9.1 $(x-a)^2 + (y-b)^2 = r^2$ krožnica

$()' = \frac{d}{dx}$ v ravnini

$a, b, r \in \mathbb{R}$

- $y = y(x)$ $\Rightarrow$ derivirajmo

$$2(x-a) + 2(y-b)y' = 0 \quad /()'$$

$$2 + 2y'y' + 2(y-b)y'' = 0 \quad /\frac{1}{y''}$$

$$\frac{2}{y''} + \frac{2(y')^2}{y''} + 2(y-b) = \quad /()'$$

$$-\frac{2y'''}{y''} + \frac{(4y'\cdot y'')y'' - 2(y')^2y'''}{(y'')^2} + 2 = 0 \quad /(y'')^2$$

$$-2y'''y'' + 4y'(y'')^2 - 2(y'^2)y''' + 2(y'')^2 = 0$$

9.2 $y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{pmatrix}$

Diferenciální rovnice

$$y'(x) = \begin{pmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{pmatrix}$$

$$\boxed{y'(x) = A\,y(x)} \quad (*)$$

Je-li v vlastní vektor matice A, tj. $Av = \lambda v$

pak $\boxed{y(x) = e^{\lambda x} \cdot v}$ je řešení rovnice $(*)$

$$y'(x) = \lambda e^{\lambda x} \cdot v = e^{\lambda x}(\lambda v) =$$
$$= e^{\lambda x} A \cdot v = A(e^{\lambda x} v) = A \cdot y(x)$$

Vlastní čísla

$$\det(A - \lambda E) = \det\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} =$$

$$= (-\lambda^3 + 1 + 1) - (-\lambda - \lambda - \lambda)$$

$$= -\lambda^3 + 3\lambda + 2 \qquad \lambda = 2 \qquad \lambda = -1$$

	-1	0	3	2
2	-1	-2	-1	0
-1	-1	-1	0	
-1	-1	0		

$$\to = -(\lambda - 2)(\lambda + 1)^2$$

$\lambda_1 = 2$:

$$A - 2E = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \Rightarrow v_1 = (1, 1, 1)$$

$$\boxed{y(x) = e^{2x}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$ je řešením soustavy

$\lambda_2 = -1$:

$$A + E = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

$v_2 = (1, -1, 0)$

$v_3 = (0, 1, -1)$

$$y(x) = e^{-x}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad y(x) = e^{-x}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Závěr: obecné řešení je

$$\boxed{y(x) = C_1 e^{2x}\begin{pmatrix}1\\1\\1\end{pmatrix} + C_2 e^{-x}\begin{pmatrix}1\\-1\\0\end{pmatrix} + C_3 e^{-x}\begin{pmatrix}0\\1\\-1\end{pmatrix}}$$

$C_1, C_2, C_3 \in \mathbb{R}$

9.3 $\quad y'(x) = A \cdot y(x)$

$$A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \qquad y(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\det(A - \lambda E) = \det\begin{pmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 4$$

$$= \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$$

$\underline{\lambda_1 = 3}$: $\quad A - 3E = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 \end{pmatrix}$

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\underline{\lambda_2 = -1}$: $\quad A + E = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \end{pmatrix}$

$$v_2 = (2, -1)$$

Obecné řešení:

$$\boxed{y(x) = C_1 e^{3x} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-x} \begin{pmatrix} 2 \\ -1 \end{pmatrix}}$$

$$y(0) = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \boxed{\begin{matrix} C_1 = 2 \\ C_2 = -1 \end{matrix}}$$

$$\left(\begin{array}{cc|c} 2 & 2 & 2 \\ 1 & -1 & 3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 4 & -4 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right)$$

Závěr: $y(x) = 2e^{3x}\begin{pmatrix}2\\1\end{pmatrix} - e^{-x}\begin{pmatrix}2\\-1\end{pmatrix}$

$$\boxed{y(x) = \begin{pmatrix}4e^{3x} - 2e^{-x}\\ 2e^{3x} + e^{-x}\end{pmatrix}}$$

Jiný pohled: $y' = \begin{pmatrix}y_1'\\y_2'\end{pmatrix} = \begin{pmatrix}1 & 4\\1 & 1\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}$

$y_1' = y_1 + 4y_2 \quad ()'$ $\qquad y_1(0) = 2$

$y_2' = y_1 + y_2 \quad / \cdot (-4)$ $\qquad y_2(0) = 3$

Eliminace: $\qquad y_1' - 4y_2' = -3y_1$

$y_1'' = y_1' + 4y_2'$ $\qquad 4y_2' = y_1' + 3y_1$

$y_1'' = y_1' + (y_1' + 3y_1)$

$y_1'' = 2y_1' + 3y_1$

$$\boxed{y_1'' - 2y_1' - 3y_1 = 0} \quad \text{2. řády}$$

$$\lambda^2 - 2\lambda - 3 = 0 \rightsquigarrow (\lambda-3)(\lambda+1)=0$$

<u>9.4</u> $y' = A\cdot y,\quad A = \begin{pmatrix}1&0&0\\-3&1&-2\\2&2&1\end{pmatrix}$

$$\det(A-\lambda E) = \det\begin{pmatrix}1-\lambda&0&0\\3&1-\lambda&-2\\2&2&1-\lambda\end{pmatrix} =$$

$$= (1-\lambda)\left[(1-\lambda)^2+4\right] = -(\lambda-1)\left[\underbrace{\lambda^2-2\lambda+5}\right]$$

