① $\sum_{n=1}^{\infty} \operatorname{arctg} \frac{1}{n}$, K/D?

$\lim_{n\to\infty} \operatorname{arctg} \frac{1}{n} = \operatorname{arctg}\left(\lim_{n\to\infty} \frac{1}{n}\right) = \operatorname{arctg} 0 = 0$

nutná podmínka konv. je splněna

$\left(\frac{1}{n^p}\right) = n^{-p}$

$$\lim_{n\to\infty} \frac{\operatorname{arctg}\frac{1}{n}}{\frac{1}{n^p}} = \left|\frac{0}{0}\right| \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n^2}}\cdot\left(-\frac{1}{n^2}\right)}{-p\,\frac{1}{n^{p+1}}} = \lim_{n\to\infty} \frac{\frac{1}{n^2+1}}{p\,\frac{1}{n^{p+1}}} = \lim_{n\to\infty} \frac{n^{p+1}}{p(n^2+1)} =$$

$p>0$

$p \le 0$, $L=0$ tam nás to nezajímá

$$= \lim \frac{n^{p+1}}{p\cdot n^2} \overset{p=1}{=} \lim_{n\to\infty} \frac{n^2}{1\,n^2} = 1 \in (0,\infty)$$

$\sum^{\infty} \frac{1}{n}$ D $\Rightarrow \sum^{\infty} \operatorname{arctg}\frac{1}{n}$ D

$\underbrace{\sum^{\infty} \operatorname{arctg}\frac{1}{n}}_{\ge 0}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\operatorname{arctg}\frac{1}{n+1}}{\operatorname{arctg}\frac{1}{n}} = \left|\frac{0}{0}\right| \overset{L'H}{=}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{(n+1)^2}}\cdot\left(-\frac{1}{(n+1)^2}\right)}{\frac{1}{1+\frac{1}{n^2}}\left(-\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2+1}}{\frac{1}{n^2+1}} = \lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1} =$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2} = 1$$ nelze rozhodnout

② $\sum_{n=1}^{\infty} \underbrace{\frac{n^n}{n!}}_{a_n \geq 0}$ KD: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{n+\cancel{1}}}{\underbrace{(n+1)!}_{\cancel{(n+1)}\cancel{n!}}} \cdot \frac{\cancel{n!}}{n^n} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n} =$

$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \boxed{e} > 1$ řada diverguje

$e^{\ln\left(1+\frac{1}{n}\right)^n}$

③ $\sum_{n=1}^{\infty} \underbrace{\frac{n^{2n}}{(2n+1)!}}_{a_n \geq 0}$

$$\lim_{n\to\infty} \frac{\frac{(n+1)^{2n+2}}{(2n+3)!}}{\frac{n^{2n}}{(2n+1)!}} = \lim_{n\to\infty} \frac{\cancel{(2n+1)!}\,(n+1)^{2n+2}}{(2n+3)(2n+2)\cancel{(2n+1)!}\,n^{2n}} =$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+3)(2n+2)} \cdot \left[\left(\frac{n+1}{n}\right)^n\right]^2 =$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}}{4\cancel{n^2}} \cdot \Big[\underbrace{\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n}_{=e}\Big]^2 = \frac{1}{4} \cdot e^2 = \left(\frac{e}{2}\right)^2 > (1)^2 = 1$$

$$\frac{e}{2} \approx \frac{2{,}71}{2} > 1$$

řada diverguje

4) $\sum_{n=1}^{\infty} \frac{n}{\underbrace{\left(\frac{n+1}{n}\right)^n}_{a_n \geq 0}}$ odmocninové krit.

