Algebra III exercises, fall semester 2021-22

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1. Describe coproduct in the category **Ab** of abelian groups and group homomorphisms. What is the relation between these and coproducts in **Grp**?
[2 points]

2. A generating set for an algebra A is a set $X \subseteq A$ such that $\langle X \rangle$. A basis is a generating set X such that for each $a \in X$, then $\langle X \setminus \{a\} \rangle \neq A$.
Prove that an algebra need not have a basis. Prove that if A is finitely generated, then it has a finite basis.
[2 points]

3. Let consider an algebra structure on $\mathbb{N}$ with one unary operation $-^{\odot}$ defined by $0^{\odot} = 0$ and for $n > 0$, $n^{\odot} = n - 1$. Prove that $\mathbb{N}$ has no basis.
[1 point]

4. Consider the set $\{a, b, c, d\}$ with no operations. How many congruences does it have? Now endow it with a unary operation defined by $a \mapsto b, b \mapsto a, c \mapsto d, d \mapsto c$. How many congruences are there now?
[1 point]

5. An algebra is simple if its only congruences are the diagonal $\Delta_A \subseteq A \times A$ and $A \times A$ itself.
Consider an algebra A with one ternary operation t defined by
$$t(x, y, z) = \begin{cases} z & \text{if } x = y \\ x & \text{if } x \neq y. \end{cases}$$
Prove that A is simple.
[1 point]

6. Given a monad T on a category $\mathcal{C}$, we know that there are two functors $F^T : \mathcal{C} \leftrightarrows \mathrm{Alg}(T) : U^T$. Prove that these form an adjunction $F^T \dashv U^T$.
[2 points]

7. A split fork is a diagram $a \overset{f_0}{\underset{f_1}{\rightrightarrows}} b \xrightarrow{e} c$ for which there exist arrows $a \xleftarrow{t} b \xleftarrow{s} c$ in such a way that $ef_0 = ef_1$, $es = 1_c$, $f_0 t = 1_b$ and $f_1 t = se$.
Prove that a split fork is always an absolute coequalizer.
[1 point]

8. Prove that the free group monad is finitely accessible.
[2 points]

9. Describe the algebras of the monad $(-)^{\mathbb{N}}$.
[1 point]

10. Prove that, if $|X| = \lambda$, then the monad $(-)^X$ is λ^+-accessible, but it is not μ-accessible for any $\mu \leq \lambda$.
[2 points]

11. There is a free point monad on sets given by $X \mapsto X \coprod \bullet$. The multiplication $X \coprod \bullet \coprod \bullet \to X \coprod \bullet$ is defined by sending X to X and both additional points to the unique additional points in the codomain. The unit is the inclusion $X \hookrightarrow X \coprod \bullet$.
Establish whether this monad is λ-accessible for some λ, and describe its algebras.
[2 points]

12. Prove that a sequence of R-modules $A \to B \to C \to 0$ is exact if and only if for every R-module N the induced sequence

$$0 \to \mathrm{Hom}(C, N) \to \mathrm{Hom}(B, N) \to \mathrm{Hom}(A, N)$$

is exact.
[2 points]

13. Prove that a sequence $A \to B \to C \to 0$ of R-modules is exact if and only if the square $\begin{array}{ccc} A & \xrightarrow{u} & B \\ \downarrow & & \downarrow v \\ 0 & \longrightarrow & C \end{array}$ is a pushout.
[2 points]

14. Find an example of a non-flat module.
[2 points]

15. Find two modules M and N with finitely generated submodules M_0 and N_0 respectively, such that there is a non-zero tensor $\sum m_i \otimes n_i \in M_0 \otimes N_0$ which is zero in $M \otimes N$.
[1 point]

16. Remember that a poset I is directed if every finite subposet I_0 admits an upper bound in I. A diagram in a category is called directed if it is indexed by a directed poset. Give an explicit description of the colimit of a directed diagram.
(Hint: if all the maps in a given directed diagram are inclusions, then the colimit is simply the union of all the involved modules.)
[2 points]

17. Show that the operation of tensoring commutes with directed colimits, i.e. if A is a module and $(B_i)_{i \in I}$ is a directed diagram, then the natural map

$$\mathrm{colim}_{i \in I}(A \otimes B_i) \to A \otimes \mathrm{colim}_{i \in I} B_i$$

is an isomorphism.
[2 points]

18. A ring is called a PID (principal ideal domain) if every ideal is of the form $I = (a)$, i.e. generated by a single element. Show that, if R is a PID, then a quotient of an injective R-module is injective.
[1 point]

19. Show that every module admits both a projective and an injective resolution.
[2 points]

20. Show that the homology and the cohomology functors are additive, i.e. for two maps $f, g : C \to D$ between chain complexes, we have $H_n(f + g) = H_n(f) + H_n(g)$ and similarly for cohomology.
[2 points]

21. Show that, for a left (resp. right) exact functor F there is a natural isomorphism $R_0 F \cong F$ (resp. $L_0 F \cong F$).
[1 point]

22. If B is a $\mathbb{Z}$-module and p is a natural number, compute the modules $\mathrm{Ext}^n(\mathbb{Z}/p, B)$ for all n. (Hint: $\mathbb{Z}$ is a PID.)
Also, compute all modules $\mathrm{Tor}^n(\mathbb{Z}/p, B)$.
[3 points]

23. Prove at least one between (a) and (b) and at least one between (c) and (d) among the following statements:

    (a) A is projective $\Leftrightarrow \forall B\ \mathrm{Ext}^1(A, B) = 0 \Leftrightarrow \forall B\ \forall n > 0\ \mathrm{Ext}^n(A, B) = 0$;
    (b) B is injective $\Leftrightarrow \forall A\ \mathrm{Ext}^1(A, B) = 0 \Leftrightarrow \forall A\ \forall n > 0\ \mathrm{Ext}^n(A, B) = 0$;
    (c) A is flat $\Leftrightarrow \forall B\ \mathrm{Tor}^1(A, B) = 0 \Leftrightarrow \forall B\ \forall n > 0\ \mathrm{Tor}^n(A, B) = 0$;
    (d) B is flat $\Leftrightarrow \forall A\ \mathrm{Tor}^1(A, B) = 0 \Leftrightarrow \forall A\ \forall n > 0\ \mathrm{Tor}^n(A, B) = 0$.

    [4 points]

24. Show that if U is an exact functor and F is right exact, then we have natural isomorphisms $U L_n F \cong L_n U F$.
[2 points]

25. Show that $\mathrm{Tor}^n(A, B)$ commutes with finite sums in both variables.
[2 points]