vnitřek koule o pol. A, středem $[0,0,0]$

① $\iiint_V f(x,y,z)\,dx\,dy\,dz = I$ V: $x^2+y^2+z^2 \le A^2$, $A>0$, $x \ge 0$, $y \ge 0$, $z \ge 0$

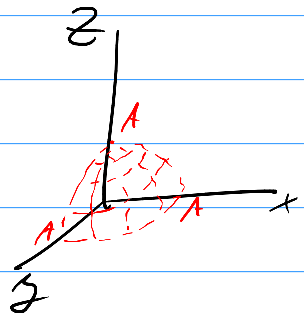
1. oktant

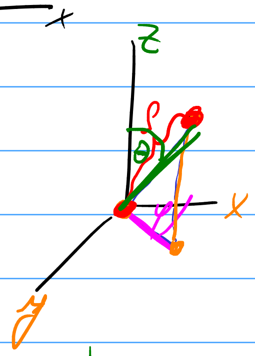
sférické souř.

$0 \le x = \rho \cos\varphi \sin\theta$ ≥ 0 obecně $\rho \ge 0$

$0 \le y = \rho \sin\varphi \sin\theta$ ≥ 0 $\varphi \in [0, 2\pi]$

$z = \rho \cos\theta$ ≥ 0 $\theta \in [0, \pi]$



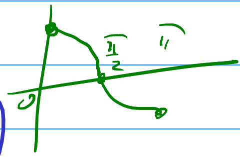


$A^2 \ge x^2+y^2+z^2 = \rho^2$

$\rho^2 \le A^2$

$\rho \le A$ $\rho \in [0, A]$

$\cos\theta \ge 0$



$\theta \in [0, \frac{\pi}{2}]$

$\cos\varphi \ge 0$ $\varphi \in [0, \frac{\pi}{2}]$

$\sin\varphi \ge 0 \rightsquigarrow$ ~~$\varphi \in [0, \pi]$~~

$$I = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{A} f(\rho\cos\varphi\sin\theta, \rho\sin\varphi\sin\theta, \rho\cos\theta) \cdot \underbrace{\rho^2 \sin\theta}_{|J|}\, d\rho\, d\theta\, d\varphi$$

$\rho\cos\theta = z \geq 0 \Rightarrow \cos\theta \geq 0$

$\theta \in [0, \frac{\pi}{2}]$

② $\iiint\limits_V \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$, $\quad V: \boxed{x^2+y^2+z^2 \leq z}$

$x^2+y^2+z^2-z \leq 0$

$z^2 - \underbrace{2\cdot A}_{=1}\cdot z + A^2 - A^2$

$x^2+y^2+\underbrace{z^2-z+\frac{1}{4}}-\frac{1}{4} \leq 0$

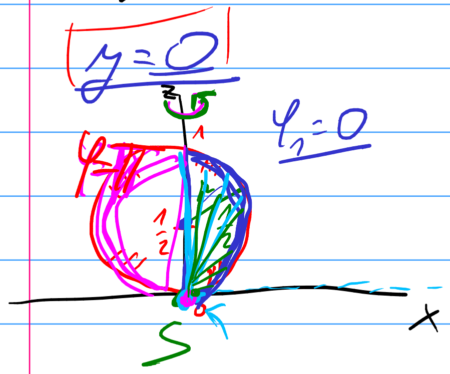
$$\boxed{x^2+y^2+\underbrace{(z-\tfrac{1}{2})^2}_{\rho^2} \leq \frac{1}{4}}$$

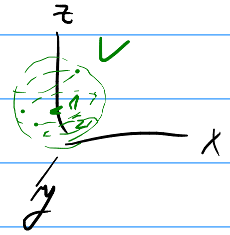
koule se stř. $[0,0,\frac{1}{2}]$

a poloměr $r = \frac{1}{2}$

$\boxed{y=0}$

$\varphi_1 = 0$



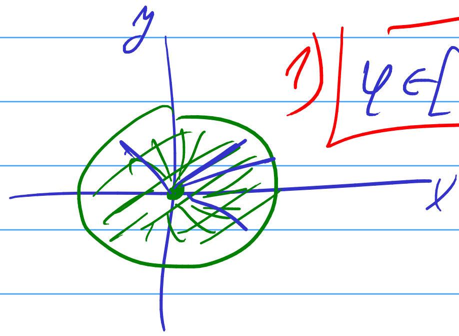


$x = \rho\cos\varphi\sin\theta$ $\quad$ ~~$\varphi\in[0,2\pi]$~~

$y = \rho\sin\varphi\sin\theta$ $\quad$ ~~$\theta\in[0,\pi]$~~

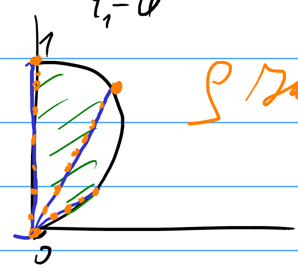
$z = \cancel{\frac{1}{2}} + \rho\cos\theta$ $\quad$ ~~$\rho\in[0,\frac{1}{2}]$~~