$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{\sqrt[n]{n} \to 1}{\underbrace{\frac{n+1}{n}}_{1+\frac{1}{n} \to 1}} = \frac{1}{1} = 1$ nevíme nic

$e^{\lim_{n\to\infty} \frac{\ln n}{n}} = \left|\frac{\infty}{\infty}\right| \text{l'h} \lim_{n\to\infty} \frac{\frac{1}{n} \to 0}{1}$
$= e^{\ldots} = e^0 = 1$

$\lim_{n\to\infty} \frac{n \to \infty}{\left(1+\frac{1}{n}\right)^n \to e} = \frac{\infty}{e} = \infty \neq 0$ nutná podmínka konv. není splněna

tedy řada diverguje

(5) $\sum_{n=1}^{\infty} \frac{n^2+1}{2^n}$ $\quad \underbrace{}_{a_n \geq 0}$

KD?: $\lim_{n\to\infty} \sqrt[n]{\frac{n^2+1}{2^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n^2+1}}{2} = \frac{1}{2} < 1$

řada konverguje

$\lim_{n\to\infty} e^{\frac{\ln(n^2+1)}{n}} \overset{l'H}{=} e^{\lim_{n\to\infty} \frac{\frac{1}{n^2+1}\cdot 2n}{1}} = e^{\lim_{n\to\infty} \frac{2}{2n}} = e^0 = 1$

$\to 0$

6 $\sum_{n=1}^{\infty} a_n$ K/D?, $a_n = \begin{cases} 2^n, & n \text{ sudé} \\ \frac{3}{(n+1)^n}, & n \text{ liché} \end{cases}$

$a_n \geq 0$ $\quad \sqrt[n]{a_n} = \begin{cases} 2, & n \text{ sudé} \\ \frac{\sqrt[n]{3}}{n+1} \to 1, & n \text{ liché} \\ \quad \downarrow \infty \end{cases}$

řada K

$\Leftarrow k < 1$

$\lim \sqrt[n]{a_n} \ \nexists$ $\quad \boxed{\limsup \sqrt[n]{a_n} = 2} > 1$

$\liminf \sqrt[n]{a_n} = 0$

$\boxed{\text{řada diverguje}}$

⑦ $\sum_{n=1}^{\infty} \frac{1}{(n+\alpha)^p}$ K/D? $\alpha \ge 0, p \in \mathbb{R}$

integrální krit.

$a_n = \frac{1}{(n+\alpha)^p} = f(n) \rightsquigarrow f(x) = \frac{1}{(x+\alpha)^p}$

1) $f(x) \ge 0$ $[N, \infty)$
2) $f(x)$ nerostoucí

1) $(x+\alpha)^p \ge 0$ ✓ $f(x) > 0$ ✓
$x \ge 1$

2) $f' = [(x+\alpha)^{-p}]' = -p\,(x+\alpha)^{-p-1} = \frac{-p}{(x+\alpha)^{p+1}}$ <0, $\underbrace{-p}_{<0}$, $\underbrace{(x+\alpha)^{p+1}}_{>0}$ >0

$p > 0$

$p \le 0$ není splněna NPK $\rightsquigarrow$ řada D

$\sum^{\infty} \underbrace{(n+\alpha)^{-p}}_{a_n}$

$\lim_{n\to\infty} a_n \begin{cases} \infty, & p<0 \\ 1, & p=0 \end{cases}$

$p>0$ $\int_N^{\infty} \frac{1}{(x+\alpha)^p}\,dx = \left| \begin{matrix} t = x+\alpha \\ dt = dx \end{matrix} \right| = \int_{N+\alpha}^{\infty} \frac{1}{t^p}\,dt = \int_{N+\alpha}^{\infty} t^{-p}\,dt =$ (*) $p=1$?

$\underset{p\neq 1}{=} \left[\frac{t^{-p+1}}{-p+1} \right]_{N+\alpha}^{\infty} = \lim_{t\to\infty} \frac{1}{(-p+1)\,t^{p-1}} - \underbrace{\frac{1}{(-p+1)(N+\alpha)^{p-1}}}_{\in \mathbb{R}}$ $p \neq 1$ $p=1$ $\frac{1}{0}$

$\lim_{t\to\infty} \frac{1}{t^{p-1}} = \begin{cases} 0 & K \quad p-1>0 \to p>1 \\ \infty & D \quad p-1<0 \to p<1 \\ \cancel{1 \quad K \quad p=1} \end{cases}$

(*) $\int_{N+\alpha}^{\infty} \frac{1}{t}\,dt = [\ln t]_{N+\alpha}^{\infty} = \lim_{t\to\infty} \ln t - \ln(N+\alpha) = \infty$ D

($\ln t \to \infty$)

