Neurčitý integrál

F je primitivní funkce k f na intervalu I, jestliže

$$F'(x) = f(x)$$

$$\int f(x)\,dx = F(x) + c$$

① $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \qquad n \in \mathbb{Z},\ n \neq -1$

② $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + c$

$$\int \frac{1}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}}\,dx = \frac{x^{-\frac{1}{2}+1}}{\frac{1}{2}} = 2\sqrt{x}$$

③ $\displaystyle\int x^\alpha\,dx = \frac{x^{\alpha+1}}{\alpha+1} + c \qquad \alpha \in \mathbb{R},\ \alpha \neq -1$

(4) $\int \sin x \, dx = -\cos x + C$

(5) $\int \cos x \, dx = \sin x + C$

(6) $\int \frac{1}{\cos^2 x} \, dx = \operatorname{tg} x + C$

(7) $\int \frac{1}{1+x^2} \, dx = \operatorname{arctg} x + C$

(8) $\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$

(9) $\int \frac{1}{\sqrt{1+x^2}} \, dx = \ln\left(x + \sqrt{x^2+1}\right) + C$

Integrování pomocí věty o substituci

Nechť funkce f definovaná na intervalu J má primitivní funkci F. Nechť φ zobrazuje interval I do intervalu J a má vlastní derivaci. Pak funkce

$$f(\varphi(x)) \cdot \varphi'(x)$$

má primitivní funkci $F(\varphi(x))$ na intervalu I.

$$\int f(\overset{t}{\varphi(x)}) \overbrace{\varphi'(x)\,dx}^{dt} = \int f(t)\,dt = F(\varphi(x))$$

$t = \varphi(x)$

Vyjpočet

$$\int x \cdot \sqrt{1-x^2}\,dx =$$

$$f(t) = \sqrt{t}$$

$$t = \varphi(x) = 1-x^2 \qquad \varphi'(x) = -2x$$

$$dt = \varphi'(x)\,dx \qquad dt = (-2x)\,dx$$

$$dx = \frac{dt}{-2x}$$

$$= \int x\sqrt{t}\,\frac{dt}{-2x} = \frac{1}{-2}\int \sqrt{t}\,dt = \left(-\frac{1}{2}\right)\frac{t^{3/2}}{\frac{3}{2}} + c =$$

$$= -\frac{1}{3}t^{3/2} + c = -\frac{1}{3}\left(1-x^2\right)^{3/2} + c$$

Príklad 2

$$\int \operatorname{tg} x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{dt}{t} = -\ln|t| + c$$

$t = \cos x \qquad dt = -\sin x \, dx$

$$= -\ln|\cos x| + c$$

Príklad 3

$$\int \cos^3 x \, dx = \int \cos x \cdot \cos^2 x \, dx =$$

$$= \int \underline{\cos x} \cdot (1 - \sin^2 x) \, \underline{dx} = \int (1 - t^2) \, dt = t - \frac{t^3}{3} + c$$

$t = \sin x$

$dt = \cos x \, dx$

$$= \sin x - \frac{\sin^3 x}{3} + c$$

Příklad 4

$$\int \frac{\ln^2 x}{x}\,dx = \int t^2\,dt = \frac{t^3}{3}+c = \frac{\ln^3 x}{3}+c$$

$t = \ln x$

$dt = \frac{1}{x}\,dx$

Jiný způsob použití věty o substituci

$$\int f(x)\,dx \qquad x = \psi(z) \qquad dx = \psi'(z)\,dz$$

$\psi^{-1}(x) = z$

$$= \underbrace{\int f(\psi(z))\,\psi'(z)\,dz}_{\text{Prim. funkce bude } G(z)} = G(z)+c = G(\psi^{-1}(x))+c$$

Příklad

$$\int \sqrt{1-x^2}\,dx =$$

$x = \sin z \qquad z \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$x \in [-1, 1]$

$\arcsin x = z$

$dx = \cos z\,dz$

$$= \int \sqrt{1-\sin^2 z}\,\cos z\,dz$$

$$= \int \sqrt{\cos^2 z}\,\cos z\,dz = \int \cos^2 z\,dz =$$

$|\cos z| = \cos z$

$$\cos 2z = \cos^2 z - \sin^2 z = 2\cos^2 z - 1$$

$$\cos^2 z = \frac{1+\cos 2z}{2}$$

$$= \frac{1}{2}\int (1+\cos 2z)\,dz = \frac{1}{2}\left(z + \frac{1}{2}\int \cos t\,dt\right) = \frac{1}{2}\left(z + \frac{1}{2}\sin t\right) + C$$