$|J| = \rho^2\sin\theta$ $\qquad$ $x^2+y^2+z^2 \neq \rho^2$



1) $\boxed{\varphi\in[0,2\pi]}$ $\qquad$ 2) $\theta\in[0,\frac{\pi}{2}]$

$\varphi_1 = 0$



ρ závisí na θ

$x^2+y^2+z^2 = \underbrace{\rho^2 \leq \rho\cos\theta}$

$\rho \leq \cos\theta$ $\qquad$ 3) $\rho\in[0,\cos\theta]$

$$I = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{\cos\theta} \underbrace{\cancel{\sqrt{\rho^2}\cdot\rho^2}}_{\text{neobsahuje } \varphi}\!\!\to\rho^3 \,\sin\theta\,d\rho\,d\theta\,d\varphi = 2\pi\int_0^{\frac{\pi}{2}}\left[\frac{\rho^4}{4}\right]_0^{\cos\theta}\sin\theta\,d\theta =$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\cos^4\theta\sin\theta\,d\theta = \frac{\pi}{2}\int_0^1 t^4\,dt = \frac{\pi}{2}\left[\frac{t^5}{5}\right]_0^1 = \frac{\pi}{10}$$

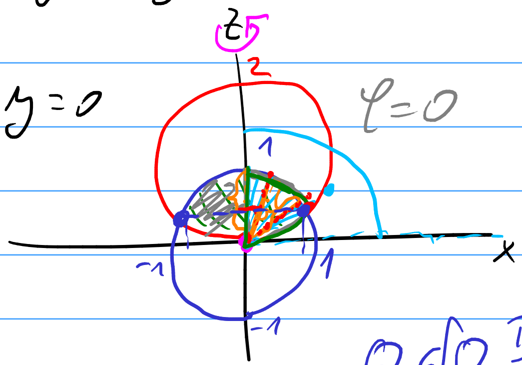
$\left| \begin{array}{l} t = \cos\theta \\ dt = -\sin\theta \\ 0 \to 1 \\ \frac{\pi}{2} \to 0 \end{array}\right.$

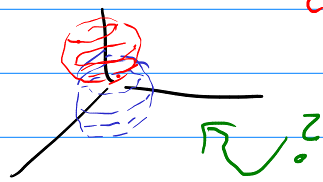
(3) $\iiint\limits_V x^2+y^2 \, dx\,dy\,dz$

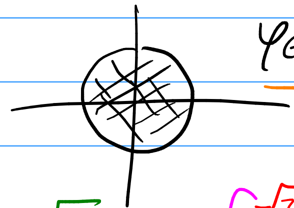
$V:$ $\boxed{x^2+y^2+z^2 \leq 1}$ → KOULE STŘED $[0,0,0]$ $r=1$

$\boxed{x^2+y^2+(z-1)^2 \leq 1}$ → KOULE STŘED $[0,0,1]$ $r=1$

$y=0$ $\quad$ $\varphi=0$







$\varphi \in [0, 2\pi]$

$\theta \in [0, \frac{\pi}{2}]$

$\rho \in [0, 1]$

$x = \rho \cos\varphi \sin\theta$

$y = \rho \sin\varphi \sin\theta$

$z = \rho \cos\theta$

$[\frac{\sqrt{3}}{2}, 0, \frac{1}{2}]$

závisí na θ

1) $\varphi \in [0, 2\pi]$
$\theta \in [0, \frac{\pi}{3}]$
$\rho \in [0, 1]$

2) $\varphi \in [0, 2\pi]$
$\theta \in [\frac{\pi}{3}, \frac{\pi}{2}]$
$\rho \in [0, 2\cos\theta]$

$\theta_1 = ?$

$x^2+z^2=1 \cap x^2+(z-1)^2=1$

$z^2 = (z-1)^2$

$\cancel{z^2} = \cancel{z^2} - 2z + 1$

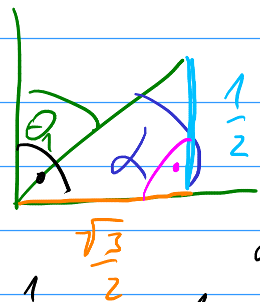
$2z = 1$

$z = \frac{1}{2}$

$x^2 = 1 - \frac{1}{4}$

$x^2 = \frac{3}{4}$

$x = \pm \frac{\sqrt{3}}{2}$



$\alpha + \theta_1 = \frac{\pi}{2}$

$\operatorname{tg}\alpha = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\alpha = \frac{\pi}{6}$

$\boxed{\theta_1} = \frac{\pi}{2} - \frac{\pi}{6} = \frac{3-1}{6}\pi = \frac{2}{6}\pi = \boxed{\frac{\pi}{3}}$