$$\lambda_1 = 1 \qquad \lambda_{1,2} = \frac{2\pm\sqrt{4-20}}{2}$$

$$\lambda_{1,2} = 1\pm 2i$$

$$A-E = \begin{pmatrix}0&0&0\\3&0&-2\\2&2&0\end{pmatrix} \sim \begin{pmatrix}3&0&-2\\1&1&0\end{pmatrix}$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$

$$\begin{pmatrix}0&3&-2\\1&1&0\end{pmatrix}$$

$$\begin{array}{ccc} x_2 & x_1 & x_3 \end{array}$$

$x_3 = 3p$

$x_1 = 2p$

$x_2 = -2p$

$$v_1 = \left(1,-1,\frac{3}{2}\right)$$

$$v_1 := (2,-2,3)$$

$$y(x) = e^x\begin{pmatrix}2\\-2\\3\end{pmatrix}$$

$$\lambda_2 = 1+2i$$

$$A-1-2i=\begin{pmatrix} \cancel{-2i} & \cancel{0} & \cancel{0} \\ 3 & -2i & -2 \\ \cancel{2} & \cancel{2} & \cancel{-2i} \\ 1 & 1 & -i \\ 1 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2i & -2 \\ 0 & 1 & -i \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -i \end{pmatrix}$$

$$\lambda_2 = 1+2i \rightsquigarrow v_2 = (0, i, 1)$$

$$\lambda_3 = 1-2i \rightsquigarrow v_3 = (0, -i, 1)$$

Obecné řešení v prostoru komplexních funkcí (s jednou reálnou proměnnou):

$$y(x) = C_1 e^x \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} + C_2 e^{(1+2i)x} \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} + C_3 e^{(1-2i)x} \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}$$

Dále najdeme reálnou bázi:

$$\underbrace{e^{(1+2i)x}}_{e^x e^{i(2x)}}\begin{pmatrix}0\\ i\\ 1\end{pmatrix}=e^x(\cos 2x+i\sin 2x)\begin{pmatrix}0\\ i\\ 1\end{pmatrix}$$

$$e^{(1-2i)x}\begin{pmatrix}0\\ -i\\ 1\end{pmatrix}=e^x(\cos 2x-i\sin 2x)\begin{pmatrix}0\\ -i\\ 1\end{pmatrix}$$

Součet: $e^x\begin{pmatrix}0\\ -\sin 2x-\sin 2x\\ 2\cos 2x\end{pmatrix}=e^x\begin{pmatrix}0\\ -2\sin 2x\\ 2\cos 2x\end{pmatrix}$

rozdíl: $e^x\begin{pmatrix}0\\ 2i\cos 2x\\ 2i\sin 2x\end{pmatrix}=ie^{2x}\begin{pmatrix}0\\ 2\cos 2x\\ 2\sin 2x\end{pmatrix}$

Obecné reálné řešení

$$\boxed{y(x)=C_1e^x\begin{pmatrix}2\\ -2\\ 3\end{pmatrix}+D_1e^x\begin{pmatrix}0\\ -\sin 2x\\ \cos 2x\end{pmatrix}+D_2e^x\begin{pmatrix}0\\ \cos 2x\\ \sin 2x\end{pmatrix}}$$

9.5 $\quad y'(x) = A \cdot y(x), \quad A = \begin{pmatrix} 17 & 9 \\ -25 & -13 \end{pmatrix}$

$$\det(A - \lambda E) = \det\begin{pmatrix} 17-\lambda & 9 \\ -25 & -13-\lambda \end{pmatrix} =$$

$$= (17-\lambda)(-13-\lambda) + 9 \cdot 25$$

$$= \lambda^2 + (-17+13)\lambda \underbrace{- 17 \cdot 13 + 9 \cdot 25}_{4 = -221+225}$$

$$= \lambda^2 - 4\lambda + 4$$

$$= (\lambda - 2)^2 \rightarrow \lambda_1 = 2 \rightarrow v_1 = (-3, 5)$$

$$A - 2E = \begin{pmatrix} 15 & 9 \\ -25 & -15 \end{pmatrix} \sim \begin{pmatrix} 5 & 3 \\ 5 & 3 \end{pmatrix} \sim \begin{pmatrix} 5 & 3 \end{pmatrix}$$

$$\boxed{y(x) = e^{2x}\begin{pmatrix} -3 \\ 5 \end{pmatrix}}$$

$$(A - 2E)v_1 = 0$$

$$(A - 2E)^2 v_2 = 0$$

$$\boxed{0 \xleftarrow{A-2E} v_1 \xleftarrow{A-2E} v_2}$$

↳ [illegible]

[illegible]

$$\underbrace{\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}}_{B}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\boxed{0 \xleftarrow{B-2E} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xleftarrow{B-2E} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$$B - 2E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Vypočet v_2: $(A-2E)\cdot v_2 = v_1$

$$\boxed{v_2 = \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}}$$

$$\begin{pmatrix} 15 & 9 \\ -25 & -15 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 15 & 9 & -3 \\ -25 & -15 & 5 \end{array}\right) \sim \left(\begin{array}{cc|c} 5 & 3 & -1 \\ 5 & 3 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 5 & 3 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

$$x_1 = 0, \quad x_2 = 1/3$$

$$y(x) = e^{2x}\left[\underbrace{\begin{pmatrix} 0 \\ 1/3 \end{pmatrix}}_{v_2} + \frac{x}{1!}\underbrace{(A-2E)\begin{pmatrix} 0 \\ 1/3 \end{pmatrix}}_{v_1}\right]$$