$t = 2z$

$dt = 2\,dz$

$$= \frac{1}{2}\left(z + \frac{1}{2}\sin 2z\right) + C = \frac{1}{2}\arcsin x + \frac{1}{2}\sin 2z$$

$$= \frac{1}{2}\arcsin x + \frac{1}{4}\, 2 \sin z \underbrace{\cos z}_{\sqrt{1-\sin^2 z}} = \frac{1}{2}\arcsin x$$

$$+ \frac{1}{2}\sin(\arcsin x)\cdot \cos(\arcsin x) + C$$

$$= \frac{1}{2}\arcsin \cancel{x} + \frac{1}{2}x\underline{\sqrt{1-x^2}} + C$$

<u>Zkusíme jinak</u>

$x = \sqrt{1-t}$

$$\int \sqrt{1-x^2}\,dx \qquad t = 1-x^2 \qquad dt = -2x\,dx$$

$$= \int \sqrt{t}\,\frac{dt}{-2x} = \int \frac{\sqrt{t}\,dt}{-2\sqrt{1-t}} = ??? \qquad dx = \frac{dt}{-2x}$$

Určitý integrál

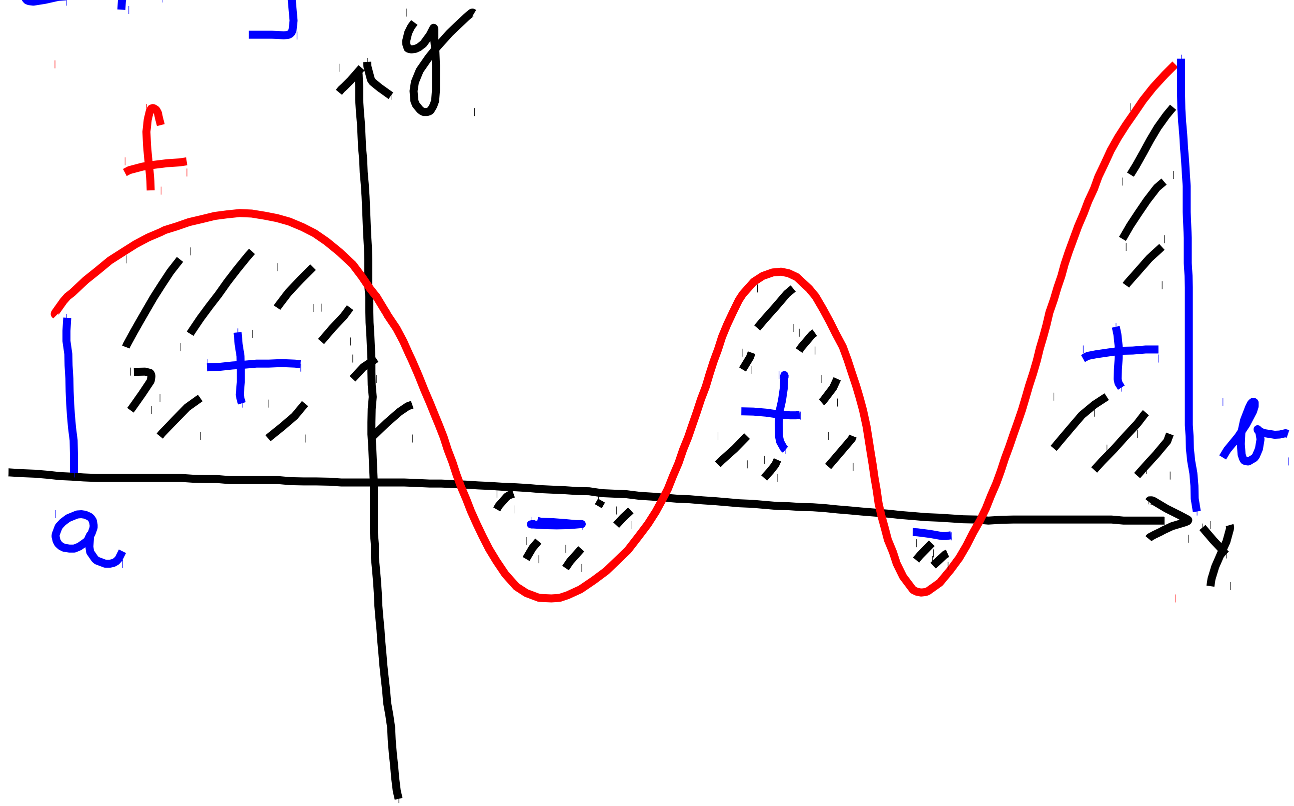
<u>Definice</u>: Nechť je funkce f definovaná na intervalu $[a,b]$ a je na něm spojitá. Určitý integrál funkce f <u>od a do b</u> je číslo

