a,b)}_{v} = \frac{2}{5}a - \frac{2}{5}b = d_{(a,b)}(1,-1)$$

$$d_{(1,2)}(1,-1) = \frac{2}{5}(1-(-1)) = \frac{4}{5} = d_u f(1,-1)$$

$d_v f(x_0, y_0)$ $\qquad v = (a, b)$

$\|$

$$(f_x, f_y)\begin{pmatrix} a \\ b \end{pmatrix} = f_x(x_0, y_0)\cdot a + f_y(x_0, y_0)\cdot b$$

Prz. 2. $\quad f(x, y) = x^4 y + x y^2 + x + 2$

- $f_x(x, y) = 4x^3 y + y^2 + 1$ $\quad\Big|\quad$ $f_x(1,1) = 6$

  $f_y(x, y) = x^4 + 2xy$ $\quad\Big|\quad$ $f_y(1,1) = 3$

- $f'(1,1) = df(1,1) = \underbrace{(6, 3)}_{\mathbb{R}^2 \to \mathbb{R}}$

- $f_{xx}(x, y) = 12x^2 y$ $\quad\Big|\quad$ $f_{xx}(1,1) = 12$

  $f_{xy}(x, y) = 4x^3 + 2y$ $\quad\Big|\quad$ $f_{xy}(1,1) = f_{yx}(1,1) = 6$

  $f_{yx}(x, y) = 4x^3 + 2y$

  $f_{yy}(x, y) = 2x$ $\quad\Big|\quad$ $f_{yy}(1,1) = 2$

$$f''(1,1) = \begin{pmatrix} f_{xx}(1,1) & f_{xy}(1,1) \\ f_{yx}(1,1) & f_{yy}(1,1) \end{pmatrix} =$$

$$= \begin{pmatrix} 12 & 6 \\ 6 & 2 \end{pmatrix}$$ Hessova matice

• Taylorův polynom: $f(x)$

$$T(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \frac{1}{3!} f'''(x_0)(x - x_0)^3 + \dots$$

• Taylorův polynom: $f(x,y)$

$$T(x,y) = f(x_0,y_0) + \left(f_x(x_0,y_0),\ f_y(x_0,y_0)\right) \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

$$+ \frac{1}{2}(x - x_0, y - y_0) \begin{pmatrix} f_{xx}(x_0,y_0) & f_{xy}(x_0,y_0) \\ f_{yx}(x_0,y_0) & f_{yy}(x_0,y_0) \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

$$+ \frac{1}{3!}\left[\dots\dots\right]$$

$$T_2(x,y) = f(1,1) +$$
$$+ f_x(1,1)(x-1) + f_y(1,1)(y-1) +$$
$$+ \frac{1}{2}\left[f_{xx}(1,1)(x-x_0)^2 + 2f_{xy}(1,1)(x-1)(y-1) + f_{yy}(1,1)(y-1)^2\right]$$

$$f(1,1) = 5$$

$$T_2(x,y) = 5 + 6(x-1) + 3(y-1)$$
$$+ \frac{1}{2}\left[12(x-1)^2 + 2 \cdot 6(x-1)(y-1) + 2(y-1)^2\right]$$

2.3 Určete lokální extrémy

(i) $f(x,y) = x^3 + y^3 - 3xy$

Stacionární body:

$$f_x(x,y) = 3x^2 - 3y = 0 \qquad y = x^2$$
$$f_y(x,y) = 3y^2 - 3x = 0 \qquad x = y^2$$

$$y = y^4$$

$$y(y^3 - 1) = 0$$

$$y(y-1)(y^2+y+1) = 0$$

$y = 0 \rightarrow x = 0$

$y = 1 \rightarrow x = 1$

$y^2 + y + 1 \neq 0$

Stac. body: $[0,0]$ a $[1,1]$

$$f_{xx}(x,y) = 6x$$
$$f_{xy}(x,y) = f_{yx}(x,y) = -3$$
$$f_{yy}(x,y) = 6y$$

$$f''(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$$

$$f''(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$$

$$6 > 0$$

$$\det(-) = 36 - 9 > 0$$

$$\Rightarrow \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix} \text{ poz. def.}$$

$\Rightarrow (1,1)$ je lokální minimum

Chování funkce $f(x,y) = x^3 + y^3 - 3xy$ v okolí bodu $(0,0)$

$\rightsquigarrow y = 0$. $f(x,0) = x^3$

$\Rightarrow$ v bodě $f(0,0)$ není extrém

(iii) $f(x,y) = x^4 + y^4 - x^2 - 2xy - y^2$

$$f_x(x,y) = 4x^3 - 2x - 2y = 0$$
$$f_y(x,y) = 4y^3 - 2x - 2y = 0$$

$$f_{xx}(x,y) = 12x^2 - 2$$
$$f_{xy}(x,y) = -2$$
$$f_{yy}(x,y) = 12y^2 - 2$$

stacionární body: $x^3 - y^3 = 0$

$$(x-y)\underbrace{(x^2 + xy + y^2)}_{} = 0$$

$$(x + \tfrac{1}{2}y)^2 - \tfrac{1}{4}y^2 + y^2$$
$$(x + \tfrac{1}{2}y)^2 + \tfrac{3}{4}y^2 \geq 0$$

$x = y$

$$4y^3 - 2x - 2y = 0$$
$$4x^3 - 4x = 0$$
$$x(x^2 - 1) = 0 \rightsquigarrow$$

[0, 0]

[1, 1]

[−1, −1]

tři stacionární body

• bod [1,1]

$$f''(1,1) = \begin{pmatrix} 10 & -2 \\ -2 & 10 \end{pmatrix}$$

$10 > 0$

$\det(\cdot) > 0$

$\Rightarrow$ lok. minimum v bodě [1,1]

• bod [−1,−1]

$$f''(-1,-1) = \begin{pmatrix} 10 & -2 \\ -2 & 10 \end{pmatrix}$$ poz. def

$\Rightarrow$ lok. min. v bodě [−1,−1]

• bod [0,0]

$$f''(0,0) = \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}$$

$-2 < 0$

$\det(\cdot) = 0$

chování funkce

$$f(x,y) = x^4 + y^4 - x^2 - 2xy - y^2$$

v okolí bodu [0,0]:

• $y = 0 \rightsquigarrow f(x,0) = x^4 - x^2$

$$= \underbrace{x^2}_{\geq 0}\underbrace{(x^2 - 1)}_{< 0} < 0$$

$$\Downarrow$$

mimo bod $[0,0]$

• $x = -y$

$$\rightsquigarrow f(x,-x) = x^4 + y^4 - x^2 + 2x^2 - x^2$$

$$= x^4 + y^4 > 0$$

$$\Downarrow$$

mimo bod $[0,0]$

V bodě $[0,0]$ není extrém.