$x^2+y^2+(z-1)^2 \leq 1$

$\rho^2 - 2z + 1 \leq 1$

$\rho^2 \leq 2z = 2\rho\cos\theta$

$\underline{\underline{\rho \leq 2\cos\theta}}$

$x^2+y^2 = \rho^2(\cos^2\varphi\sin^2\theta + \sin^2\varphi\sin^2\theta)$

$= \rho^2\sin^2\theta(\underbrace{\cos^2\varphi + \sin^2\varphi}_{=1})$

$$I=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{1}\rho^4\sin^3\theta\, d\rho\, d\theta\, d\varphi+\int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^{2\cos\theta}\rho^4\sin^3\theta\, d\rho\, d\theta\, d\varphi$$

$$=2\pi\int_0^1\rho^4\,d\rho\int_0^{\frac{\pi}{3}}\sin^3\theta\,d\theta+2\pi\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\sin^3\theta\,\frac{2^5\cos^5\theta}{5}\,d\theta=$$

$t=\cos\theta$

$dt=-\sin\theta\,d\theta$

$\theta\to1$

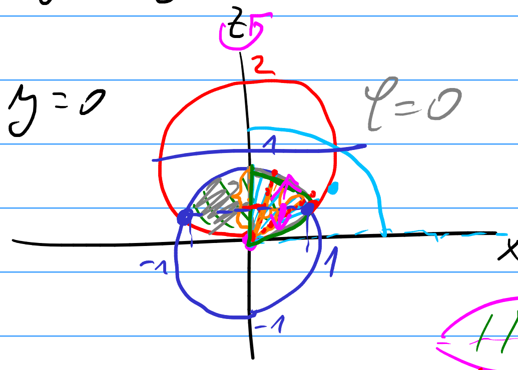
$\frac{\pi}{3}\to\frac{1}{2}$

$$=2\pi\left[\frac{\rho^5}{5}\right]_0^1\int_{\frac{1}{2}}^{1}1-t^2\,dt+\frac{2^6}{5}\pi\int_0^{\frac{1}{2}}\underbrace{t^5(1-t^2)}_{t^5-t^7}\,dt=$$

$$=\frac{2\pi}{5}\left[t-\frac{t^3}{3}\right]_{\frac{1}{2}}^{1}+\frac{2^6}{5}\pi\left[\frac{t^6}{6}-\frac{t^8}{8}\right]_0^{\frac{1}{2}}=$$

$$=\frac{2\pi}{5}\left[1-\frac{1}{3}-\frac{1}{2}+\frac{1}{24}\right]+\frac{\cancel{2^6}\pi}{5}\cdot\frac{1}{\cancel{2^6}}\left[\frac{1}{6}-\frac{1}{32}\right]=\ldots$$

$$\iiint_V x^2+y^2\,dx\,dy\,dz$$



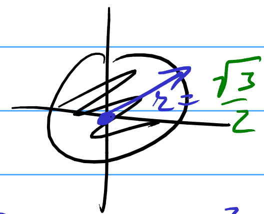
$x=\rho\cos\varphi$

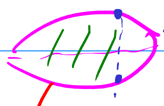
$y=\rho\sin\varphi$

$\varphi\in[0,2\pi]$

$\rho\in\left[0,\frac{\sqrt3}{2}\right]$

z





$x^2+y^2+z^2=1$

$z=\sqrt{1-x^2-y^2}$

$x^2+y^2+(z-1)^2=1$

$z=1-\sqrt{1-x^2-y^2}$

$$1-\sqrt{1-x^2-y^2}\le z\le\sqrt{1-x^2-y^2}$$

$$1-\sqrt{1-\rho^2}\le z\le\sqrt{1-\rho^2}$$

$$2\pi \quad \int_0^{2\pi}\int_0^{\frac{\sqrt{3}}{2}}\int_{1-\sqrt{1-\rho^2}}^{\sqrt{1-\rho^2}} \rho^3 \, dz \, d\rho \, d\varphi = 2\pi \int_0^{\frac{\sqrt{3}}{2}} \rho^3 \, 2\sqrt{1-\rho^2} - \rho^3 \, d\rho =$$

$$\sqrt{1-\rho^2} - 1 + \sqrt{1-\rho^2}$$

$$= 4\pi \int_0^{\frac{\sqrt{3}}{2}} \underbrace{\rho^3 \sqrt{1-\rho^2}}\, d\rho - 2\pi \left[\frac{\rho^4}{4}\right]_0^{\frac{\sqrt{3}}{2}}$$

$t = \sqrt{1-\rho^2}$

$t^2 = 1-\rho^2$

$\rho^2 = 1-t^2$

$2\rho \, d\rho = -2t \, dt$

$0 \rightsquigarrow 1$

$\frac{\sqrt{3}}{2} \rightsquigarrow \frac{1}{2}$

$$= 4\pi \int_{\frac{1}{2}}^{1} \underbrace{(1-t^2) t \cdot t}_{t^2 - t^4} \, dt - \frac{2\pi}{4} \cdot \frac{9}{16} =$$

$$= 4\pi \left[\frac{t^3}{3} - \frac{t^5}{5}\right]_{\frac{1}{2}}^{1} - \frac{9\pi}{32} = \ldots$$