řada pro $p \le 1$ diverguje pro $p > 1$ konverguje

8) $\sum_{n=2}^{\infty} \underbrace{\frac{\ln n}{n^2}}_{a_n}$ K/D? dle integrální krit. řada K

$$f(x) = \frac{\ln x}{x^2} \geq 0 \qquad x \geq 3$$

$$f' = \frac{\frac{1}{x} x^2 - 2x \ln x}{x^4} = \frac{\overbrace{1 - 2\ln x}^{<0}}{\underbrace{x^3}_{>0}} < 0 \text{ na } (3,\infty)$$

f je klesající

$$\int_3^{\infty} \frac{\ln x}{x^2}\,dx = \left| \begin{matrix} u = \ln x & u' = \frac{1}{x} \\ v' = \frac{1}{x^2} & v = -\frac{1}{x} \end{matrix} \right| =$$

$$= \left[-\frac{\ln x}{x}\right]_3^{\infty} + \int_3^{\infty} \frac{1}{x^2}\,dx = -\lim_{x\to\infty} \frac{\ln x}{x} \overset{\frac{\infty}{\infty}}{\to 0} + \frac{\ln 3}{3} - \left[\frac{1}{x}\right]_3^{\infty} =$$

$$= \frac{\ln 3}{3} - \lim_{x\to\infty} \frac{1}{x} + \frac{1}{3} = \frac{\ln 3 + 1}{3} < \infty \qquad \frac{\frac{1}{x}}{1} \to 0$$

integrál K

9) $\sum_{n=1}^{\infty} \frac{n!\, n^{-1}}{\underbrace{q(q+1)\cdot\ldots\cdot(q+n)}_{\prod_{i=0}^{n}(q+i)}}$ $(n-1)!$ K/D?

$q>0$

$$a_n = \frac{(n-1)!}{\prod_{i=0}^{n}(q+i)} > 0$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n!}{\prod_{i=0}^{n+1}(q+i)}}{\frac{(n-1)!}{\prod_{i=0}^{n}(q+i)}} = \frac{n\cdot\cancel{(n-1)!}\,\cancel{\prod_{i=0}^{n}(q+i)}}{\cancel{(n-1)!}\,(q+n+1)\cdot\cancel{\prod_{i=0}^{n}(q+i)}} = \boxed{\frac{n}{q+n+1}}$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{1}{1} = 1$$ nevíme rozhodnout

raabeho krit.

$$\lim_{n\to\infty} n\left(1-\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} n\left(1-\frac{n}{q+n+1}\right) =$$

$$= \lim_{n\to\infty} n\,\frac{q\cancel{+n}+1\cancel{-n}}{q+n+1} = \lim_{n\to\infty}\frac{n(q+1)}{q+n+1} = \left|\frac{\infty}{\infty}\right| = \frac{q+1}{1} =$$

$$= q+1 > 1$$

podle raabeho krit. řada $\underline{\underline{K}}$

řada konverguje

(10) $$\sum_{n=2}^{\infty} \underbrace{\frac{1}{n^3 \ln n}}_{a_n \geq 0} \qquad K/D$$

a_n je nerostoucí ✓
$n^3 \ln n$ je součin ros., tedy roste
$\Rightarrow \frac{1}{n^3 \ln n}$ klesá

$$\lim_{n\to\infty} \frac{\frac{1}{n^3 \ln n}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{1}{\ln n} = 0 \qquad K$$

$$\sum^{\infty} \frac{1}{n^3} \; K$$

$$a_n \rightarrow 2^k a_{2^k}$$

$$\sum_{k=1}^{\infty} 2^k a_{2^k} = \sum_{k=1}^{\infty} \frac{1}{(2^k)^3 \ln 2^k} 2^k = \frac{1}{\ln 2} \sum_{k=1}^{\infty} \frac{1}{2^{3k} \cdot k} \quad K/D ?$$

konv.

$$\lim_{k\to\infty} \sqrt[k]{\frac{1}{2^{3k} \cdot k}} = \lim_{k\to\infty} \frac{1}{8 \cdot \sqrt[k]{k}} = \frac{1}{8} < 1 \text{ řada } K$$

$$\sqrt[k]{k} \rightarrow 1$$