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

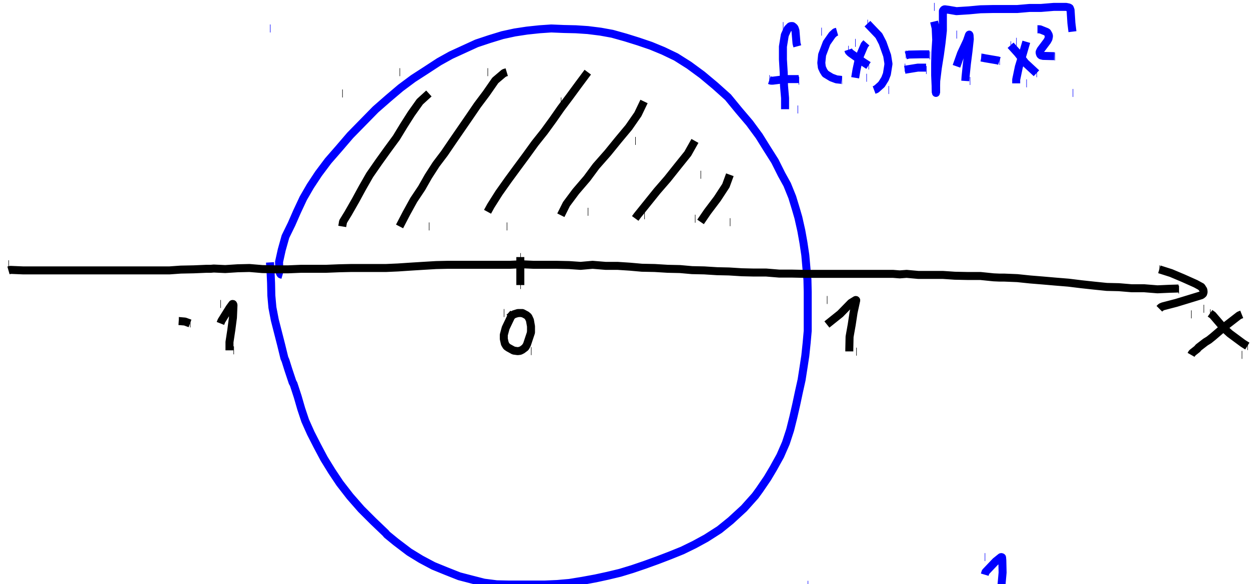
kde F je primitivní funkce k f.

Často
$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

Geometrický význam $\int_a^b f(x)\,dx$ je obsah oblasti mezi grafem funkce f a osou x nad intervalem $[a, b]$



Příklad: obsah kruhu ..... obsah půlkruhu o poloměru 1



$f(x) = \sqrt{1-x^2}$

$x^2 + y^2 = 1$

$y^2 = 1 - x^2$

$y = \sqrt{1-x^2}$

$$S = \int_{-1}^{1} \sqrt{1-x^2}\, dx = \left[\frac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right) \right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\arcsin 1 + 1\sqrt{1-1^2} - \arcsin(-1) - 1\sqrt{1-(-1)^2}\right]$$

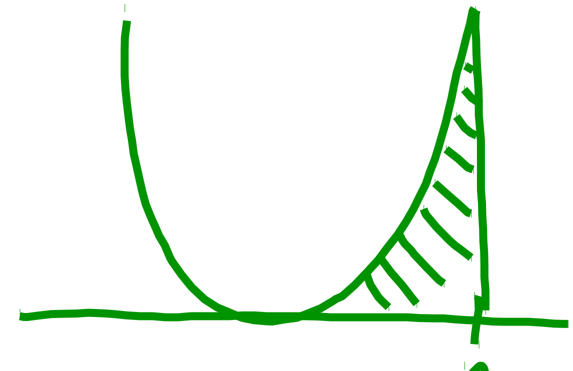
$$= \frac{1}{2}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{2}$$

Kruh o poloměru r

$$\int \sqrt{r^2 - x^2}\, dx \qquad \text{substituce} \quad x = r \sin z$$

Příklady

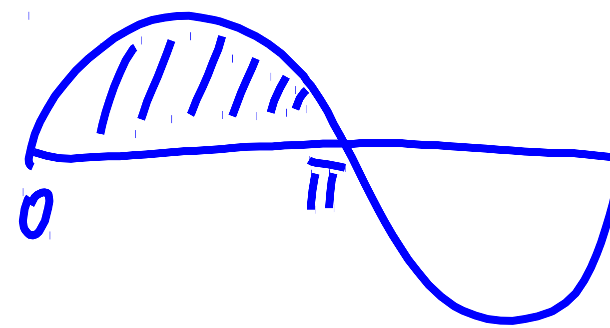
① $$\int_0^a x^2\, dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$



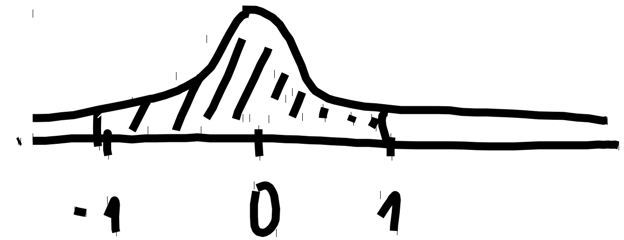
② $$\int_0^\pi \sin x\, dx =$$

$$= \Big[-\cos x\Big]_0^\pi = -\cos\pi - (-\cos 0)$$

$$= 1 + 1 = 2$$



③ $$\int_{-1}^{1} \frac{1}{x^2+1}\,dx$$



$$= \Big[\operatorname{arctg} x\Big]_{-1}^{1} = \operatorname{arctg} 1 - \operatorname{arctg}(-1) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \underline{\underline{\frac{\pi}{2}}}$$

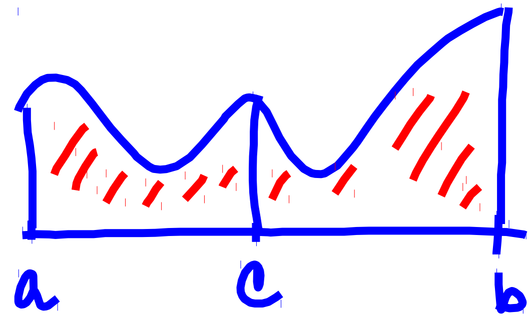
Vlastnosti určitého integrálu

① $$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

② $$\int_a^b k\cdot f(x)\,dx = k\int_a^b f(x)\,dx$$

③ $a < c < b$

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$



④ $f(x) \geq 0$

$$\int_a^b f(x)\,dx \geq 0$$

⑤ $f(x) \geq g(x)$

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

Per partes pro určité integrály

Věta: Nechť u a v mají spojité derivace na intervalu $[a,b]$. Potom

$$\int_a^b u'(x)v(x)\,dx = \Big[u(x)v(x)\Big]_a^b - \int_a^b u(x)v'(x)\,dx$$

Příklad

$$\int_1^e x^3 \ln x\,dx = \left[\frac{x^4}{4}\cdot\ln x\right]_1^e - \int_1^e \frac{x^4}{4}\cdot\frac{1}{x}\,dx$$

$u'(x) = x^3$, $v(x) = \ln x$

$u(x) = \frac{x^4}{4}$

$$= \left[\frac{x^4}{4}\cdot\ln x\right]_1^e - \left[\frac{1}{16}x^4\right]_1^e$$

$$= \frac{e^4}{4} - \frac{1}{16}\left(e^4 - 1\right) = \frac{3e^4+1}{16}$$

Věta o substituci pro určitý integrál

Nechť $f(t)$ je spojitá na intervalu $[a,b]$,
nechť $t=\varphi(x)$ má spojitou derivaci na intervalu $[\alpha,\beta]$ a zobrazuje tento interval do $[a,b]$.
Pak platí

$$\int_{\alpha}^{\beta} f(\varphi(x))\,\varphi'(x)\,dx = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(t)\,dt$$

$$\int_{1}^{-2} f(t)\,dt = -\int_{-2}^{1} f(t)\,dt$$

Jestliže $\varphi(\alpha) > \varphi(\beta)$, pak počítáme

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(t)\,dt = F(\varphi(\beta)) - F(\varphi(\alpha)) = -\left(F(\varphi(\alpha)) - F(\varphi(\beta))\right) = -\int_{\varphi(\beta)}^{\varphi(\alpha)} f(t)\,dt$$

Příklad

$$\int_0^5 \frac{x}{\sqrt{1+3x}}\,dx =$$

$t = \sqrt{1+3x}$ $\quad [\alpha, \beta] = [0, 5]$ (x)

$t \in [a, b] = [1, 4]$

$t^2 = 1+3x$

$x = \dfrac{t^2-1}{3}$ $\qquad dt = \frac{1}{2}\dfrac{1}{\sqrt{1+3x}}\,3\,dx$

$$= \int_1^4 \frac{t^2-1}{3}\cdot \frac{2}{3}\,dt = \frac{2}{9}\int_1^4 (t^2-1)\,dt = \frac{2}{9}\left[\frac{t^3}{3} - t\right]_1^4 =$$

$$= \frac{2}{9}\left(\frac{64}{3} - 4 - \frac{1}{3} + 1\right) =$$

$$= \frac{2}{9}\,\frac{64-12-1+3}{3} = \frac{2}{9}\,\frac{54}{3} = 2\cdot 2 = \underline{\underline{4}}$$